1. Give a bijection between the rings $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q}[i]$. Explain why there is no isomorphism between them.

2. Suppose that $f : G \rightarrow H$ is a homomorphism of rings and that it’s onto. If $H$ has $m$ elements and the kernel has $k$ elements, how many elements does $G$ have? Why?

3. Suppose that $h : \mathbb{Q}[X] \rightarrow \mathbb{C}$ is the ring homomorphism such that $h(q) = q$ for every $q \in \mathbb{Q}$ and $h(X) = -3i$.
   
   (a) What is $h(b_0 + b_1X + \ldots + b_tX^t)$ for any polynomial $Q(X) = b_0 + b_1X + \ldots b_tX^t$?
   
   (b) Give the kernel $K$ and the image $h(\mathbb{Q}[X])$ of this homomorphism.
   
   (c) Give an isomorphism between $\mathbb{Q}[X]/K$ and a subfield of $\mathbb{C}$.

4. Which of the following are subrings of $\mathbb{R}[X]$? Which are ideals? Briefly justify your answers.
   
   (a) $\mathbb{Q}[X]$.
   
   (b) The set of all polynomials of even degree, including 0.
   
   (c) The set of all polynomials $P(X)$ such that $P(1 - 2i) = 0$.

5. Suppose that $p$ and $q$ are distinct prime numbers. Show that $\mathbb{Z}_p \times \mathbb{Z}_q$ is isomorphic to $\mathbb{Z}_{pq}$. Show that $\mathbb{Z}_p \times \mathbb{Z}_q$ is not isomorphic to $\mathbb{Z}_{p^2}$.

6. There are subfields of $\mathbb{C}$ which are isomorphic to $\mathbb{Q}[X]/(X^4 - 2)$. How many? Describe them explicitly, and in each case give an isomorphism. You need not justify.

7. Suppose that each of $R$ and $S$ is a ring with at least 2 automorphisms. Show that there are at least 4 automorphisms of $R \times S$.

8. List all the maximal ideals in $\mathbb{Z}_{140}$. For each maximal ideal $I$, give a number $m$ (depending on $I$) such that $\mathbb{Z}_{140}/I$ is isomorphic to $\mathbb{Z}_m$. Justify briefly.

9. Show that

   $$\left\{ \begin{pmatrix} a & b & c \\ 0 & d & f \\ 0 & 0 & g \end{pmatrix} : a, b, c, d, f, g \in \mathbb{Z}_3 \right\}$$

   is a ring with unity. Is it commutative? How many elements does it have? How many units (invertible elements) does it have? Justify everything.

10. Factor the polynomial

    $$4X^8 + 28X^7 + 61X^6 + 42X^5 + 25X^4 + 112X^3 + 244X^2 + 168X + 36$$

    over $\mathbb{Q}$, over $\mathbb{R}$ and over $\mathbb{C}$; give the roots with multiplicity in all three cases.
McGILL UNIVERSITY
FACULTY OF SCIENCE

FINAL EXAMINATION
(Alternate Version)

MATHEMATICS 189-235A

BASIC ALGEBRA I

Examiner: Professor J. Loveys
Associate Examiner: Professor J. Labute

Date: December 1, 1999

INSTRUCTIONS

Calculators are neither needed nor permitted.

This exam comprises the cover and one page of questions.