1. (i) (8 marks) State and prove a theorem about the interchange of limit and integral.

Let $f(t) = \displaystyle\int_{x=-1}^{1} \sin(tx^2)\, dx$.

(ii) (2 marks) Write down a series expansion for $\sin(tx^2)$ in powers of t. Justification is not required.

(iii) (8 marks) Using the expansion you have written down in (ii), find a valid power series expansion for $f(t)$ about $t = 0$. Justify all steps.

(iv) (2 marks) Find the radius of convergence of the power series expansion you have found in (iii).

2. (i) (4 marks) Define the term *metric space*.

Let X be a metric space with distance function d and let E be a nonempty subset of X. Define a mapping $d_E : X \longrightarrow [0, \infty[$ by $d_E(x) = \inf\{d(x, e); e \in E\}$.

(ii) (4 marks) Show that $d_E(x) \le d(x, x') + d_E(x')$.

(iii) (4 marks) Deduce that $|d_E(x) - d_E(x')| \le d(x, x')$.

(iv) (4 marks) Deduce that $d_E : X \longrightarrow [0, \infty[$ is a continuous function.

(v) (4 marks) If in addition E is a closed subset of X, show that $d_E(x) = 0 \Leftrightarrow x \in E$.

3. (i) (5 marks) State a theorem giving a condition for $\displaystyle\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} a_{p,q} = \sum_{q=1}^{\infty}\sum_{p=1}^{\infty} a_{p,q}$ where $a_{p,q}$ are real numbers of either sign.

In this question, you may assume without proof that $\displaystyle\sum_{p=1}^{q}(-1)^{p-1}(2p-1) = (-1)^{q-1}q$ and that $\displaystyle\sum_{q=p}^{\infty} \frac{1}{q(q+1)(q+2)} = \frac{1}{2p(p+1)}$. Let $a_{p,q} = \begin{cases} (-1)^{p-1}\dfrac{2p-1}{q(q+1)(q+2)} & \text{if } q \ge p, \\ 0 & \text{if } q < p. \end{cases}$

(ii) (5 marks) Show that the hypotheses of the theorem you have stated in (i) are not adequate to show directly that $\displaystyle\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} a_{p,q} = \sum_{q=1}^{\infty}\sum_{p=1}^{\infty} a_{p,q}$.

(iii) (5 marks) Let $b_{p,q} = a_{2p-1,q} + a_{2p,q}$. Write down explicit formulae for $b_{p,q}$ in each of the three cases $1 \le q < 2p - 1$, $q = 2p - 1$ and $q \ge 2p$. Show that the hypotheses of the theorem you have stated in (i) are valid for $b_{p,q}$. Deduce that $\displaystyle\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} b_{p,q} = \sum_{q=1}^{\infty}\sum_{p=1}^{\infty} b_{p,q}$

(iv) (5 marks) Show that $\displaystyle\sum_{p=1}^{\infty}(-1)^{p-1}\frac{2p-1}{2p(p+1)} = \sum_{q=1}^{\infty}(-1)^{q-1}\frac{1}{(q+1)(q+2)}$ in spite of (ii) above.

4. (i) (6 marks) Describe Riemann's Criterion for Integrability.
   (ii) (7 marks) If f is a Riemann Integrable function on $[0,1]$ show that the function $|f|$ defined by $|f|(x) = |f(x)|$ is also Riemann Integrable on $[0,1]$.
   (iii) (7 marks) Give an example where $|f|$ is Riemann Integrable on $[0,1]$, but f is not. Justify your example.

5. Let $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$. For each of the following statements, either provide a proof that the statement necessarily holds, or an example of a specific instance where it does not.
   (i) (7 marks) $\sum_{n=1}^{\infty} n a_n^2 < \infty$.
   (ii) (7 marks) $\liminf_{n\to\infty} n a_n = 0$.
   (iii) (6 marks) $\sum_{n=1}^{\infty} a_n e^{a_n} < \infty$.

6. For each of the following sequences of functions defined on $\mathbb{R}$ determine (a) if a pointwise limit exists everywhere on $\mathbb{R}$, (b) if a uniform limit exists on each bounded subset of $\mathbb{R}$ and (c) if a uniform limit exists on $\mathbb{R}$.
   (i) (7 marks) $f_n(x) = \dfrac{nx}{1+n^2x^2}$.
   (ii) (6 marks) $f_n(x) = ne^{-n|x|}$.
   (iii) (7 marks) $f_n(x) = xe^{-nx^2}$.

Justify your answer.

7. (i) (3 marks) State the Fundamental Theorem of Calculus.
   (ii) (3 marks) State a theorem about differentiating under the integral sign.
   (iii) (3 marks) State a theorem about changes of variables in integrals.

   Let $f(x) = \left(\int_{t=0}^{x} e^{-t^2}\,dt\right)^2$ and $g(x) = \int_{t=0}^{1} \frac{e^{-x^2(t^2+1)}}{t^2+1}\,dt$.
   (iv) (7 marks) Show that $f'(x) + g'(x) = 0$.
   (v) (4 marks) Deduce that $\lim_{x\to\infty} \int_{t=0}^{x} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$.

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FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-255B

Analysis II

Examiner: Professor S. W. Drury
Associate Examiner: Professor K. N. GowriSankaran

Date: Friday, 14 April 2000
Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

All seven questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
No calculators are allowed.

All questions are of equal weight; each is worth 20 marks.
The exam will be marked out of a total of 140 marks
and subsequently scaled to a percentage.

This exam comprises the cover and 2 pages of questions.