1. Let $A$ be the matrix
\[
\begin{pmatrix}
1 & 3 & 5 & 7 \\
9 & 11 & 13 & 15 \\
17 & 19 & 21 & 23
\end{pmatrix}.
\]
Find a basis for the row space, for the column space, and for the null space of $A$.

2. (a) Suppose that $\lambda$ is an eigenvalue for the linear transformation $T$ and $\vec{v}$ is a corresponding eigenvector. Let $f(x) = a_2x^2 + a_1x + a_0$. Show that $\vec{v}$ is also an eigenvector of $f(T)$; give the corresponding eigenvalue.

(b) Show that the difference between two solutions of the system $A\vec{v} = \vec{b}$ is a solution to the corresponding homogeneous system.

3. Let $V$ be the vector space of polynomials of degree $\leq 2$ and let $T$ be differentiation:
\[T(a_2x^2 + a_1x + a_0) = 2a_2x + a_1.\]
(a) Show that
\[\{1, 1 + x, 1 + x + x^2\}\]
is a basis for $V$.

(b) Verify that $T$ is a linear transformation. Show also that it is nilpotent.

(c) Find the matrix of $T$ with respect to the given basis.

(d) Show that the set of polynomials of degree $\leq 1$ is a $T$-invariant subspace.

4. (a) Find a basis of eigenvectors for the matrix $A$ below and find a diagonal matrix similar to $A$. Display the change of base matrix.
\[A = \begin{pmatrix}
2 - i & -i \\
-1 + i & 1 + i
\end{pmatrix},\]

(b) Find a matrix $B$ such that $B^2 = A$. (Hint: First do it for the diagonal matrix; you may leave your final answer as a product of matrices.)

(c) Find $A^{20}$. (N.B. $2^{20} = 1048576$, but you may leave your answer as a product of matrices.)
5. Consider the quadratic form $q(x, y, z) = 3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$.

   (a) Find a symmetric matrix $A$ so that

   $$q(x, y, z) = (x, y, z)A\begin{pmatrix}x \\
y \\z\end{pmatrix}.$$ 

   Find an orthogonal matrix $P$ so that $P^tAP$ is in diagonal form; identify the general shape of $q(x, y, z) = 1$.

   (b) Polarize $q$ to find the associated inner product on $\mathbb{R}^3$.

6. Consider the matrix

   $$A = \begin{pmatrix}0 & 0 & -1 \\
1 & 0 & 1 \\
0 & 1 & 1\end{pmatrix}.$$ 

   (a) Find polynomials $a(t)$ and $b(t)$ so that $a(t)(t - 1)^2 + b(t)(t + 1) = 1$.

   (b) Find the projection matrices $a(A)(A - I)^2$ and $b(A)(A + I)$.

   (c) Find a matrix $P$ so that $P^{-1}AP$ is in Jordan form.

7. (a) Use Cramer’s rule to solve the system

   $$\begin{align*}
   2x + (2 + i)y &= 7 - i \\
(2 - i)x + 2y &= -1.
   \end{align*}$$

   (b) Show that, if $A$ is a square matrix with two equal rows, $\det A = 0$. 

2