McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-255B

ANALYSIS II

Examiner: Professor D. Jakobson
Associate Examiner: Professor S. Drury

Date: Monday, April 29, 2002
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Answer all questions.
Each question is worth 20 points.

This exam comprises the cover and 2 pages of questions.

FINAL EXAM

Do all the problems. Every problem is worth 20 points.

Problem 1. Establish the convergence/divergence for the series whose nth term is given by

a) **(6 points.)** $(n!)^2/(2^{n^2})$.
b) **(7 points.)** $2^n \cdot n!/(n^n)$ and $3^n \cdot n!/(n^n)$.
c) **(7 points.)**

$$\left(\frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot (2n)}\right)^p, \qquad p=1 \textit{ and } p=3.$$

Problem 2. Determine whether the following sequences of functions converge uniformly or pointwise (or neither) in the regions indicated. Determine the pointwise limits (where they exist); are the limiting functions continuous/differentiable, (in the latter case, do the derivatives converge uniformly)?

a) **(7 points.)**

$$f_n(x) = \begin{cases} \frac{\sin nx}{nx}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

for $x \in [-\pi, \pi]$.
b) **(6 points.)** $f_n(x) = x^2/(3+2nx^2)$ for $x \in [0,1]$.
c) **(7 points.)** $f_n(x) = e^{-nx}/n$ for $x \in [0,\infty)$.

Problem 3. Let $n_1 < n_2 < n_3 < \ldots$ be the numbers that don't use the digit 7 in their decimal expansion. Prove that

$$\sum_{k=1}^{\infty} \frac{1}{n_k}$$

converges.

Problem 4. Suppose that both the series

$$F(x) = \sum_{n=0}^{\infty} f^{(n)}(x)$$

and the series

$$G(x) = \int_0^x f(t_1)dt_1 + \int_0^x dt_1 \int_0^{t_1} f(t_2)dt_2 + \ldots$$

converge uniformly on some interval. What can you say about the function represented by the series $H(x) := F(x) + G(x)$?

Problem 5.

a) **(5 points.)** State the Lebesgue's integrability criterion.
b) **(5 points.)** Define sets of measure 0 (null sets) in $\mathbf{R}$.

c) **(5 points.)** Prove that if $g : [a,b] \to [c,d]$ is Riemann integrable on $[a,b]$, and if f is continuous on $[c,d]$, then $h(x) := f(g(x))$ is also Riemann integrable on $[a,b]$.

d) **(5 points.)** Give an example of a Riemann integrable function f on $[0,1]$ such that $\operatorname{sgn}(f(x))$ is not Riemann integrable.

Problem 6.

a) **(10 points.)** Let a_n be a sequence of real numbers such that $\sum a_n^2$ converges. Prove that $\sum(|a_n|/n)$ also converges.

b) **(10 points.)** Let $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$ be a monotone decreasing sequence of nonnegative numbers, and let $\sum_{n=1}^{\infty} a_n$ converge. Prove that

$$\lim_{n\to\infty} na_n = 0.$$