Conventions: throughout this exam, F is a field with $0 \neq 1$, V is a vector space over F, W is a subspace of V, and T is a linear operator on V; also n is a natural number, and p is a prime number. Each problem is worth 10%

1. Here, $F = \mathcal{Z}_7$. Let $A = \begin{pmatrix} 1 & 2 & 4 & 2 \\ 3 & 6 & 4 & 1 \\ 5 & 3 & 2 & 0 \end{pmatrix}$; $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

   Find a basis for the row space , column space and kernel of A. Find the set of solutions to $A\vec{v} = \vec{b}$.

2. Let V be the set of those polynomials P of degree ≤ 4 over $\mathcal{R}$, and $T(P) = P' + 2P$. What are the eigenvalues and eigenvectors of T? Give the matrix A of T with respect to the usual basis $\{1, x, x^2, x^3, x^4\}$. Give the Jordan canonical form J of T, and a matrix Q such that $Q^{-1}AQ = J$.

3. Solve the following system of differential equations:

$$\begin{array}{rcrcrcr} y_1' & = & y_1 & + & y_2 & + & -6y_3 \\ y_2' & = & -4y_1 & - & 3y_2 & + & 13y_3 \\ y_3' & = & & & & & -y_3 \end{array} .$$

4. Show that if F is finite, it cannot be algebraically closed; conclude that the algebraic closure of a finite field is infinite, but countable. [Hint: Recall that if F has $q = p^n$ elements, then it is the splitting field over $\mathcal{Z}_p$ of $x^q - x$.]

5. Let $\vec{v} \in V$ be any nonzero vector. Show that there is a subspace $W \leq V$ maximal such that:

   (1) $\vec{v} \notin W$, and (2) W is T-invariant.

   Give an example to show that in general W need not be a maximal proper subspace of V.

6. Show that if W is T-invariant and T_1 is another linear operator on V such that $T \circ T_1 = T_1 \circ T$, then the image $T_1(W)$ is also T-invariant. Give an example to show that it need not be T-invariant if we don't assume $T \circ T_1 = T_1 \circ T$.

7. Let $< \cdot | \cdot >$ be an inner product on V (over $F = \mathcal{R}$ for this problem). Suppose that $S = \{\vec{v}_i : i \in I\}$ is a collection of nonzero vectors in V which are pairwise orthogonal (i.e., $\vec{v}_i \perp \vec{v}_j$ whenever $i \neq j$). Show that S is independent.

8. Recall that a matrix B over the complex numbers $\mathcal{C}$ is called *Hermitian* if $B = \bar{B}^t$, and let's call C *skew-Hermitian* if $C = -\bar{C}^t$. Show that, for any matrix A over $\mathcal{C}$, $A + \bar{A}^t$ is Hermitian. Show that any matrix over $\mathcal{C}$ can be represented uniquely as the sum of a Hermitian and a skew-Hermitian matrix.

9. Recall that $< f, g >= \int_{-1}^{1} f(t)g(t)dt$ is an inner product on the set of real-valued polynomials of degree ≤ 3. Use the Gram-Schmidt process to change the standard basis $\{1, t, t^2, t^3\}$ into an orthonormal basis. (Orthonormal meaning that the vectors are pairwise orthogonal and each vector has norm one.)

10. A *permutation matrix* is square and has exactly one 1 in each row and column, all the other entries being 0. Show that, under matrix multiplication, the set of $n \times n$ permutation matrices forms a group isomorphic to the symmetric group S_n.