1. (a) Let

$$f(x) = \begin{cases} 0\,, & -1 \le x \le 0, \\ 1\,, & 0 < x \le 1, \end{cases}$$

and let

$$F(x) = \int_{-1}^{x} f(t)dt, \; x \in [-1, 1].$$

Is F differentiable on $[-1,1]$? Does f have an antiderivative on $[-1,1]$?

(b) Suppose that f is a continuous function on $[0, \infty)$ such that $f(x) \neq 0$ for all $x > 0$. Show that if

$$(f(x))^2 = 2\int_0^x f \text{ for all } x > 0$$

then $f(x) = x$ for all $x \geq 0$.

2. (a) State the Riemann Criterion for integrability.

(b) Let $[a, b]$ be a closed and bounded interval and let $f : [a, b] \to \mathbb{R}$ be a bounded function with a finite number of discontinuities. Show that f is integrable on $[a, b]$.

3. (a) Let $A \subseteq \mathbb{R}$ and let (f_n) be a sequence of bounded functions on A. Show that if (f_n) converges uniformly on A to a function f, then f is bounded on A.

(b) Let $f_n(x) = \dfrac{nx}{1 + nx^2}$, $x \in [0, +\infty)$. Show that (f_n) is not uniformly convergent on $[0, +\infty)$.

(c) Let $\varepsilon > 0$. Show that the sequence (f_n) in (b) is uniformly convergent on $[\varepsilon, +\infty)$.

4. (a) State the Weierstrass M-test.

(b) Show that the series of functions

$$\sum_{n=1}^{\infty} \frac{x^2}{x^2 + n^2}$$

is uniformly convergent on any bounded subset of $\mathbb{R}$.

(c) Is the series in (b) uniformly convergent on $\mathbb{R}$?

5. Given the power series $\sum_{n=0}^{\infty} (n+1)x^n$,

(a) find the radius of convergence.

(b) Find the sum of the power series on the interval of convergence.

(c) Is the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ convergent? If yes, calculate the sum.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-243B

REAL ANALYSIS

Examiner: X. Zhang | Date: Tuesday, April 25, 2000
Associate Examiner: Professor I. Klemes | Time: 9:00 A.M. - 12:00 Noon.

INSTRUCTIONS

Each question is worth 20 marks.

This exam comprises the cover and one page of questions.