1. (i) (4 marks) State the Bernoulli Inequality.
   (ii) (4 marks) State the Cauchy–Schwarz Inequality
   (iii) (12 marks) Prove by induction on $n$ that

$$1+\sum_{j=1}^{n} a_j \leq \prod_{j=1}^{n}(1+a_j)$$

whenever $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ are nonnegative real numbers.

2. (i) (4 marks) State a theorem about the convergence of monotone sequences of real numbers.
   (ii) (4 marks) State the Nested Intervals Theorem.
   (iii) (12 marks) Let $c > 0$. A sequence satisfies $x_1 > 0$ and $x_{n+1} = \sqrt{c + x_n}$. Show that $(x_n)_{n=1}^{\infty}$ is a monotone sequence and that it converges to $\dfrac{1+\sqrt{1+4c}}{2}$. Be sure to explain why the hypotheses of the theorem you have stated in (i) above apply.

3. (i) (4 marks) What is meant by a Cauchy sequence of real numbers?
   (ii) (4 marks) Let $f : ]0, \infty[ \longrightarrow \mathbb{R}$ be a uniformly continuous function. If $(x_n)$ is a Cauchy sequnce in $]0, \infty[$ show that $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$.
   (iii) (12 marks) Let $f : ]0, \infty[ \longrightarrow \mathbb{R}$ be a uniformly continuous function. What can be said about $\lim_{x \to 0+} f(x)$? Justify your answer in detail.

4. (i) (4 marks) Define the term "subsequence" of a sequence of real numbers
   (ii) (4 marks) State the Bolzano-Weierstrass Theorem.

Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers and let $s = \sup\{x_n; n \in \mathbb{N}\}$. Suppose that $s \neq x_n$ for all $n \in \mathbb{N}$.

   (iii) (6 marks) Show that for each natural number $N$ we have

$$s = \sup\{x_n; n \in \mathbb{N}, n > N\}.$$

   (iv) (6 marks) Show that $(x_n)_{n=1}^{\infty}$ has a subsequence converging to $s$.

5. (i) (4 marks) State the Bolzano Intermediate Value Theorem.
(ii) (8 marks) Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous, $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in [a, b]$. Show that there necessarily exists $x \in [a, b]$ such that

$$f(x) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

(iii) (8 marks) Show that the equation

$$\frac{1}{\sqrt{x + x^2}} + x^2 = 2x$$

has at least two solutions in $]0, \infty[$.

6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$.
(i) (4 marks) What does it mean for $f$ to be continuous?
(ii) (4 marks) What does it mean for $f$ to be uniformly continuous?
(iii) (12 marks) If $f$ is continuous and $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ both exist and are finite, show that $f$ is uniformly continuous. State carefully any theorem that you use.

7. (i) (4 marks) State Rolle's Theorem.
(ii) (4 marks) State the Mean Value Theorem.
(iii) (6 marks) Show that for all $c \in \mathbb{R}$, the equation $x^3 - 3x = c$ does not have two distinct solutions in $[0, 1]$.
(iv) (6 marks) Show that for all $a, b \in \mathbb{R}$, the equation $x^4 - 6x^2 = ax + b$ does not have three distinct solutions in $[0, 1]$.

⋆ ⋆ ⋆ ⋆ ⋆

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-242A

Analysis I

Examiner: Professor S. W. Drury
Associate Examiner: Professor K. N. GowriSankaran

Date: Friday, December 10, 1999
Time: 9: 00 am. – 12: 00 noon

INSTRUCTIONS

**All seven questions should be attempted for full credit.**

**This is a closed book examination.**
**Write your answers in the booklets provided.**
**No calculators are allowed.**

**All questions are of equal weight; each is worth 20 marks.**
**The exam will be marked out of a total of 140 marks**
**and subsequently scaled to a percentage.**

This exam comprises the cover and 2 pages of questions.