1 (a) Define:
   (i) Least upper bound of a bounded set $S \subset \mathbb{R}$;
   (ii) $X = (x_k), x_k \in \mathbb{R}$, is a convergent sequence;
   (iii) $X = (x_k), x_k \in \mathbb{R}$ is a Cauchy sequence
   (iv) A function $f$ defined on $S \subset \mathbb{R}$ is uniformly continuous on $S$.

(b) State the Least Upper Bound Axiom.

2. Let $A \subset \mathbb{R}, B \subset \mathbb{R}$ be two non-empty bounded sets. Show that the set
   
   $C = \{c \mid c = a + b, \quad a \in A \quad b \in B\}$

   is bounded and $\text{Sup } C = \text{Sup } A + \text{Sup } B$

3. (a) Let $a > 0$, prove that
   
   \[ \lim_{n \to \infty} \sqrt[n]{a} = 1 \]

(b) Let $a_1, a_2, \ldots, a_{k-1}, a_k$ be $k$ positive numbers. Prove that

   \[ \lim_{n \to \infty} \left( a_1^n + a_2^n + \cdots + a_{k-1}^n + a_k^n \right)^{\frac{1}{n}} \max(a_1, a_2, \ldots, a_k) \]

4 (a) Show that every increasing bounded sequence $(a_n)$ is convergent.

(b) Let $f$ be defined and increasing on the interval $(a, b)$. Prove that for all $c \in (a, b)$ we have that \( \lim_{x \to c^+} f \) and \( \lim_{x \to c^-} f \) exist.

5. (a) Let $f$ be defined on the punctured neighborhood $N = \{x : 0 < |x - a| < \lambda\}$

   If $\lim_{x \to a^+} f(x) = A$, prove that $f$ is bounded on a punctured neighborhood $\{x : 0 < |x - a| < \beta \leq \lambda\}$.

(b) Let $g$ be defined and positive on $\{x : 0 < |x| < \lambda\}$

   Suppose that $\lim_{x \to 0} \left( g(x) + \frac{1}{g(x)} \right) = 2$.

   Prove that $\lim_{x \to 0} g(x)$ exists and it is equal to 1.
6. (a) State and prove the Intermediate Value Theorem
   (b) Let \( f \) be defined and continuous on \([0,1]\). If \( 0 \leq f \leq 1 \), show that there exists a \( \zeta \in [0,1] \) such that \( f(\zeta) = \zeta \).

7. (a) State Rolle’s Theorem.
   (b) Prove the Mean Value Theorem.
   (c) Let \( f \) be continuous on \([0,1]\) and differentiable on \((0,1)\). Suppose that \( f(0) = f(1) = 0 \) and that there is an \( x_0 \in (0,1) \) such that \( f(x_0) = 1 \). Prove that \( |f'(c)| \geq 2 \) for some \( c \epsilon (0,1) \).
   (d) (This question is for extra points.) Show that in (c) the strict inequality \( |f'(c)| > 2 \) is true.

8. (a) State Taylor’s Theorem, with the Lagrange remainder.
   (b) Establish the inequality
   \[
   1 + rx + \frac{r(r-1)}{2}x^2 \leq (1 + x)^r
   \]
   if \( x \geq 0 \) and \( r \geq 2 \).