1. (a) Let 
\[ f(x) = \begin{cases} 
0, & -1 \leq x \leq 0, \\
1, & 0 < x \leq 1, 
\end{cases} \]
and let 
\[ F(x) = \int_{-1}^{x} f(t) \, dt, \quad x \in [-1, 1]. \]
Is \( F \) differentiable on \([-1, 1]\)? Does \( f \) have an antiderivative on \([-1, 1]\)?

(b) Suppose that \( f \) is a continuous function on \([0, \infty)\) such that \( f(x) \neq 0 \) for all \( x > 0 \).
Show that if 
\[(f(x))^2 = 2 \int_{0}^{x} f \text{ for all } x > 0 \]
then \( f(x) = x \) for all \( x \geq 0 \).

2. (a) State the Riemann Criterion for integrability.

(b) Let \([a, b]\) be a closed and bounded interval and let \( f : [a, b] \to \mathbb{R} \) be a bounded function with a finite number of discontinuities. Show that \( f \) is integrable on \([a, b]\).

3. (a) Let \( A \subseteq \mathbb{R} \) and let \( (f_n) \) be a sequence of bounded functions on \( A \). Show that if \( (f_n) \) converges uniformly on \( A \) to a function \( f \), then \( f \) is bounded on \( A \).

(b) Let \( f_n(x) = \frac{nx}{1 + nx^2}, \quad x \in [0, +\infty) \). Show that \( (f_n) \) is not uniformly convergent on \([0, +\infty)\).

(c) Let \( \varepsilon > 0 \). Show that the sequence \( (f_n) \) in (b) is uniformly convergent on \([\varepsilon, +\infty)\).

4. (a) State the Weierstrass M-test.

(b) Show that the series of functions 
\[ \sum_{n=1}^{\infty} \frac{x^2}{x^2 + n^2} \]
is uniformly convergent on any bounded subset of \( \mathbb{R} \).

(c) Is the series in (b) uniformly convergent on \( \mathbb{R} \)?

5. Given the power series \( \sum_{n=0}^{\infty} (n + 1)x^n \),

(a) find the radius of convergence.

(b) Find the sum of the power series on the interval of convergence.

(c) Is the series \( \sum_{n=1}^{\infty} \frac{n}{2^n} \) convergent? If yes, calculate the sum.
McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-243B

REAL ANALYSIS

Examiner: X. Zhang Date: Tuesday, April 25, 2000
Associate Examiner: Professor I. Klemes Time: 9:00 A.M. - 12:00 Noon.

INSTRUCTIONS

Each question is worth 20 marks.

This exam comprises the cover and one page of questions.