

Mathematics 242 Analysis I  
McGILL UNIVERSITY

FINAL EXAMINATION  
No computers, notes or books allowed.  
All questions counted equally.

Thursday December 13, 2007

Time: 2:00 PM-5:00 PM

1. Define:

- (a) Least upper bound of a set  $S \subset \mathbb{R}$ ;
- (b)  $X = (x_k)$ ,  $x_k \in \mathbb{R}$ , is a convergent sequence;
- (c) (i)  $\lim_{x \rightarrow c} f = L$ , (ii)  $\lim_{x \rightarrow \infty} g = K$ , (iii)  $f$  is continuous at  $c$ ;
- (d) the real number  $m$  is the derivative of  $f$  at  $x = c$ .

2. (a) Prove that  $\lim_{n \rightarrow \infty} (\sqrt[3]{n+1} - \sqrt[3]{n}) = 0$ .

- (b) Let  $S = \{\sqrt[3]{n+1} - \sqrt[3]{m} : n, m \in \mathbb{N}\}$ . If  $x$  and  $y$  are any real numbers with  $x < y$ , show that there exists an  $s \in S$  such that  $x < s < y$ .

3. (a) Prove that  $\lim_{n \rightarrow \infty} n^{1/n} = 1$ .

- (b) State and prove the Squeeze Theorem for sequences.
- (c) Find  $\lim_{n \rightarrow \infty} (n!)^{1/n^2}$ . Justify your answer.

4. (a) Let  $f, g$  be functions defined on an interval  $(-c, c)$ ,  $c > 0$ , and satisfying  $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 0$ . Suppose that for all  $x \in (-c, c)$  the identity

$$a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (a_n + f(x))x^n = b_0 + b_1x + \cdots + b_{n-1}x^{n-1} + (b_n + g(x))x^n,$$

holds. Prove that  $a_0 = b_0, a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$ .

- (b) If the polynomial  $p(x) = \sum_{k=0}^n a_k x^k = 0$  for all  $x \in (-c, c)$ , determine the coefficients  $a_k$ . Justify your answer.

5. (a) State and prove the "Maximum-Minimum" Theorem.  
 (b) Let  $f$  be continuous on  $[0, \infty)$ . If  $f(x) \geq 0$  and  $\lim_{x \rightarrow \infty} f(x) = 0$  prove that there exists a  $\zeta \in [0, \infty)$  such that  $f(x) \leq f(\zeta)$  for all  $x \in [0, \infty)$ .
6. (a) Define what is meant by the statement that a function is uniformly continuous on an interval.  
 (b) Let  $f$  be continuous on  $\mathbb{R}$ . Suppose  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$  both are finite, prove that  $f$  is uniformly continuous on  $\mathbb{R}$ .  
 (c) Let  $f$  be continuous on  $[0, 1]$ . Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{2n} (-1)^k f\left(\frac{k}{2n}\right) = 0.$$

7. (a) State and prove Rolle's Theorem.  
 (b) Let  $f$  be differentiable on the interval  $[a, b]$ ,  $(-\infty < a < b < +\infty)$ . Suppose that  $f(a) = 0$ ,  $f(b) > 0$  and  $f'(b) < 0$ . Prove that there exists an  $c \in (a, b)$  such that  $f'(c) = 0$ .
8. (a) State the "Cauchy Mean Value Theorem".  
 (b) Let  $f$  and  $g$  be differentiable in an interval  $I$ . If for all  $x \in I$ ,  $f'(x) \neq 0$  and  $|f'(x)| \geq |g'(x)|$ , prove that

$$|f(y) - f(x)| \geq |g(y) - g(x)|$$

for all pairs  $x, y \in I$ .