1. (i) (8 marks) State and prove a theorem about the interchange of limit and integral. Let \( f(t) = \int_{x=-1}^{1} \sin(tx^2) \, dx \).

(ii) (2 marks) Write down a series expansion for \( \sin(tx^2) \) in powers of \( t \). Justification is not required.

(iii) (8 marks) Using the expansion you have written down in (ii), find a valid power series expansion for \( f(t) \) about \( t = 0 \). Justify all steps.

(iv) (2 marks) Find the radius of convergence of the power series expansion you have found in (iii).

2. (i) (4 marks) Define the term metric space. Let \( X \) be a metric space with distance function \( d \) and let \( E \) be a nonempty subset of \( X \). Define a mapping \( d_E : X \rightarrow [0, \infty] \) by \( d_E(x) = \inf\{d(x, e); e \in E\} \).

(ii) (4 marks) Show that \( d_E(x) \leq d(x, x') + d_E(x') \).

(iii) (4 marks) Deduce that \( |d_E(x) - d_E(x')| \leq d(x, x') \).

(iv) (4 marks) Deduce that \( d_E : X \rightarrow [0, \infty] \) is a continuous function.

(v) (4 marks) If in addition \( E \) is a closed subset of \( X \), show that \( d_E(x) = 0 \iff x \in E \).

3. (i) (5 marks) State a theorem giving a condition for \( \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{p,q} = \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} a_{p,q} \)

where \( a_{p,q} \) are real numbers of either sign.

In this question, you may assume without proof that \( \sum_{p=1}^{q} (-1)^{p-1}(2p-1) = (-1)^{q-1}q \) and

\[
\sum_{q=p}^{\infty} \frac{1}{q(q+1)(q+2)} = \frac{1}{2p(p+1)}.
\]

Let \( a_{p,q} = \begin{cases} (-1)^{p-1} \frac{2p-1}{q(q+1)(q+2)} & \text{if } q \geq p, \\ 0 & \text{if } q < p. \end{cases} \)

(ii) (5 marks) Show that the hypotheses of the theorem you have stated in (i) are not adequate to show directly that \( \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{p,q} = \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} a_{p,q} \).

(iii) (5 marks) Let \( b_{p,q} = a_{2p-1,q} + a_{2p,q} \). Write down explicit formulae for \( b_{p,q} \) in each of the three cases \( 1 \leq q < 2p-1 \), \( q = 2p-1 \) and \( q \geq 2p \). Show that the hypotheses of the theorem you have stated in (i) are valid for \( b_{p,q} \). Deduce that \( \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} b_{p,q} = \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} b_{p,q} \).

(iv) (5 marks) Show that \( \sum_{p=1}^{\infty} (-1)^{p-1} \frac{2p-1}{2p(p+1)} = \sum_{q=1}^{\infty} (-1)^{q-1} \frac{1}{(q+1)(q+2)} \) in spite of (ii) above.
4. (i) (6 marks) Describe Riemann’s Criterion for Integrability.
(ii) (7 marks) If $f$ is a Riemann Integrable function on $[0, 1]$ show that the function $|f|$ defined by $|f|(x) = |f(x)|$ is also Riemann Integrable on $[0, 1]$.
(iii) (7 marks) Give an example where $|f|$ is Riemann Integrable on $[0, 1]$, but $f$ is not. Justify your example.

5. Let $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$. For each of the following statements, either provide a proof that the statement necessarily holds, or an example of a specific instance where it does not.
   (i) (7 marks) $\sum_{n=1}^{\infty} na_n^2 < \infty$.
   (ii) (7 marks) $\liminf_{n \to \infty} na_n = 0$.
   (iii) (6 marks) $\sum_{n=1}^{\infty} a_n e^{a_n} < \infty$.

6. For each of the following sequences of functions defined on $\mathbb{R}$ determine (a) if a pointwise limit exists everywhere on $\mathbb{R}$, (b) if a uniform limit exists on each bounded subset of $\mathbb{R}$ and (c) if a uniform limit exists on $\mathbb{R}$.
   (i) (7 marks) $f_n(x) = \frac{nx}{1 + n^2 x^2}$.
   (ii) (6 marks) $f_n(x) = ne^{-n|x|}$.
   (iii) (7 marks) $f_n(x) = xe^{-nx^2}$.
   Justify your answer.

7. (i) (3 marks) State the Fundamental Theorem of Calculus.
(ii) (3 marks) State a theorem about differentiating under the integral sign.
(iii) (3 marks) State a theorem about changes of variables in integrals.

Let $f(x) = \left( \int_{t=0}^{x} e^{-t^2} \, dt \right)^2$ and $g(x) = \int_{t=0}^{1} \frac{e^{-x^2(t^2+1)}}{t^2 + 1} \, dt$.

(iv) (7 marks) Show that $f'(x) + g'(x) = 0$.

(v) (4 marks) Deduce that $\lim_{x \to \infty} \int_{t=0}^{x} e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2}$.
FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-255B

Analysis II

Examiner: Professor S. W. Drury  Date: Friday, 14 April 2000
Associate Examiner: Professor K. N. GowriSankaran  Time: 2:00 pm. – 5:00 pm.

INSTRUCTIONS

All seven questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
No calculators are allowed.

All questions are of equal weight; each is worth 20 marks.
The exam will be marked out of a total of 140 marks
and subsequently scaled to a percentage.

This exam comprises the cover and 2 pages of questions.