1. (i) (10 marks) State and prove the Cauchy–Schwarz inequality.
(ii) (10 marks) Let \( a_1, a_2, \ldots, a_n \) be positive numbers. By writing \( a_1 = (a_1 a_2^{-\frac{1}{2}}) a_2^{\frac{1}{2}} \),
\[
a_2 = (a_2 a_3^{-\frac{1}{2}}) a_3^{\frac{1}{2}}, \ldots, a_n = (a_n a_1^{-\frac{1}{2}}) a_1^{\frac{1}{2}}
\] or otherwise, show that
\[
a_1 + a_2 + \cdots + a_{n-1} + a_n \leq a_1^2 a_2^{-1} + a_2^2 a_3^{-1} + \cdots + a_{n-1}^2 a_n^{-1} + a_n^2 a_1^{-1}
\]

2. (i) (6 marks) Describe Riemann’s Criterion for Integrability.
(ii) (7 marks) If \( f \) is a Riemann Integrable function on \([0, 1]\) show that the function \(|f|\) defined by \(|f| (x) = |f(x)|\) is also Riemann Integrable on \([0, 1]\).
(iii) (7 marks) Let
\[
g(x) = \begin{cases} 
0 & \text{if } x \text{ is irrational}, \\
\frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms with } p \text{ and } q \text{ integers}.
\end{cases}
\]

Is \( g \) Riemann Integrable on \([0, 1]\)? Justify your answer.

3. (i) (5 marks) Let \( f(x) = e^{x^2} \int_0^x e^{-t^2} \, dt \). How is it possible to assert on theoretical grounds that \( f \) has a power series expansion about \( x = 0 \) with infinite radius?
(ii) (5 marks) Show that \( f'(x) = 1 + 2xf(x) \).
(iii) (5 marks) Find the power series expansion of \( f \) about \( x = 0 \) as far as the term in \( x^7 \).
(iv) (5 marks) Use the ratio test to verify that the radius of the series you have found is indeed infinite.

4. For each of the following sequences of functions defined on \( \mathbb{R} \) determine (a) if a pointwise limit exists everywhere on \( \mathbb{R} \), (b) if a uniform limit exists on each bounded subset of \( \mathbb{R} \) and (c) if a uniform limit exists on \( \mathbb{R} \).
(i) (7 marks) \( f_n(x) = \left(1 + \frac{x}{n}\right)^n \).
(ii) (6 marks) \( f_n(x) = \frac{x}{1 + nx^2} \).
(iii) (7 marks) \( f_n(x) = \cos(nx^2) \).

Justify your answers.
5. Let \( a_n > 0 \) and \( \sum_{n=1}^{\infty} a_n < \infty \). For each of the following statements, either provide a proof that the statement necessarily holds, or an example of a specific instance where it does not.

   (i) (7 marks) \( \sum_{n=1}^{\infty} n^2 a_n^3 < \infty \).

   (ii) (7 marks) \( \liminf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq 1 \).

   (iii) (6 marks) \( \sum_{n=1}^{\infty} \frac{a_n}{1 + a_n^2} < \infty \).

6. (i) (6 marks) State the Fundamental Theorem of Calculus.

(ii) (7 marks) Let \( g \) and \( h \) be two differentiable functions such that

   - \( g(0) = h(0) \)
   - \( g'(x) \leq h'(x) \) for \( x > 0 \)

   Show that \( g(x) \leq h(x) \) for \( x \geq 0 \).

(iii) (7 marks) Suppose that \( f \) is a differentiable function such that \( f(0) = 0 \) and \( 0 < f'(x) \leq 1 \) for all \( x > 0 \). Show that for \( x \geq 0 \)

\[
\int_{0}^{x} \left( f(t) \right)^3 \, dt \leq \left( \int_{0}^{x} f(t) \, dt \right)^2.
\]

Hint: Apply (ii) twice (at least).
INSTRUCTIONS

All six questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
No calculators are allowed.

All questions are of equal weight; each is worth 20 marks.
The exam will be marked out of a total of 120 marks
and subsequently scaled to a percentage.

This exam comprises the cover and 2 pages of questions.