1. (10 marks) Find the distance of $(1, 2, 3)$ from the range of $T_A$ where $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

2. (25 marks) Let $T \in L(P_3)$ be defined by $Tp(t) = D^2p(t) + tDp(t) - p(t)$, where $D$ denotes the operator of differentiation with respect to t.

   (a) Find the matrix of $T$ with respect to the standard basis of $P_3$.

   (b) Find the characteristic polynomial of $T$ and verify that $\sigma(T) = \{-1, 0, 1, 2\}$.

   (c) Find a basis for $P_3$ consisting of eigenvectors of $T$ and write down the matrix of $T$ with respect to that basis. What is the relationship of that matrix with respect to the basis of eigenvectors. Write down the relationship to the matrix you have calculated in (a).

   (d) Evaluate $[T^3 + 2T^2]t^2$.

   (e) Evaluate $\det(2I + 3T)^2$.

   (f) Find the solution set of $T^2p(t) = 1 + t^2$.

3. (25 marks) Let

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

   (a) Diagonalize A using an orthogonal matrix.

   (b) Find a symmetric matrix C such that $2C^2 + 4C = A$.

   (c) For which values of the real parameter $\alpha$ is $A^5 + \alpha A + 4I$ invertible?

   (d) Write out the quadratic function $q(x) = Ax \cdot x$ and describe the nature of the critical point.

   (e) Find the maximum of $q(x) = Ax \cdot x$ on the sphere $|x| = 3$ and find all points where this is achieved.

4. (10 marks) Suppose it is known that $1 + 2i$ and $5$ are eigenvalues of $A \in R^{4 \times 4}$ and that $\det A$ = -50.

   (a) Find the trace of $A$.

   (b) Decide whether $A$ is diagonalizable, justifying your answer.

   (c) Find expressions for $A^{-1}$ and $A^6 + A^3$ as polynomials in A of degree smaller than 4.

5. (10 marks) Suppose that $-1+2i$ and 3 are eigenvalues of a given real $3 \times 3$ matrix $A$ corresponding to eigenvectors $(1, 1+i, i)$ and $(1, 0, 1)$.

   (a) Write down a basis for the real solutions of $\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t)$.

   (b) Find the solution corresponding to the initial values $\mathbf{x}(0) = (2, 1, 4)$.

   (c) For which real initial vectors does the corresponding solution tend to 0 as $t$ tends to $\infty$?

6. (20 marks) Answer 4 of the following 5 questions:

   (a) Let $T \in L(V)$ be invertible and $\{v^1, \cdots, v^n\}$ be independent. Show that $\{Tv^1, \cdots, Tv^n\}$ is again independent.

   (b) Let $\mathbf{x}$ be a non-zero vector in $R^n$. Show that if we write $\mathbf{x}$ as a row, then $P = \mathbf{x}^t\mathbf{x}$ is the matrix of orthogonal projection onto Span(x)

   (c) Suppose that $T$ is diagonalizable with eigenvalues 1 and 4. Find all values of the parameters $\alpha$ and $\beta$ for which $\alpha I + \beta T$ is a projection.

   (d) Suppose $A \in R^{n \times n}$ is skew symmetric (so that $A^t = -A^t$). Show that the eigenvalues are purely imaginary and that eigenvectors corresponding to different eigenvalues are orthogonal with respect to the standard inner product on $C^n$.

   (e) Let $A \in R^{m \times n}$. Show that $(I + A^tA)$ is invertible and that the eigenvalues of $(I + A^tA)^{-1}$ lie between 0 and 1.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-247B

LINEAR ALGEBRA

Examiner: Professor G. Schmidt — Date: Wednesday, April 28, 1999
Associate Examiner: Professor S. Zlobec — Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

**Answer all questions.**
**Calculators are NOT permitted.**

This exam comprises the cover and 2 pages of questions.