1. (i) (4 marks) Define the term metric space.
(ii) (4 marks) Define the term open subset of a metric space.
(iii) (4 marks) Define the term closed subset of a metric space.
(iv) (8 marks) Show from first principles that a subset of a metric space is closed if and only if its complement is open.

2. For each of the following series, determine whether the series converges. Justify your answer.
   (i) (5 marks) \( \sum_{n=1}^{\infty} \left( \sqrt{n+1} - \sqrt{n} \right) \).
   (ii) (5 marks) \( \sum_{n=1}^{\infty} \sin\left( \frac{n^2 + 1}{n} \right) \).
   (iii) (5 marks) \( \sum_{n=1}^{\infty} \frac{3^n n!}{n^n} \).
   (iv) (5 marks) \( \sum_{n=3}^{\infty} (\ln n)^{-\ln n} \).

3. (i) (4 marks) Define the term Riemann partition.
(ii) (4 marks) Define the upper and lower Riemann sums \( U(P,f) \) and \( L(P,f) \) for a Riemann partition \( P \).
Let \( f : [0,1] \rightarrow [-1,1] \) be defined by \( f(x) = (-1)^k \) if \( x \in [2^{-1(k+1)}, 2^{-k}] \) and \( f(0) = 0 \).
(iii) (8 marks) Given \( \epsilon > 0 \) find explicitly a Riemann partition \( P \) of \([0,1]\) with \( U(P,f) - L(P,f) < \epsilon \). Justify your answer. What is the significance of what you have just shown?
(iv) (4 marks) What is the value of \( \int_0^1 f(x)dx \)?

4. For each of the following sequences of functions defined on \([0, \infty]\) determine the point-wise limit. Determine also whether convergence is uniform on \([0, \infty]\). Justify your answer.
   (i) (10 marks) \( f_n(x) = \frac{\lceil nx \rceil}{n \lfloor x \rfloor} \).
   Note: The notation \( \lceil x \rceil \) denotes the unique integer \( k \) such that \( k - 1 < x \leq k \).
   (ii) (10 marks) \( f_n(x) = \frac{\sin(nx)}{nx} \).
5. (20 marks) Consider the power series 

\[ f(x) = x - \frac{3}{4} x^4 + \frac{3^2}{4 \cdot 7} x^7 - \frac{3^3}{4 \cdot 7 \cdot 10} x^{10} + \frac{3^4}{4 \cdot 7 \cdot 10 \cdot 13} x^{13} - \cdots \]

(i) (4 marks) What is the radius of convergence \( \rho \) of this series?
(ii) (6 marks) Show that \( f'(x) + 3x^2 f(x) = 1 \) for \( |x| < \rho \). Outline briefly the theorems that you are using.
(iii) (6 marks) Let \( g(x) = e^{-x^3} \int_{u=0}^{x} e^{u^3} \, du \). Show that \( g'(x) + 3x^2 g(x) = 1 \) for all real \( x \). Outline briefly the theorems that you are using.
(iv) (4 marks) Deduce that \( f(x) = g(x) \) for \( |x| < \rho \).

6. (i) (4 marks) If \( p > 0 \) and \( n \) is a nonnegative integer, show that the function \( x \mapsto x^p \left( - \ln(x) \right)^n \) is continuous on \([0, 1] \).
(ii) (4 marks) Show that the series \( \sum_{n=0}^{\infty} \frac{1}{n!} \left( - x \ln(x) \right)^n \) converges uniformly on \([0, 1] \).
(iii) (4 marks) State a theorem about the integral of a uniform limit.
(iv) (2 marks) Show that \( \int_{0}^{1} x^{-x} \, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{0}^{1} \left( - x \ln(x) \right)^n \, dx \).
(v) (4 marks) Let \( p > 0 \) and let \( n \) be a nonnegative integer. Show by induction that \( \int_{0}^{1} x^p \left( - \ln(x) \right)^n \, dx = \frac{n!}{(p + 1)^{n+1}} \).
(vi) (2 marks) Deduce that \( \int_{0}^{1} x^{-x} \, dx = \sum_{n=0}^{\infty} (n + 1)^{-(n+1)} \).

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INSTRUCTIONS

All six questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
No calculators are allowed.

All questions are of equal weight; each is worth 20 marks.
The exam will be marked out of a total of 120 marks and subsequently scaled to a percentage.

This exam comprises the cover and 2 pages of questions.