1. Give a bijection between the rings $\mathcal{Q} \times \mathcal{Q}$ and $\mathcal{Q}[i]$. Explain why there is no isomorphism between them.

2. Suppose that $f : G \longrightarrow H$ is a homomorphism of rings and that it's onto. If H has m elements and the kernel has k elements, how many elements does G have? Why?

3. Suppose that $h : \mathcal{Q}[X] \longrightarrow \mathcal{C}$ is the ring homomorphism such that $h(q) = q$ for every $q \in \mathcal{Q}$ and $h(X) = -3i$.

   (a) What is $h(b_0 + b_1 X + \ldots b_\ell X^\ell)$ for any polynomial $Q(X) = b_0 + b_1 X + \ldots b_\ell X^\ell$?

   (b) Give the kernel K and the image $h(\mathcal{Q}[X])$ of this homomorphism.

   (c) Give an isomorphism between $\mathcal{Q}[X]/K$ and a subfield of $\mathcal{C}$.

4. Which of the following are subrings of $\mathcal{R}[X]$? Which are ideals? *Briefly* justify your answers.

   (a) $\mathcal{Q}[X]$.

   (b) The set of all polynomials of even degree, including 0.

   (c) The set of all polynomials $P(X)$ such that $P(1 - 2i) = 0$.

5. Suppose that p and q are distinct prime numbers. Show that $\mathcal{Z}_p \times \mathcal{Z}_q$ is isomorphic to $\mathcal{Z}_{pq}$. Show that $\mathcal{Z}_p \times \mathcal{Z}_p$ is *not* isomorphic to $\mathcal{Z}_{p^2}$.

6. There are subfields of $\mathcal{C}$ which are isomorphic to $\mathcal{Q}[X]/(X^4 - 2)$. How many? Describe them explicitly, and in each case give an isomorphism. You need not justify.

7. Suppose that each of R and S is a ring with at least 2 automorphisms. Show that there are at least 4 automorphisms of $R \times S$.

8. List all the maximal ideals in $\mathcal{Z}_{140}$. For each maximal ideal I, give a number m (depending on I) such that $\mathcal{Z}_{140}/I$ is isomorphic to $\mathcal{Z}_m$. Justify *briefly.*

9. Show that

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & d & f \\ 0 & 0 & g \end{pmatrix} : a, b, c, d, f, g \in \mathcal{Z}_3 \right\}$$

   is a ring with unity. Is it commutative? How many elements does it have? How many units (invertible elements) does it have? Justify everything.

10. Factor the polynomial

$$4X^8 + 28X^7 + 61X^6 + 42X^5 + 25X^4 + 112X^3 + 244X^2 + 168X + 36$$

   over $\mathcal{Q}$, over $\mathcal{R}$ and over $\mathcal{C}$; give the roots with multiplicity in all three cases.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION
(Alternate Version)

MATHEMATICS 189-235A

BASIC ALGEBRA I

Examiner: Professor J. Loveys
Associate Examiner: Professor J. Labute

Date: December 1, 1999

INSTRUCTIONS

Calculators are neither needed nor permitted.

This exam comprises the cover and one page of questions.