FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH 255

Analysis 2

Examiner: Professor S. W. Drury | Date: Tuesday, April 26, 2005

Associate Examiner: Professor K. N. GowriSankaran | Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

All six questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
No calculators are allowed.

All questions are of equal weight; each is worth 20 marks.
The exam will be marked out of a total of 120 marks
and subsequently scaled to a percentage.

This exam has 6 questions and 4 pages

1. (i) (4 points) Define the term *metric space*.

(ii) (4 points) For A a subset of a general metric space (X, d), define the closure of A.

(iii) (4 points) Define the term *normed space* and explain how every normed space can be interpreted as a metric space.

(iv) (8 points) Let V be a general normed space with norm $\| \ \|$. Show that the closure of the subset $\{x \in V; \|x\| < 1\}$ is exactly $\{x \in V; \|x\| \leq 1\}$.

2. Consider the rearrangement of the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \qquad (\mathbf{A})$$

into repeated blocks of 3 terms, each consisting of 2 positive terms followed by 1 negative term, where terms of the same sign are taken in the same relative order as in the original series (**A**). This rearrangement would commence

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \cdots \qquad (\mathbf{B})$$

Consider also the corresponding bracketted rearranged series

$$\left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \left(\frac{1}{9} + \frac{1}{11} - \frac{1}{6}\right) + \left(\frac{1}{13} + \frac{1}{15} - \frac{1}{8}\right) + \cdots \qquad (\mathbf{C})$$

(i) (4 points) What is the sum of the original series (**A**)? State a theorem that would allow you to prove this, given that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \ln(1+x)$ for $-1 < x < 1$.

(ii) (4 points) For k a positive integer, show the sum of the first $3k$ terms (i.e. k blocks) of the rearranged series (**B**) is given by

$$\sum_{n=1}^{2k}(-1)^{n-1}\frac{1}{n} + \sum_{n=k+1}^{2k}\frac{1}{2n-1}. \qquad (\mathbf{D})$$

(iii) (4 points) Use the method of the integral test to determine upper and lower bounds for the second sum in (**D**).

(iv) (4 points) Deduce that the bracketted rearranged series (**C**) converges and find its sum explicitly.

(v) (4 points) Prove, using the method of bracketting, that the rearranged series (**B**) converges to the same sum.

3. Let f be a bounded real valued function on the interval $[a, b]$.

(i) (2 points) Define the Riemann sum of f corresponding to a tagged partition.

(ii) (2 points) Define the Upper sum of f corresponding to a partition.

(iii) (2 points) Define the Lower sum of f corresponding to a partition.

(iv) (4 points) State Riemann's Criterion for integrability.

(v) (10 points) Suppose that f is a Riemann integrable function on $[a, b]$ and let $\epsilon > 0$. Show that there exists a continuous $g : [a, b] \longrightarrow \mathbb{R}$ such that $\int_a^b |f(x) - g(x)|dx < \epsilon$.

Hint: For a suitable partition P, consider defining g to be the function that is linear on each interval of the partition P and agrees with f at the endpoints of each such interval.

4. (i) (6 points) State the Fundamental Theorem of Calculus (the version that deals with continuous integrands).

(ii) (7 points) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Show that

$$\int_0^x (x - u)f(u)du = \int_0^x \left(\int_0^u f(t)dt \right) du.$$

(iii) (7 points) Deduce that

$$\int_0^x uf(u)du = \int_0^x \left(\int_u^x f(t)dt \right) du.$$

5. For each of the following sequences of functions $(f_n)_{n=1}^{\infty}$ defined on $\mathbb{R}$ determine (a) if a pointwise limit exists everywhere on $\mathbb{R}$, (b) if a uniform limit exists on each bounded subset of $\mathbb{R}$ and (c) if a uniform limit exists on $\mathbb{R}$. Justify your answers.

(i) (5 points) $f_n(x) = n \sin\left(\frac{x}{n}\right)$.

(ii) (5 points) $f_n(x) = (\cos(x))^{2n}$.

(iii) (5 points) $f_n(x) = xe^{-nx^2}$.

(iv) (5 points) $f_n(x) = \Big(\sin(nx)\Big)^n$.

6. (i) (3 points) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a convergent numerical series and that $0 < |t| < |x|$. Prove that $\sum_{n=0}^{\infty} a_n t^n$ is a convergent numerical series.

(ii) (2 points) Write down the power series expansion for $\sin x$ at $x = 0$, stating the general term and the radius of convergence.

(iii) (2 points) Write down the power series expansion for $\cos x$ at $x = 0$, stating the general term and the radius of convergence.

(iv) (2 points) Write down the power series expansion for $\sqrt{1-u}$ at $u = 0$. Although you need not write a formula for the general term it should be clear from your answer what the general term is. State also the radius of convergence.

(v) (4 points) Write down the statement of a general theorem that will allow you to conclude from (ii) and (iv) above that the function $\sqrt{1-\sin(x)}$ has a power series expansion about $x = 0$ with strictly positive radius of convergence.

(vi) (2 points) Find explicitly the power series expansion of $\sqrt{1-\sin(x)}$ about $x = 0$. Note: You are required to give a formula for the coefficient of x^n. There is no credit for finding the first few terms of the expansion. Hint: Consider $\left(\cos(\frac{1}{2}x) - \sin(\frac{1}{2}x)\right)^2$.

(vii) (2 points) Find the radius of convergence ρ of the series you have found in (vi). Justify your answer.

(viii) (3 points) If $g(x)$ is the sum of the power series expansion that you have found in (vi) above (defined for $|x| < \rho$), determine the largest open interval containing 0 on which $g(x) = \sqrt{1-\sin(x)}$. Justify your answer.

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