1. Let A be the matrix

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 9 & 11 & 13 & 15 \\ 17 & 19 & 21 & 23 \end{pmatrix}.$$

Find a basis for the row space, for the column space, and for the null space of A.

2. (a) Suppose that λ is an eigenvalue for the linear transformation T and $\vec{v}$ is a corresponding eigenvector. Let $f(x) = a_2x^2 + a_1x + a_0$. Show that $\vec{v}$ is also an eigenvector of $f(T)$; give the corresponding eigenvalue.

   (b) Show that the difference between two solutions of the system $A\vec{v} = \vec{b}$ is a solution to the corresponding homogeneous system.

3. Let V be the vector space of polynomials of degree ≤ 2 and let T be differentiation:

$$T(a_2x^2 + a_1x + a_0) = 2a_2x + a_1.$$

   (a) Show that

$$\{1, 1+x, 1+x+x^2\}$$

   is a basis for V.

   (b) Verify that T is a linear transformation. Show also that it is nilpotent.

   (c) Find the matrix of T with respect to the given basis.

   (d) Show that the set of polynomials of degree ≤ 1 is a T-invariant subspace.

4. (a) Find a basis of eigenvectors for the matrix A below and find a diagonal matrix similar to A. Display the change of base matrix.

$$A = \begin{pmatrix} 2-i & -i \\ -1+i & 1+i \end{pmatrix},$$

   (b) Find a matrix B such that $B^2 = A$. (Hint: First do it for the diagonal matrix; you may leave your final answer as a product of matrices.)

   (c) Find A^{20}. (N.B. $2^{20} = 1048576$, but you may leave your answer as a product of matrices.)

5. Consider the quadratic form $q(x, y, z) = 3x^2 + 2y^2 + 3z^2 - 2xy - 2yz.$

   (a) Find a symmetric matrix A so that

   $$q(x, y, z) = (x,\ y,\ z)A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

   Find an orthogonal matrix P so that P^tAP is in diagonal form; identify the general shape of $q(x, y, z) = 1$.

   (b) Polarize q to find the associated inner product on $\mathcal{R}^3$.

6. Consider the matrix

   $$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

   (a) Find polynomials $a(t)$ and $b(t)$ so that $a(t)(t-1)^2 + b(t)(t+1) = 1$.

   (b) Find the projection matrices $a(A)(A - I)^2$ and $b(A)(A + I)$.

   (c) Find a matrix P so that $P^{-1}AP$ is in Jordan form.

7. (a) Use Cramer's rule to solve the system

   $$\begin{array}{rcrcr} 2x & + & (2+i)y & = & 7-i \\ (2-i)x & + & 2y & = & -1. \end{array}$$

   (b) Show that, if A is a square matrix with two equal rows, $\det A = 0$.