1. Let $A$ be a $(n \times n)$-matrix over the field $F$.
   
   (a) Define the nullspace $N_A$ of $A$ and the column space $C_A$ of $A$.
   
   (b) Show that if $A^2 = 0$, then $C_A \subseteq N_A$.
   
   (c) Show that if $A^2 = A$, then $C_A \cap N_A = 0$ and $F^n = C_A + N_A$.

2. Let

   \[ A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ -1 & 1 & -2 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & -1 \end{pmatrix}. \]

   With $N_A$ and $C_A$ as in question 1, find $\dim N_A$ and $\dim C_A$. Determine a basis for $C_A \cap N_A$.

3. Let $V$ be the vector space of complex polynomials of degree $\leq 2$. For $z \in \mathbb{C}$, define

   \[ \varepsilon_z : V \to \mathbb{C} \]

   by $\varepsilon_z(p) = p(z)$.

   (a) Show that $\varepsilon_z$ is a linear map.

   (b) Let $z_1 = 1$, $z_2 = -1$ and $z_3 = i$. Let $\lambda_i = \varepsilon_{z_i}$. Show that $\{\lambda_1, \lambda_2, \lambda_3\}$ is a basis of the vector space $\hat{V}$ of linear maps from $V$ to $\mathbb{C}$.

   (c) Express $\varepsilon_0$ as a linear combination of $\lambda_1, \lambda_2, \lambda_3$.

4. Let $A$ be the $(n \times n)$-matrix

   \[ A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \]

   (A has entries $1$ just above and below the diagonal, i.e. $a_{ij} = 1$ if $|i - j| = 1$, and all other entries are zero.) Compute $\det(A)$ (in terms of $n$).
5. Let

\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

(a) Determine the characteristic polynomial \( \chi_A \), the eigenvalues of \( A \) and bases for the eigenspaces.

(b) Find the minimal polynomial \( \mu_A \) and determine the Jordan canonical form of \( A \). Justify your answer.

(c) Compute \( A^{10} \). Hint: \( A^r = ((A - I) + I)^r \).

6. Let \( V \) be \( \mathbb{R}^4 \) with the standard inner product. Let \( U \) be the subspace generated by \((1, 1, 1, 1)\), and \((0, 1, 1, 1)\).

(a) Find orthonormal bases for \( U \) and for \( U^\perp \).

(b) Find the vector \( u \in U \) that is closest to \( e_1 = (1, 0, 0, 0) \).

7. Find an orthogonal matrix \( P \) that diagonalizes the quadratic form

\[ Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3. \]

8. Let

\[ A = \begin{pmatrix} 0 & 0 & 3 + 4i \\ 0 & 0 & 0 \\ 3 - 4i & 0 & 0 \end{pmatrix}. \]


(b) Find a unitary matrix \( U \) so that \( U^*AU \) is diagonal. Give an a priori reason why this is possible.