p-adic Deformation of Shintani Cycles

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Abstract

The goal of this thesis is to generalize to elliptic curves a classical formula of Hecke. This is chiefly achieved through studying p-adic deformation of certain Shintani cycles attached to a real quadratic field K and an elliptic curve $E/\mathbb{Q}$ of conductor N. Assume that all the prime divisors of N are split in K. Also suppose that E has multiplicative reduction at a prime p, and denote by $f_{_E}$ the newform attached to E via Shimura-Taniyama, and finally fix a quadratic unramified character $\mathcal{G}$ of $\mathrm{Gal}(\overline{K}/K)$. We

- construct a p-adic analytic function $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ which interpolates a $\mathcal{G}$-twisted sum of Shintani Cycles associated to the weight-k-specializations of a Hida family $\mathfrak{h}_E$ arising from Hida's theory applied to $f_{_E}$;

- give a factorization of $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)^2$ into the product of two Mazur-Kitagawa p-adic L-functions. We will prove that $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ vanishes at the point $\kappa = 2$ to order at least two, provided further that $E/\mathbb{Q}$ has at least two primes of multiplicative reduction. Our main result expresses $\mathcal{L}_p''(E/K, \mathcal{G}; 2)$, up to a non-zero rational fudge factor, as the product of the formal group logarithms of two global points on E defined over the narrow Hilbert class-field of K;

- show finally, in the spirit of Gross and Zagier, that the two points just alluded to are both of infinite order if and only if the complex L-function $L(E/K, \mathcal{G}; s)$ vanishes to order exactly two at the central point $s = 1$.

Résumé

Le but de cette thèse est de généraliser aux courbes elliptiques une formule classique de Hecke. Ceci nécessite principalement l'étude des déformations p-adiques de cycles de Shintani associés à un corps quadratique réel K et une courbe elliptique $E/\mathbb{Q}$ de conducteur N. Supposons que tous les diviseurs premiers de N soient scindés dans K et que E admette réduction multiplicative en p. Notons f_{E} la forme primitive associée à E via Shimura-Taniyama et choisissons également un caractère quadratique non-ramifié $\mathcal{G}$ de $\mathrm{Gal}(\overline{K}/K)$. Nous

- construisons une fonction analytique p-adique $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ qui interpole une somme $\mathcal{G}$-tordue de cycles de Shintani associés aux spécialisations de poids k d'une famille de Hida $\mathfrak{h}_E$ provenant de la théorie de Hida appliquée à f_{E};

- donnons une factorisation de $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)^2$ en produit de deux fonctions L p-adiques de Mazur-Kitagawa. Nous prouvons que $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ a un zéro d'ordre au moins deux en $\kappa = 2$, si, de plus, E a réduction multiplicative en au moins deux premiers. Notre résultat principal exprime $\mathcal{L}_p''(E/K, \mathcal{G}; 2)$, à un facteur rationnel non nul près, comme le produit des logarithmes formels de deux points globaux sur E définis sur le corps de Hilbert au sens restreint de K;

- montrons finalement, dans l'esprit de Gross et Zagier, que les deux points mentionnés plus haut sont tous deux d'ordre infini si et seulement si la fonction L complexe $L(E/K, \mathcal{G}; s)$ a un zéro d'ordre deux au point central $s = 1$.

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[1] This thesis was typeset using LaTeX.

Contents

Chapter 1

Prologue

1.1 Motivation

The main objective of this thesis may be regarded as a generalization to elliptic curves of a classical limit formula of Hecke (see the formula (1.5) below). Hecke's result is analogous to Kronecker's solution to Pell's equation in terms of special values of the eta-function of Dedekind. It expresses a sum involving the integral of the logarithm of the eta-function over certain geodesic cycles—twisted by a character $\mathcal{G}$ of special type—as the product of the natural logarithms of two fundamental units of certain quadratic fields. Involved implicitly in Hecke's formula (1.5) is the weight-two Eisenstein series $E_2(z)$ through a line integral called the Shintani cycle.

Our goal is to replace $E_2(z)$ by an ordinary newform f of weight 2 on

$\Gamma_0(N)$ with integer Fourier coefficients, and study the p-adic variations of $\mathcal{G}$-twisted sums of the Shintani cycles attached to the weight-k-specializations of a Hida family interpolating f. It turns out that these Shintani cycles are interpolated by a p-adic analytic function of k, whose second derivative at $k = 2$ gives the product of the p-adic formal group logarithms of two global points on the elliptic curve attached to the newform f. (See Theorems A and B stated in Section 1.3.) In order to motivate the reader and also to place the work done here in a broader picture, we begin by describing the classical work of Hecke. We remark that in preparation of the next section we have very closely followed Siegel's exposition in [22].

1.2 The Work of Hecke

Suppose that K is a real quadratic field with *fundamental discriminant* $\Delta > 0$ and write $\mathcal{H}_K^+$ for the *narrow Hilbert class-field* of K. We shall write

$$\alpha \mapsto \alpha'$$

for the non-trivial automorphism of K over $\mathbb{Q}$. A *genus character* of K is by definition a quadratic character of the *narrow ideal class-group* $\mathcal{C}_K^+$ of K. By global class field theory one knows that the group $\mathcal{C}_K^+$ is canonically isomorphic to $\mathrm{Gal}(\mathcal{H}_K^+/K)$; the isomorphism is the one induced by the *Artin* *map*

$$\mathfrak{p} \mapsto \sigma_{\mathfrak{p}} \tag{1.1}$$

which sends a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ to the associated Frobenius $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(\mathcal{H}_K^+/K)$ (For more on genus characters the reader is referred to Section 2.1.) Any non-trivial genus character $\mathcal{G}$, viewed as a character of $\mathrm{Gal}(\mathcal{H}_K^+/K)$, cuts out a quadratic extension $K_{\mathcal{G}} = \mathbb{Q}(\sqrt{\mathfrak{d}_1}, \sqrt{\mathfrak{d}_2})$ of K—biquadratic over $\mathbb{Q}$—where $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are two relatively prime fundamental discriminants whose product is Δ. Write χ_1, χ_2 and χ_K respectively for the primitive quadratic Dirichlet characters associated to $\mathbb{Q}(\sqrt{\mathfrak{d}_1})$, $\mathbb{Q}(\sqrt{\mathfrak{d}_2})$ and K. We say that $\mathcal{G}$ is *even* (respectively *odd*) if the field $K_{\mathcal{G}}$ is totally real (respectively totally imaginary).

From hereafter until the end of this section we assume that the genus character $\mathcal{G}$ is non-trivial and even. Given an element A of $\mathcal{C}_K^+$, choose a representative $\mathfrak{a}$ of A and without loss of generality assume that $\mathfrak{a}$ has a $\mathbb{Z}$-basis of the form $\{1, \tau\}$. We fix once and for all an embedding of K into $\mathbb{R}$ and let ϵ_K be the *fundamental unit of K of positive norm* by which we mean that ϵ_K is either the fundamental unit $u_K > 1$ of K or the square of that according as $\mathrm{Norm}_{K/\mathbb{Q}}(u_K) > 0$ or $\mathrm{Norm}_{K/\mathbb{Q}}(u_K) < 0$. On writing

$$\gamma_A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{where} \quad \epsilon_K \tau = a\tau + b, \quad \epsilon_K = c\tau + d, \tag{1.2}$$

one readily verifies that the matrix[1] γ_A is a *hyperbolic* element of $\mathbf{SL_2}(\mathbb{Z})$. The fixed points of the associated Möbius transformation

$$z \mapsto \frac{az+b}{cz+d},$$

acting on the Riemann sphere $\mathbb{P}_1(\mathbb{C})$, are easily seen to be τ and τ'.

Let η be the *Dedekind eta-function*

$$\eta(z) := e^{2\pi i z/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}),$$

defined for z in the Poincaré upper half-plane $\boldsymbol{\mathcal{H}}$. To η and the hyperbolic matrix γ_A we associate the integral

$$\mathbf{C}(\eta, \gamma_A) := \int_{z_0}^{\gamma_A z_0} \log(|\sqrt[4]{Q_A(z)}\eta(z)|^2) Q_A(z)^{-1} dz, \tag{1.3}$$

where the base point z_0 is an arbitrary point on the geodesic joining τ' to τ in $\boldsymbol{\mathcal{H}}$, and where

$$Q_A(z) = Q_A(z,1) := \frac{(z-\tau)(z-\tau')}{\mathrm{Norm}(\mathfrak{a})} \in \mathbb{Q}[z], \tag{1.4}$$

is a quadratic polynomial in z. Since γ_A fixes τ and τ', it follows that the homogenized quadratic form $Q_A(X,Y)$ attached to $Q_A(z)$ is fixed by the action of γ_A in the sense of (2.13), i.e., we have

$$Q_A(aX + bY, cX + dY) = Q_A(X, Y).$$

[1]The assignment $A \to \gamma_A$ can be used to set up a bijection between the narrow ideal classes of K and the set of conjugacy classes of the optimal embeddings of K of level *one*. For the definition of an optimal embedding of K of a general level N and the description of such bijection, see Section 2.7.

The complex number $\mathbf{C}(\eta, \gamma_{_A})$, which is independent of the choice of z_0, is called the *Shintani cycle* associated to η and $\gamma_{_A}$. The motivation to bring up all this is that one has (cf. [22], Chap.II, §3) the following analogue of Kronecker's solution to Pell's equation in the setting of real quadratic fields:

Theorem 1.2.1. (Hecke) *Let ϵ_1 and h_1 (respectively ϵ_2 and h_2) be the fundamental unit and the narrow class-number of the field $\mathbb{Q}(\sqrt{\mathfrak{d}_1})$ (respectively $\mathbb{Q}(\sqrt{\mathfrak{d}_2})$). Then*

$$\sum_{A} \mathcal{G}(A)\mathbf{C}(\eta, \gamma_{_A}) = \frac{2h_1h_2}{\sqrt{\Delta}} \log \epsilon_1 \log \epsilon_2, \tag{1.5}$$

where A runs over all the elements of the narrow ideal class-group of K. □

It is known (cf. [14], Chap. III, §2) that $\eta(z)$ is intimately related to the weight-two Eisenstein series

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z},$$

where $\sigma_1(n)$ is the sum of all positive divisors of n. More precisely, $E_2(z)$ is, up to a constant, the *logarithmic derivative* of $\eta(z)$, viz.

$$\mathrm{dlog}\ \eta(z) := \frac{\eta'(z)}{\eta(z)}\ \mathrm{d}z = \frac{2\pi i}{24} E_2(z)\ \mathrm{d}z.$$

1.3 This Thesis

The goal of the present thesis is to replace the Eisenstein series $E_2(z)$ with a cusp form $f(z)$ of weight 2 and then investigate the p-adic variations of the

Shintani cycles attached to K and various higher weigh specialisations of a Hida family interpolating f.

To begin with, let us first spell out what we mean by a Shintani cycle associated to a cusp form. Thus, to fix ideas assume that $f(z)$ is a cuspidal modular form of even weight $k \geq 2$ on the congruent subgroup $\Gamma_0(N)$ and let $\gamma \in \Gamma_0(N)$ be a hyperbolic matrix. The *Shintani cycle* attached to f and γ is the integer

$$\mathbf{C}(f,\gamma) := \frac{1}{\Omega_f}\mathbf{Real}\left(2\pi i\int_{z_0}^{\gamma z_0} f(z)Q_\gamma(z,1)^{\frac{k-2}{2}}dz\right), \tag{1.6}$$

where Ω_f is a suitable period[2] depending only on f (introduced in Section 2.3) and $Q_\gamma(x,y)$ is a binary quadratic form that is fixed by γ, i.e., the roots of $Q_\gamma(z,1)$ are the fixed points of γ (see also (1.4).) The quantities $\mathbf{C}(f,\gamma)$ are known to carry interesting arithmetic information. For instance, they are related (see §6 of [18]) to central critical values of the L-series of f over the splitting field of $Q(z,1)$ (i.e., the real quadratic field generated over $\mathbb{Q}$ by the fixed points of γ.) This very fact may be exploited to study the p-adic variation of the quantities $\mathbf{C}(f,\gamma)$ as f varies in a "family" (see for instance [3], in particular Theorem 3.5 therein.) Concretely speaking, we fix a prime

[2] To be more accurate, we should say that there are two such periods, and as a result, one can define two Shintani cycles attached to f and γ. See Subsection 2.3.2 for the details.

number p and suppose that

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \tag{1.7}$$

is a normalised newform of weight 2 on $\Gamma_0(N)$ whose p-th Fourier coefficient a_p is a p-adic unit. Whenever this latter condition holds one says that f is *ordinary at* p. In a canonical way, Hida's theory associates to f a family $\{f_k\}$ of normalised eigenforms of varying weight k on $\Gamma_0(N)$, for all even integers $k \geq 2$ in a suitable p-adic neighbourhood U of 2 (see Section 2.4 for the details). In "good" circumstances one can prove that the assignment

$$k \mapsto \mathbf{C}(f_k, \gamma)$$

extends to a p-adic analytic function[3] of $\kappa \in U$. It is therefore natural to consider the special values of the resulting function as well as its successive derivatives and wonder if one could describe those values in terms of other arithmetic objects.

We assume henceforward that f arises, via Shimura–Taniyama–Weil as in Subsection 2.2.1, as the newform attached to an elliptic curve $E/\mathbb{Q}$ of conductor N. For that reason we shall write f_E instead of f and remark that f_E has integer Fourier coefficients. We further assume that the prime p

[3]Throughout the thesis we have consistently used the letter κ for a "generic" p-adic variable; however, if that variable happens to be in $\mathbb{Z}$, we will denote it by the letter k.

divides N *exactly*:

$$p|N \qquad \text{but} \qquad p^2 \not| N. \tag{1.8}$$

This amounts to saying that E has *multiplicative reduction* at p, and therefore implies in particular that

$$a_p = \pm 1.$$

We shall write

$$N = pM,$$

and remark that $p \not| M$, due to the running assumption (1.8). Before stating the main results, some further notations are to be introduced.

We let $q \in p\mathbb{Z}_p$ be the *Tate period*[4] attached to E, and write

$$\Phi_{\mathrm{Tate}} : \mathbb{Q}_p^\times / q^{\mathbb{Z}} \longrightarrow E(\mathbb{Q}_p) \tag{1.9}$$

for the (p-adic) *Tate uniformisation* whose existence is guaranteed by the fact that p is a prime of multiplicative reduction for E. Let

$$\log_p : \mathbb{Q}_p^\times \longrightarrow \mathbb{Q}_p \tag{1.10}$$

stand for the branch of p-adic logarithm which satisfies $\log_p(q) = 0$. The two maps Φ_{Tate} and $\log_p$ are used to define the (p-adic) *formal group logarithm*

$$\log_E : E(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$$

[4] Needles to say that the Tate period has nothing to do with the q appearing in (1.7).

by the rule

$$\log_E(P) := \log_p(\Phi_{\mathrm{Tate}}^{-1}(P)). \tag{1.11}$$

Note that this is well-defined since $q^{\mathbb{Z}} \subseteq \ker(\log_p)$, thanks to the chosen branch of $\log_p$. One easily sees that the mapping $\log_E$ is a group homomorphism.

For each divisor $d|N$ with $\gcd(d, \frac{N}{d}) = 1$, we let w_d denote the eigenvalue of the *Fricke involution* W_d acting on f_E, namely,

$$(W_d|f_E)(z) := f_E|\begin{pmatrix} 0 & -1 \\ d & 0 \end{pmatrix}(z) = w_d \cdot f_E(z). \tag{1.12}$$

For $j = 1, 2$, we write $(E(\mathbb{Q}(\sqrt{\mathfrak{d}_j})) \otimes \mathbb{Q})^{\chi_j}$ for the subspace of the Mordell-Weil group $E(\mathcal{H}_K^+) \otimes \mathbb{Q}$ on which the group $\mathrm{Gal}(\mathbb{Q}(\sqrt{\mathfrak{d}_j})/\mathbb{Q})$ acts via the character χ_j.

By adapting the techniques of [3] and [2] developed by Bertolini and Darmon and based on ideas and computations of Greenberg and Stevens in [11], we prove the following as the first main result of this thesis.

Theorem A. *Suppose that all the prime divisors of N are split in K. Then there exists a p-adic analytic function $\mathcal{L}_p(\kappa)$ of κ, defined over a suitable p-adic neighbourhood $U \subseteq \mathbb{Z}_p$ of 2, which interpolates the $\mathcal{G}$-twisted sums of Shintani cycles*

$$\sum_A \mathcal{G}(A)\mathbf{C}(f_k, \gamma_A), \tag{1.13}$$

for all $k \in U \cap \mathbb{Z}^{\geq 2}$.

The mere existence of the function $\mathcal{L}_p(\kappa)$ is curious enough in its own, but the more striking fact concerning $\mathcal{L}_p(\kappa)$ is that it encodes global arithmetic information about the elliptic curve E.

Theorem B. *In the setting of Theorem A, suppose further that the elliptic curve* $E/\mathbb{Q}$ *has, apart from* p, *at least one other prime of multiplicative reduction, that* $\chi_1(-N) = w_N$ *and that* $\chi_1(p) = a_p$. *Then:*

1. $\mathcal{L}_p(\kappa)$ *vanishes to order at least two at* $k = 2$;
2. *There exist global points* $P_1 \in (E(\mathbb{Q}(\sqrt{\mathfrak{d}_1}))\otimes\mathbb{Q})^{\chi_1}$ *and* $P_2 \in (E(\mathbb{Q}(\sqrt{\mathfrak{d}_2}))\otimes\mathbb{Q})^{\chi_2}$ *and a non-zero rational number* t *such that*
$$\frac{d^2}{d\kappa^2}\mathcal{L}_p(\kappa)\Big|_{k=2} = t\log_E(P_1)\log_E(P_2); \tag{1.14}$$
3. *The two points* P_1 *and* P_2 *are simultaneously of infinite order if and only if*
$$L''(E/K,\mathcal{G};1) \neq 0,$$
where $L(E/K,\mathcal{G};s)$ *is the complex* L*-series of* E *over* K *whose definition is recalled in Subsection 2.2.2.*

Remark 1.3.1. Note the strong analogy between (1.14) and (1.5). In fact, our results illuminate another instance of the close parallel between units of number fields and points on elliptic curves. The Dirichlet unit theorem is analogous to the Mordell-Weil theorem; Stark units are number field counterparts of Stark-Heegner points on elliptic curves; to name just a few examples.

Remark 1.3.2. For a semistable elliptic curve $E/\mathbb{Q}$, the technical condition of an extra prime of multiplicative reduction is automatically verified if the sign in the functional equation satisfied by the L-function $L(E,s)$ is minus one. For, such curve must have an even number of primes of multiplicative reduction; a fact which easily follows from Theorem 3.17 of [8].

1.4 An Outline of the Proof

In this section we sketch the main lines of the proof of Theorems A and B. Before doing so, however, it will be useful to explain how the Formula (1.5) is proven. This also makes the analogy between the work of Hecke and ours more transparent.

The Formula (1.5) is established by piecing together the following four ingredients.

1. The identity

$$L_K(\mathcal{G},s) = \sum_{A\in\mathcal{C}_K^+} \mathcal{G}(A)\zeta_K(A,s),$$

   which expresses the generalised Dedekind zeta-function $L_K(\mathcal{G},s)$ of K, as the summation, twisted by $\mathcal{G}$, of the partial zeta-functions $\zeta_K(A,s)$'s (cf. Section 2.1 for the details.)

2. The following limit formula

$$\lim_{s\to 1}\left(\zeta_{K}(A,s)-\frac{2\log\epsilon_{K}}{\sqrt{\Delta}}\frac{1}{s-1}\right)=\frac{4C\log\epsilon_{K}}{\sqrt{\Delta}}+2\mathbf{C}(\eta,\gamma_{A}),$$

concerning a real quadratic field and due to Hecke, which is analogous to the limit formula of Kronecker in the setting of imaginary quadratic fields. Here $\mathbf{C}(\eta,\gamma_{A})$ is the Shintani cycle defined as in Equation (1.3) and C is the Euler constant

$$C := \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right).$$

Notice that this limit formula of Hecke together with the expression of $L_K(\mathcal{G},s)$ appearing in Step 1, on account of the earlier assumption that $\mathcal{G}$ be non-trivial, immediately imply that

$$L_K(\mathcal{G},1)=2\sum_{A\in\mathcal{C}_K^+}\mathcal{G}(A)\mathbf{C}(\eta,\gamma_A).$$

3. The factorisation formula (2.4) of Section 2.1, due to Kronecker,

$$L_K(\mathcal{G},s)=L(\chi_1,s)L(\chi_2,s),$$

which expresses the $\mathcal{G}$-twisted Dedekind zeta-function $L_K(\mathcal{G},s)$ as the product of the two Dirichlet L-functions $L(\chi_1,s)$ and $L(\chi_2,s)$ attached to χ_1 and χ_2 respectively.

4. And finally, the last ingredient is provided by the celebrated Dirichlet

class-number formula

$$L(\chi_1, 1) = \frac{2h_1 \log \epsilon_1}{\sqrt{\mathfrak{d}_1}}, \qquad L(\chi_2, 1) = \frac{2h_2 \log \epsilon_2}{\sqrt{\mathfrak{d}_2}},$$

where h_j, ϵ_j and $\mathfrak{d}_j$ (for $j = 1, 2$) are defined as in Section 1.2.

The Formula (1.5) is an immediate consequence of the four steps sketched above as follows:

$$\begin{aligned}
\sum_A \mathcal{G}(A)\mathbf{C}(\eta, \gamma_A) &= \frac{1}{2} L_K(\mathcal{G}, 1) \\
&= \frac{1}{2} L(\chi_1, 1) L(\chi_2, 1) \\
&= \frac{1}{2} \frac{2h_1 \log \epsilon_1}{\sqrt{\mathfrak{d}_1}} \frac{2h_2 \log \epsilon_2}{\sqrt{\mathfrak{d}_2}} \\
&= \frac{2h_1 h_2}{\sqrt{\Delta}} \log \epsilon_1 \log \epsilon_2.
\end{aligned}$$

We shall now sketch the work done in this thesis and give some details about the organization of the rest of the text. We hope that the following will support the assertion that the main steps underlying the proof of Theorems A and B (to be addressed shortly) run parallel to their counterparts just described.

In Chapter 2, entitled "Foundations", we summarize the definitions and basic properties of the necessary tools which will be needed afterwards.

Section 2.1 introduces genus characters (Definition 2.1.1), their associated zeta-functions in the spirit of Dedekind (as defined in (2.3)), the corresponding pairs of Dirichlet characters (uniquely characterized by the relations (2.1)) and the fundamental identity (2.4) already seen in Step 2 above.

Section 2.2 attaches two types of L-series to an eigenform g—one over $\mathbb{Q}$ and twisted by a Dirichlet character (in Subsection 2.2.1), and one over a quadratic field K and twisted by a genus character (in Subsection 2.2.2)—and then the precise relationship between the two types of L-series will be spelled out.

In Subsection 2.3.1 we will discuss the general notion of a modular symbol $\mathfrak{m}\{r \to s\}$. The description of the modular symbol $\widetilde{\mathbf{I}}_g\{r \to s\}$ attached to a cusp form g of weight k (à la Eichler and Shimura) is the subject of Subsection 2.3.2. These are linear functionals on the space of homogeneous polynomials of degree $k-2$ in two variables with complex coefficients. In particular, we will record an important result of Shimura concerning the algebraicity properties of $\widetilde{\mathbf{I}}_g$ once modified by a pair of suitable periods. Then, in the final subsection of the aforesaid section, we see how these modified modular symbols $\mathbf{I}_g^+$ and $\mathbf{I}_g^-$ are utilised to give an explicit formula for the central critical values of the L-series of the form g.

Our task in Section 2.4 will be to explain how every "ordinary" classical eigenform of weight 2 gives rise in a canonical way to a family $\{f_k\}$ of cusp forms, called a Hida family, denoted by $\mathfrak{h}_E$ and indexed by even integers k which are p-adically close enough to 2, i.e., indexed by the members of the set $\mathbb{Z}^{\geq 2} \cap U$ where $U \subset \mathbb{Z}_p$ is a suitable p-adic neighbourhood of 2. Corresponding to each weight-k-specialisation f_k of the Hida family $\mathfrak{h}_E$ and as in the preceding section, we then obtain two modular symbols $\mathbf{I}_k$ and $\hat{\mathbf{I}}_k$.

In the subsequent section, namely Section 2.5, we will take on the task of surveying the basics of a p-adic integration theory which describes p-adic measures against which $\mathbb{Z}_p$-valued homogeneous functions defined over $\mathbb{Z}_p$-sublattices of $\mathbb{Q}_p^2$ can be integrated, with a special emphasis on those measures μ on the $\mathbb{Z}_p$-lattice $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ which are $\Gamma_0(M)$-invariant with respect to a certain group action. There will be also constructed a family of mappings ρ_k, for each integer $k \geq 2$, going from measure-valued modular symbols to modular symbols with values in the linear dual to the space of homogeneous polynomials of degree $k-2$ in two variables. The main objective therein will be to state a crucial result of Greenberg and Stevens, which guarantees the existence of a certain measure μ_* whose image under ρ_k is the modular symbol $\mathbf{I}_k$ up to rescaling by a p-adic non-zero number $\lambda(k)$. Crucial to the computations of this thesis is also the construction (after Bertolini, Darmon and Iovita [4]) of a family of measure-valued modular symbols $\mu_L\{r \to s\}$, indexed by the $\mathbb{Z}_p$-lattices $L \subset \mathbb{Q}_p^2$, which interact nicely (see the identity (2.33) for the precise meaning) with a certain group action.

Included in Foundations is also a section concerning those p-adic L-

functions which come to play a role in this thesis. In Subsection 2.6.1 we will record the definition and some of the basic properties of the Mazur-Swinnerton-Dyer one-variable p-adic L-function $L_p(g, s)$ attached to any ordinary eigenform g. The task in Subsection 2.6.2 will be to introduces a two-variable p-adic L-function $L_p(\mathfrak{h}_E; \kappa, s)$ constructed first by Mazur and Kitagawa. This two-variable p-adic L-function associated to the Hida family $\mathfrak{h}_E$ interpolates, for each $k \in \mathbb{Z}^{\geq 2} \cap U$, the p-adic L-function $L_p(f_k, s)$ (as in the relation (2.36)) as well as the special values of the complex L-function $L(f_k, s)$ (as in the relation (2.37)).

The final section of the next chapter will give us a firm handle on the concept of optimal embeddings (of level N) of a quadratic field K into the matrix algebra $\mathbf{M_2}(\mathbb{Q})$ (see Definition 2.7.1). The set of equivalence classes of such embeddings (under conjugation by elements of $\mathbf{GL_2}(\mathbb{Q})$) is endowed with a transitive action of the narrow ideal class-group of K making it into a principal homogeneous space. This will make it possible to view $\mathcal{G}$ as a function on the set of such embeddings. To any optimal embedding we will also associate a $\mathbb{Z}_p$-lattice, and hence through that a modular symbol. This lattice and its associated modular symbol will turn out to play a significant role in the construction of our p-adic analytic function alluded to in Theorem A.

Whereas Chapter 2 collects only the statements of results which are already in the literature, and hence will contain no proofs, Chapter 3, which bears the title of this thesis, strives to give complete proofs for almost everything, with a few exceptions where exact references have been provided. This being said, let us now go section by section and draw a general picture on how the proof of the main results of this thesis are achieved.

Inspired by the lattice functions introduced in [2], we will begin Section 3.1 by defining two modular symbols c_k and $\hat{c}_k$. The former operates as a function $c_k(L)$ on the set of $\mathbb{Z}_p$-lattices $L \subset \mathbb{Q}_p^2$, whereas the latter operates as a function $\hat{c}_k(L_1, L_2)$ on the set of pairs of $\mathbb{Z}_p$-lattices $L_1 \supset L_2$, one contained in the other with index p. The Main Identity (3.6) of that section will then relate these two lattice functions; an identity which is the key toward the construction of the desired p-adic analytic function.[5]

The main object of this thesis, namely the p-adic analytic function $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$, will be introduced in Section 3.2. In order to do so, we will have to first construct another p-adic analytic function $\mathcal{L}_p(\Psi, \kappa)$ attached to any optimal embedding Ψ of K. We will see (cf. Remark 3.2.3) that the definition of $\mathcal{L}_p(\Psi, \kappa)$ depends only on the equivalence class represented by

[5]Theorems A and B were initially proved under the simplifying hypothesis $h_K^+ = 1$. The proofs in the general case were achieved by carrying over the technical computations done in [2] to the current setting. And in doing so, it was the key identity (3.6) that enabled the author to overcome the obstructions he was facing.

Ψ and not on Ψ itself. Having constructed these p-adic analytic functions, one for each class, we then define $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ as the twisted sum

$$\mathcal{L}_p(E/K, \mathcal{G}; \kappa) = \sum_{\Psi} \mathcal{G}(\Psi)\mathcal{L}_p(\Psi, \kappa).$$

Note that this identity very much resembles the one which appears in Step 1 in the work of Hecke. Therefore, the "partial p-adic functions $\mathcal{L}_p(\Psi, \kappa)$" are the building blocks for $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ as the partial zeta-functions $\zeta_{_K}(A, s)$ are so for $L_K(\mathcal{G}, s)$.

In the section to follow we will prove one of the salient properties of $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$, namely an interpolation formula justifying the name "p-adic L-function". More precisely, we will see in Proposition 3.3.1 that, for each $k \in \mathbb{Z}^{\geq 2} \cap U$ and *up to an Euler-type factor at* p,

$$\mathcal{L}_p(E/K, \mathcal{G}; k) = \sum_{\Psi} \mathcal{G}(\Psi)\mathbf{C}(\hat{f}_k, \Psi(\epsilon_{_K})),$$

where $\hat{f}_k$ is a newform of the same weight k but lower level $M = N/p$ from which the form f_k arises as in Remark 2.4.5. This, together with an important result of Popa [18], will yield a variant (cf. Proposition 3.3.2) of the interpolation formula which is to be viewed as analogous to the limit formula which appeared in Step 2 above concerning the "special values" of the partial zeta-functions at the critical point $s = 1$. This finishes the proof of Theorem A.

The goal of Section 3.4 is to prove the following Factorisation Formula

$$\mathcal{L}_p(E/K,\mathcal{G};\kappa)^2 = \Delta^{\frac{\kappa-2}{2}} L_p(\mathfrak{h}_E,\chi_1;\kappa,\kappa/2)L_p(\mathfrak{h}_E,\chi_2;\kappa,\kappa/2). \tag{1.15}$$

Once again note that this formula—akin to the factorisation of $L_K(\mathcal{G},s)$ appearing in Step 3 above—displays the analogy between the work of Hecke and ours.

And finally, the proof of Theorem B will be achieved in the fifth section of Chapter 3. One of the main ingredients to which we will appeal therein is a fundamental result of Bertolini and Darmon which for a given Dirichlet character χ of conductor $\mathfrak{d}$ and under certain conditions, to be spelled out precisely in Theorem 3.5.1, asserts that the Mazur-Kitagawa p-adic L-function restricted to the *central critical line* $\kappa = 2s$ vanishes to order at least two at the point $k = 2$, namely, that

$$L_p(\mathfrak{h}_E,\chi;\kappa,\kappa/2)\Big|_{k=2} = \frac{d}{d\kappa}L_p(\mathfrak{h}_E,\chi;\kappa,\kappa/2)\Big|_{k=2} = 0,$$

and that there exists a global point $\mathbf{P}_\chi \in (E(\mathbb{Q}(\sqrt{\mathfrak{d}}))\otimes\mathbb{Q})^\chi$ such that

$$\frac{d^2}{d\kappa^2}L_p(\mathfrak{h}_E,\chi;\kappa,\kappa/2)\Big|_{k=2} \doteq \log_E^2(\mathbf{P}_\chi),$$

where the symbol $\doteq$ denotes the equality up to an explicit non-zero fudge factor. This equality plays in our proof of Theorem B the very role played by Dirichlet class-number formula of Step 4 in the proof of Hecke's result. As we shall see the equality (1.14) will be a consequence of the steps thus described. We shall also see that the last part of Theorem B follows from other parts of Theorem 3.5.1.

1.5 The Work of Bertolini and Darmon

In contrast with the present work where the prime p is assumed to split in K, the earlier work of Bertolini and Darmon [2, 3] is largely concerned with the case where p is inert in K. In this section we briefly recall the main result of [3] in order to contrast it with the main results in this thesis.

So, *in this section only we assume that the prime p is inert in K* (while we continue to assume that all the prime divisors of $M = \dfrac{N}{p}$ are still split in K) and also for the sake of simplicity we assume that K has narrow class-number one. Retaining the other hypotheses as before, Bertolini and Darmon prove in [3] the existence of a global point $\mathbf{P} \in E(K)$ such that

$$\frac{d}{d\kappa}\mathbf{C}(f_k, \gamma_{_A})\Big|_{k=2} \doteq \log_E(\mathbf{P}).$$

They also prove that the point $\mathbf{P}$ is of infinite order if and only if

$$L'(E/K, 1) \neq 0.$$

As is seen, this result differs from our work in that it involves only the first derivative whereas Formula (1.14) involves the second derivative. This stems from the fact that if one assumes that p is inert in K (the situation explored by Bertolini and Darmon in [3]) then it turns out that exactly one out of the two Mazur-Kitagawa functions appearing in Bertolini-Darmon's factorisation formula stated as Theorem 3.6 in [3]—which is very similar to (1.15) in appearance—vanishes to order at least two at $k = 2$ and the other L-function is indeed non-vanishing, whereas if p is supposed to split in K (our case) then at the point $k = 2$ both of the Mazur-Kitagawa functions appearing in (1.15) vanish to order at least two. This discrepancy is reflected in "looking at the first derivative" in the former case versus "looking at the second derivative" in the latter case.

In spite of this essential difference, the present work builds on some constructions, results and ideas borrowed from [3] as well as [2].

Chapter 2

Foundations

In this chapter we survey the basic concepts and tools that we will need afterwards. Since a comprehensive survey goes well beyond the scope of this work, we only discuss those which are closely related to the content of this thesis.

2.1 Genus Characters

Recall that an integer Δ is called a *fundamental discriminant* if it is the discriminant of the maximal order $\mathcal{O}_K$ of a quadratic field K. A fundamental discriminant is called a prime discriminant if it is divisible by only one prime (i.e., $\Delta = -4,\ +8,\ -8,\ p$ with p a prime $\equiv 1 \pmod 4$, $-p$ with p a prime $\equiv -1 \pmod 4$). It can be shown (cf. [22], p. 59) that every fundamental discriminant is a product of prime discriminants. For a quadratic number field K with fundamental discriminant Δ, we will denote by $\chi_{_K}$ the primitive quadratic Dirichlet character modulo $|\Delta|$ satisfying

$$\chi_{_K}(p) = \begin{cases} \left(\frac{\Delta}{p}\right) & p \text{ an odd prime;} \\ (-1)^{\frac{\Delta-1}{4}} & p = 2,\ \Delta \text{ odd;} \\ 0 & p = 2,\ \Delta \text{ even;} \\ \operatorname{sgn} \Delta & p = -1, \end{cases}$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

Suppose now that $\chi_{_K}(-1) = +1$ (i.e., that K is real), and let as before the generator of $\mathrm{Gal}(K/\mathbb{Q})$ be denoted by the mapping $\alpha \mapsto \alpha'$. We shall write $\mathcal{C}_K$ (respectively h_K) for the ideal class-group (respectively the class-number) of K. Two nonzero fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be equivalent in the *narrow sense* if there exists an[1]

$$\alpha \in K \quad \text{with} \quad \mathrm{Norm}_{K/\mathbb{Q}}(\alpha) > 0$$

such that $\alpha\mathfrak{a} = \mathfrak{b}$. The equivalence classes equipped with multiplication form the narrow ideal class-group $\mathcal{C}_K^+$ of K whose order, denoted by h_K^+, is called the *narrow class-number* of K. If K contains a unit of norm -1, this brings nothing new; but if all the units have positive norm, then each ideal class is

[1]Note that for a general number field, the condition "α having a positive norm" has to be replaced by the condition "α being totally positive". In our setting, however, these two conditions are equivalent.

the union of two narrow ideal classes. In other words, one has

$$h_K^+ = \begin{cases} h_K & \text{if } O_K \text{ has a unit of negative norm;} \\ 2h_K & \text{otherwise.} \end{cases}$$

Global class field theory canonically identifies $\mathcal{C}_K^+$—by means of the Artin map (1.1)—with the Galois group over K of the maximal abelian extension $\mathcal{H}_K^+$ of K in which all prime ideals of K are unramified. The field $\mathcal{H}_K^+$ is called the *narrow Hilbert class-field of* K.

Definition 2.1.1. *By a genus character of K we understand a quadratic character*

$$\mathcal{G} : \mathcal{C}_K^+ \longrightarrow \{\pm 1\}$$

of the narrow ideal class-group $\mathcal{C}_K^+$ of K.

Remark 2.1.2. *If a rational prime ℓ splits in K as $\ell\mathcal{O}_K = \boldsymbol{\lambda}\boldsymbol{\lambda}'$, then since $\mathcal{G}$ is quadratic, it easily follows that $\mathcal{G}(\boldsymbol{\lambda}) = \mathcal{G}(\boldsymbol{\lambda}')$. This is true since $\boldsymbol{\lambda}\boldsymbol{\lambda}'$ may represent the identity class in $\mathcal{C}_K^+$.*

By setting

$$\mathcal{G}(\sigma_{\boldsymbol{\lambda}}) = \mathcal{G}(\boldsymbol{\lambda}),$$

where $\sigma_{\boldsymbol{\lambda}} \in \mathrm{Gal}(\mathcal{H}_K^+/K)$ is the Frobenius automorphism associated to the prime ideal $\boldsymbol{\lambda}$, any such $\mathcal{G}$ may as well be viewed as a quadratic character of $\mathrm{Gal}(\mathcal{H}_K^+/K)$ or equivalently as a quadratic character of $\mathrm{Gal}(\overline{K}/K)$ which is unramified at all finite places of K. Any non-trivial genus character cuts out a biquadratic extension $K_{\mathcal{G}} = \mathbb{Q}(\sqrt{\mathfrak{d}_1}, \sqrt{\mathfrak{d}_2})$ of $\mathbb{Q}$ fitting into the following diagram:

$$\begin{tikzcd}
 & H_K^+ \arrow[d, dash] & \\
 & K_{\mathcal{G}} = \mathbb{Q}(\sqrt{\mathfrak{d}_1}, \sqrt{\mathfrak{d}_2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{\mathfrak{d}_1}) \arrow[dr, dash] & K = \mathbb{Q}(\sqrt{\Delta}) \arrow[d, dash] & \mathbb{Q}(\sqrt{\mathfrak{d}_2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

where $\Delta = \mathfrak{d}_1\mathfrak{d}_2$. In fact the genus characters of K are in one-to-one correspondence with the factorisations of Δ into a product of two relatively prime fundamental discriminants $\mathfrak{d}_1$ and $\mathfrak{d}_2$, or what amounts to the same thing, in one-to-one correspondence with the unordered pairs (χ_1, χ_2) of primitive quadratic Dirichlet characters of coprime conductors satisfying $\chi_1\chi_2 = \chi_K$(cf. [22], Chap.II, §1.) Under such bijection, the trivial genus character $\mathcal{G}_{\text{trivial}}$ corresponds to the factorisation $\Delta = 1 \cdot \Delta$ (or equivalently, to the pair $(\chi_{\text{trivial}}, \chi_K)$). The pair (χ_1, χ_2) is uniquely characterised by the relations

$$\mathcal{G}(\sigma_{\boldsymbol{\lambda}}) = \mathcal{G}(\boldsymbol{\lambda}) = \chi_1(\ell) = \chi_2(\ell), \tag{2.1}$$

valid for any rational prime ℓ which splits in K as $\ell\mathcal{O}_K = \boldsymbol{\lambda}\boldsymbol{\lambda}'$. The analogous identity at the infinite place is

$$\mathcal{G}(\mathfrak{c}) = \chi_1(-1) = \chi_2(-1), \tag{2.2}$$

where $\mathfrak{c}$ is the image in $\mathrm{Gal}(\mathcal{H}_K^+/K)$ (by restriction) of the complex conjugation. A genus character is called even (respectively odd) if $\mathcal{G}(\mathfrak{c}) = 1$ (respectively $\mathcal{G}(\mathfrak{c}) = -1$). Equivalently, $\mathcal{G}$ is even (respectively odd) if and only if it cuts out a totally real (respectively totally imaginary) quadratic extension of K. It therefore follows from (2.2) that if $\mathcal{G}$ is even (respectively odd), then so are both characters χ_1 and χ_2. *We fix once and for all a genus character $\mathcal{G}$, giving ourselves a unique pair (χ_1, χ_2) of Dirichlet characters attached to it.*

To the character $\mathcal{G}$ we now attach the following *$\mathcal{G}$-twisted Dedekind zeta-function*

$$L_K(\mathcal{G}, s) := \sum_{\mathfrak{a}} \mathcal{G}(\mathfrak{a})(\mathrm{Norm}(\mathfrak{a}))^{-s}, \tag{2.3}$$

the summation being taken over all the non-zero ideals of $\mathcal{O}_K$. We remark that, due to the multiplicative nature of $\mathcal{G}$, the function $L_K(\mathcal{G}, s)$ acquires the following Euler product representation

$$L_K(\mathcal{G}, s) = \prod_{\mathfrak{p}} (1 - \mathcal{G}(\mathfrak{p})\mathrm{Norm}(\mathfrak{p})^{-s})^{-1},$$

where $\mathfrak{p}$ runs over all the non-zero prime ideals of $\mathcal{O}_K$. It is also useful to introduce the *partial zeta-function* $\zeta_K(A, s)$, associated to any (narrow) ideal class A, by the formula

$$\zeta_K(A, s) := \sum_{\mathfrak{a} \in A} (\mathrm{Norm}(\mathfrak{a}))^{-s}.$$

The significance (one among many!) of these partial zeta-functions stems from the following identity

$$L_K(\mathcal{G}, s) = \sum_{A \in \mathcal{C}_K^+} \mathcal{G}(A) \zeta_K(A, s).$$

Finally we remark that, in light of the Euler product of $L_K(\mathcal{G}, s)$ above, one may exploit the crucial identity (2.1) as the key to the proof of the following result of Kronecker (cf. page 62 of [22].)

Theorem 2.1.3. (Kronecker) *We have*

$$L_K(\mathcal{G}, s) = L(\chi_1, s) L(\chi_2, s), \tag{2.4}$$

where the two L-functions on the right are the Dirichlet L-functions associated to the characters χ_1 and χ_2 respectively. □

2.2 Complex L-Functions

In this section we recall the definitions and some basic properties of those complex L-functions which are pertinent to the considerations of this thesis.

2.2.1 L-Functions over $\mathbb{Q}$

The (complex) *Hasse-Weil* L-function of the elliptic curve E of Section 1.3 is defined by the infinite Euler product

$$L(E, s) := \prod_{\ell | N} (1 - a_\ell(E) \ell^{-s})^{-1} \prod_{\ell \nmid N} (1 - a_\ell(E) \ell^{-s} + \ell^{1-2s})^{-1} =: \sum_{n=1}^{\infty} a_n n^{-s}$$

where[2]

$$a_\ell(E) = a_\ell := \ell + 1 - E(\mathbb{F}_\ell), \qquad \text{if} \quad \ell \nmid N.$$

A deep conjecture formulated by Shimura, Taniyama and Weil and established by Wiles and his school asserts that the L-series $L(E,s)$ is the *Mellin transform* of a newform of weight two on $\Gamma_0(N)$. To explain what exactly this means, we need to recall the definition of the L-series of a modular form. Thus, let

$$g(z) = \sum_{n=1}^{\infty} a_n(g) q^n$$

be a cusp form of weight k on $\Gamma_0(N)$. The L-function $L(g,s)$ of g begins life as the Dirichlet series

$$L(g,s) := \sum_{n=1}^{\infty} a_n(g) n^{-s} = \frac{2\pi i}{\Gamma(s)} \int_0^{\infty} g(it) t^s \frac{dt}{t}, \tag{2.5}$$

where s is a complex variable. It is originally defined over the half-plane $\mathrm{Re}(s) > \dfrac{k+1}{2}$. However, one knows (cf. [20], Chap. 3) that $L(g,s)$ has an analytic continuation to the whole complex plane. $L(g,s)$ enjoys an infinite Euler product representation

$$L(g,s) = \prod_{\ell | N} (1 - a_\ell(g)\ell^{-s})^{-1} \prod_{\ell \nmid N} (1 - a_\ell(g)\ell^{-s} + \ell^{k-1-2s})^{-1}$$

[2]For the definition of $a_\ell(E)$ in the remanning case $\ell | N$, the reader is referred to [8], page 7.

if and only if g is a normalized eigenform for all the Hecke operators T_ℓ where $\ell \nmid N$. If g in addition is an eigenvector of the Fricke involution W_N, then it also satisfies a symmetrical functional equation with respect to the transformation "$s \to k - s$".

The Shimura-Taniyama-Weil conjecture simply (well, simply!) states that

$$f_E := \sum_{n=1}^{\infty} a_n(E) q^n = \sum_{n=1}^{\infty} a_n q^n$$

is a newform in $S_2(\Gamma_0(N))$ whose Mellin transform defined as in (2.5) equals the L-series of E. In other words, the two objects E and f_E correspond[3] to one another via the equality of their L-series

$$L(E, s) = L(f_E, s).$$

Moreover, this L-series satisfies the functional equation

$$\Lambda(E, 2 - s) = \operatorname{sign}(E, \mathbb{Q}) \Lambda(E, s), \tag{2.6}$$

where

$$\Lambda(E, s) := N^{s/2} (2\pi)^{-s} L(E, s).$$

The number $\operatorname{sign}(E, \mathbb{Q}) = \pm 1$ appearing in the functional equation is known to be equal to the opposite of the eigenvalue w_N of the Fricke involution W_N of (1.12) acting on f_E.

[3]Not in a one-to-one fashion!

More generally, to a form $g \in S_k(\Gamma_0(N))$ and a Dirichlet character χ of conductor prime to N one may associate the χ-twisted L-function

$$L(g, \chi; s) := \sum_{n=1}^{\infty} \chi(n) a_n(g) n^{-s},$$

which enjoys the same properties as $L(g, s)$ does, with of course the appropriate modifications. For instance, the sign in the functional equation satisfied by $L(E, \chi; s)$ is equal to $-w_{_N}\chi(-N)$.

2.2.2 L-Functions over K

Though defined over $\mathbb{Q}$, the curve E may also be viewed as an elliptic curve over K, and one may likewise define its L-function $L(E/K, s)$ over K by defining the local Euler factors at each prime ideal $\boldsymbol{\lambda}$ of $\mathcal{O}_K$ as follows.

For simplicity, we shall denote by $|\boldsymbol{\lambda}|$ the norm of a non-zero prime ideal $\boldsymbol{\lambda}$, and then we define the local Euler factor of $L(E/K, s)$ at a prime $\boldsymbol{\lambda}$ of good reduction[4] to be

$$(1 - a_{\boldsymbol{\lambda}} |\boldsymbol{\lambda}|^{-s} + |\boldsymbol{\lambda}|^{1-2s})^{-1},$$

where $a_{\boldsymbol{\lambda}} := |\boldsymbol{\lambda}| + 1 - \#E(\mathbb{F}_{|\boldsymbol{\lambda}|})$.

Remark 2.2.1. It is very important that the coefficient $a_{\boldsymbol{\lambda}}$ is associated to the ideal $\boldsymbol{\lambda}$ and not to its norm $|\boldsymbol{\lambda}|$. In other words, we would not get the

[4]The definition of $a_{\boldsymbol{\lambda}}$ is exactly the same is that of a_ℓ (albeit not given!) if E has bad reduction at $\boldsymbol{\lambda}$.

right L-function if we had defined the local Euler factor of $L(E/K, s)$ at $\boldsymbol{\lambda}$ using the relation "$a_{|\boldsymbol{\lambda}|} = |\boldsymbol{\lambda}| + 1 - \#E(\mathbb{F}_{|\boldsymbol{\lambda}|})$". This would only give the correct factor(s)—as will be discussed below—if $\boldsymbol{\lambda}$ lies over some rational prime ℓ which is *split* in K.

We now give a second description of the L-series of E over K, which is more useful for practical purposes, as follows. We first recall the definition of the *twist* E' of E by the quadratic character $\chi_{_K}$. Choosing a Weierstrass equation for $E/\mathbb{Q}$ of the form $y^2 = p(x)$ with integer coefficients, the elliptic curve E' can be described by the Weierstrass equation

$$E' \ : \ \Delta y'^2 = p(x').$$

The regular map

$$E' \longrightarrow E, \qquad (a, b) \longmapsto (a, \sqrt{\Delta} b),$$

shows that E' is in fact isomorphic to E over $K = \mathbb{Q}(\sqrt{\Delta})$. The same is true over the finite field $\mathbb{F}_\ell$ if Δ is a square mod ℓ. That is to say, the reduction of E' at ℓ is isomorphic to that of E if ℓ is split in K, i.e., if $\chi_{_K}(\ell) = 1$. On writing

$$1 - a_\ell(E)\ell^{-s} + \ell^{1-2s} = (1 - \alpha_\ell \ell^{-s})(1 - \beta_\ell \ell^{-s}) \tag{2.7}$$

and

$$1 - a_\ell(E')\ell^{-s} + \ell^{1-2s} = (1 - \alpha'_\ell \ell^{-s})(1 - \beta'_\ell \ell^{-s}), \tag{2.8}$$

(for primes of good reduction) and examining the relations

$$\#E(\mathbb{F}_{\ell^k}) = 1 + \ell^k - {\alpha_\ell}^k - {\beta_\ell}^k \quad \text{and} \quad \#E'(\mathbb{F}_{\ell^k}) = 1 + \ell^k - {\alpha'_\ell}^k - {\beta'_\ell}^k,$$

valid for $k = 1, 2, 3, ...$, one verifies that

$$\alpha'_\ell = \chi_{_K}(\ell)\alpha_\ell \qquad \text{and} \qquad \beta'_\ell = \chi_{_K}(\ell)\beta_\ell. \tag{2.9}$$

It is now an easy matter to check, one prime at a time, that if $\ell \not| N\Delta$, then

$$\prod_{\boldsymbol{\lambda}|\ell}(1 - a_{\boldsymbol{\lambda}}|\boldsymbol{\lambda}|^{-s} + |\boldsymbol{\lambda}|^{1-2s}) = (1 - a_\ell(E)\ell^{-s} + \ell^{1-2s})(1 - a_\ell(E')\ell^{-s} + \ell^{1-2s}),$$

where the product on the left is taken over the prime ideals $\boldsymbol{\lambda}$ of $\mathcal{O}_K$ which divide ℓ. For, if $\ell\mathcal{O}_K = \boldsymbol{\lambda}\boldsymbol{\lambda}'$ splits in K, then this is true since

$$|\boldsymbol{\lambda}| = |\boldsymbol{\lambda}'| = \ell \qquad \text{and} \quad a_\ell(E) = \alpha_\ell + \beta_\ell = \alpha'_\ell + \beta'_\ell = a_\ell(E').$$

And if $\ell\mathcal{O}_K = \boldsymbol{\lambda}$ is inert in K, then $|\boldsymbol{\lambda}| = \ell^2$ and by virtue of the relations (2.7), (2.8) and (2.9) we are done again! So, we infer that[5]

$$L(E/K, s) = L(E, s)L(E', s). \tag{2.10}$$

Far more generally, to any pair consisting of an eigenform $g \in S_k(\Gamma_0(N))$ and a genus character $\mathcal{G}$ one can attach a complex L-function $L(g/K, \mathcal{G}; s)$ which, similar to the L-series $L(E/K, s)$, admits the factorisation

$$L(g/K, \mathcal{G}; s) = L(g, \chi_{_1}; s)L(g, \chi_{_2}; s), \tag{2.11}$$

[5] One also needs to take care of the bad primes; something which is left to the interested reader!

where (χ_1, χ_2) are the quadratic Dirichlet characters associated to $\mathcal{G}$. This last equality, which may be verified likewise at the level of local Euler factors and one prime at the time, will reduce to the relation (2.10) if g is the newform f_E attached to E and if $\mathcal{G}$ is the trivial genus character.

2.3 Modular Symbols

The aim of this section is to introduce modular symbols in general, to define a special type of modular symbol associated to modular forms, and to address the rule played by them in expressing the central critical values of L-series of modular forms.

2.3.1 Generalities on Modular Symbols

We begin with the definition of a modular symbol.

Definition 2.3.1. *Let $\mathcal{D}_0$ denote the group of degree zero divisors on the rational projective line $\mathbb{P}_1(\mathbb{Q})$. For an abelian group A, any element of $Hom_{\mathbb{Z}}(\mathcal{D}_0, A)$ is called an A-valued modular symbol. In other words, a modular symbol $\mathfrak{m}$ is an assignment which to any divisor of the form $D = (r) - (s) \in \mathcal{D}_0$, where $r, s \in \mathbb{P}_1(\mathbb{Q})$, attaches an element $\mathfrak{m}\{r \to s\}$ of A, and satisfies the relation*

$$\mathfrak{m}\{r \to s\} + \mathfrak{m}\{s \to t\} = \mathfrak{m}\{r \to t\},$$

for all $r, s, t \in \mathbb{P}_1(\mathbb{Q})$. We denote by $\mathcal{MS}(A)$ the set of all such modular symbols.

If A is endowed with a left action of a subgroup G of $\mathbf{GL_2}(\mathbb{Q})$, then one may let G act on $\mathcal{MS}(A)$ on the right by the rule

$$(\mathfrak{m}, \gamma) \longrightarrow \mathfrak{m}|\gamma,$$

$$(\mathfrak{m}|\gamma)\{r \to s\} := \gamma^{-1}\mathfrak{m}\{\gamma r \to \gamma s\},$$

where G is acting on $\mathbb{P}_1(\mathbb{Q})$ on the left by the usual Möbius transformations. Thus, $\mathcal{MS}(A)$ has the structure of a right G-module.

It is readily seen from the definition that a modular symbol $\mathfrak{m}$ is G-invariant (i.e., $\mathrm{Stab}_G(\mathfrak{m}) = G$) if

$$\gamma^{-1}\mathfrak{m}\{\gamma r \to \gamma s\} = \mathfrak{m}\{r \to s\}, \quad \text{for all} \quad \gamma \in G,$$

or equivalently,

$$\mathfrak{m}\{\gamma r \to \gamma s\} = \gamma\mathfrak{m}\{r \to s\}, \quad \text{for all} \quad \gamma \in G.$$

The set of all G-invariant modular symbols will be denoted by $\mathcal{MS}_G(A)$.

2.3.2 Modular Symbols Attached to Modular Forms

Germane to the investigations of the present thesis are modular symbols attached to modular forms. Let

$$g = \sum_{n=1}^{\infty} a_n(g) q^n \in S_k(\Gamma_0(N)) \tag{2.12}$$

be a normalized eigenform of even weight $k \geq 2$ on $\Gamma_0(N)$, and denote by K_g the field generated over $\mathbb{Q}$ by the Fourier coefficients $a_n(g)$. It can be proved that K_g is a totally real number field (cf. [9], Chapter 6).

Whenever necessitated by the context, we will regard the field K_g as a complex field (respectively as a p-adic field) by fixing once and for all an embedding of K_g into $\mathbb{C}$ (respectively into $\mathbb{C}_p$.)

Let now F be a subfield of $\mathbb{C}$ or $\mathbb{C}_p$. For a given $k \geq 2$, let $\mathbf{P}_k(F)$ be the space of homogeneous polynomials of degree $k-2$ in two variables with coefficients in F. The rule

$$(P|\gamma)(X,Y) := P(aX+bY, cX+dY), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.13}$$

endows $\mathbf{P}_k(F)$ with a right linear (i.e., $(P_1+P_2)|\gamma = P_1|\gamma + P_2|\gamma$) action of $\mathbf{GL_2}(F)$.

We also denote by $\mathbf{V}_k(F)$ the F-linear dual to $\mathbf{P}_k(F)$. The rule

$$(\gamma \cdot \Phi)(P) := \Phi(P|\gamma), \tag{2.14}$$

where $\Phi \in \mathbf{V}_k(F), \gamma \in \mathbf{GL_2}(F)$ and $P \in \mathbf{P}_k(F)$, also shows that $\mathbf{V}_k(F)$ inherits a left linear action by the same group.

The motivation to bring up these spaces is the following well known construction of Eichler and Shimura which to the form g associates a $\mathbf{V}_k(\mathbb{C})$-valued modular symbol $\widetilde{\mathbf{I}}_g$ by assigning to $(r,s) \in \mathbb{P}_1(\mathbb{Q})$ the functional

$\widetilde{\mathbf{I}}_g\{r \to s\}$ which takes any homogeneous polynomial $P(X,Y)$ in $\mathbb{C}[X,Y]$ of degree $k-2$ to the complex number

$$\widetilde{\mathbf{I}}_g\{r \to s\}(P) := 2\pi i \int_r^s g(z)P(z,1)dz, \tag{2.15}$$

where the integral is taken along the geodesic in the upper half-plane joining r to s. One notices that since g is assumed to be a cusp form, the integral used to define $\widetilde{\mathbf{I}}_g$ is convergent. The invariance property of the holomorphic differential $2\pi i g(z)(dz)^{\frac{k}{2}}$ under the group $\Gamma_0(N)$ also implies, after a simple change of variables, that the modular symbol $\widetilde{\mathbf{I}}_g$ enjoys an invariance property under the action of the same group. That is to say, for any $\gamma \in \Gamma_0(N)$, we have

$$\widetilde{\mathbf{I}}_g\{\gamma^{-1}r \to \gamma^{-1}s\}(P|\gamma) = \widetilde{\mathbf{I}}_g\{r \to s\}(P).$$

In other words, $\widetilde{\mathbf{I}}_g$ belongs to $\mathcal{MS}_{\Gamma_0(N)}(\mathbf{V}_k(\mathbb{C}))$.

One decomposes this latter space as a direct sum of two eigen-subspaces of the linear involution

$$\mathcal{MS}_{\Gamma_0(N)}(\mathbf{V}_k(\mathbb{C})) \longrightarrow \mathcal{MS}_{\Gamma_0(N)}(\mathbf{V}_k(\mathbb{C})), \qquad \mathfrak{m} \longmapsto \mathfrak{m}|\iota,$$

induced by the matrix

$$\iota := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is made possible by exploiting the fact that ι normalizes the group $\Gamma_0(N)$, i.e. $\iota^{-1}\Gamma_0(N)\iota = \Gamma_0(N)$, and that $\iota^2 = \mathbf{1}_2$. This gives rise to the decomposition of $\mathcal{MS}_{\Gamma_0(N)}(\mathbf{V}_k(\mathbb{C}))$ as the direct sum of the two eigen-subspaces corresponding respectively to the two eigenvalues $+1$ and -1. With respect to this decomposition, we shall write respectively $\mathfrak{m}^+$ and $\mathfrak{m}^-$ for the plus and minus eigen-components of any $\Gamma_0(N)$-invariant modular symbol $\mathfrak{m}$ with values in $\mathbf{V}_k(\mathbb{C})$. This being done, we have the following theorem of Shimura (cf. [21]).

Theorem 2.3.2. (Shimura) *There exist complex periods Ω_g^+ and Ω_g^- with the property that the modular symbols*

$$\mathbf{I}_g^+ := \frac{1}{\Omega_g^+}(\widetilde{\mathbf{I}}_g)^+ \quad \text{and} \quad \mathbf{I}_g^- := \frac{1}{\Omega_g^-}(\widetilde{\mathbf{I}}_g)^-$$

belong to $\mathcal{MS}_{\Gamma_0(N)}(\mathbf{V}_k(\mathcal{O}_g))$, where $\mathcal{O}_g$ is the ring of integers of the number field K_g. These periods can be chosen so as to satisfy

$$\Omega_g^+\Omega_g^- = \langle g, g\rangle.$$

Here $\langle\ ,\ \rangle$ is the Petersson inner product defined on $S_k(\Gamma_0(N))$ by the rule

$$\langle g_1, g_2\rangle := \frac{1}{[\Gamma(1):\Gamma_0(N)]}\int_{\Gamma_0(N)\backslash\mathcal{H}} g_1(z)\overline{g_2(z)}y^k\frac{dxdy}{y^2},$$

where $\Gamma(1) = \mathbf{SL_2}(\mathbb{Z})$. □

We recall the fixed genus character $\mathcal{G}$ and the corresponding pair of

Dirichlet characters (χ_1, χ_2) of Section 2.1, and set

$$(\Omega_g, \mathbf{I}_g) := \begin{cases} (\Omega_g^+, \mathbf{I}_g^+) & \text{if } \mathcal{G}(\mathfrak{c}) = +1; \\ (\Omega_g^-, \mathbf{I}_g^-) & \text{if } \mathcal{G}(\mathfrak{c}) = -1. \end{cases}$$

2.3.3 Modular Symbols and Special Values of L-Functions

For the duration of this section, let χ be a primitive Dirichlet character of conductor $\mathfrak{d}$. We wish to study, by means of the modular symbol $\mathbf{I}_g$, the central critical values $L(g, \chi; j)$ $(1 \leq j \leq k-1)$ of the Hecke L-function $L(g, \chi, s)$ associated to the form g. So, for any integer a, set

$$\mathbf{I}_g[j, a] := \mathbf{I}_g\{\infty \to \frac{a}{\mathfrak{d}}\}(P_a(X, Y)),$$

where $P_a(X, Y)$ is the degree $k-2$ homogeneous polynomial

$$P_a(X, Y) := \left(X - \frac{a}{\mathfrak{d}} Y\right)^{j-1} Y^{k-j-1},$$

and where the even integer $k \geq 2$ is as before the weight of the form g. By exploiting the invariance property of $\mathbf{I}_g$ under the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, one readily verifies that $\mathbf{I}_g[j, a]$ is $\mathfrak{d}$-periodic, i.e., its value, as a function of a, depends only on the class of a modulo $\mathfrak{d}$.

Before stating the relationship between the critical values of $L(g, \chi; s)$ and $\mathbf{I}_g$, let us recall the definition of the Gauss sum $\tau(\chi)$ attached to χ. It should be pointed out that for the following definition χ need not be primitive; we stick to this assumption, though, because it is needed in the proposition below. The Gauss sum associated to χ is by definition the quantity

$$\tau(\chi) := \sum_{a=1}^{\mathfrak{d}} \chi(a) e^{\frac{2\pi i a}{\mathfrak{d}}},$$

and it satisfies the well known relation

$$|\tau(\chi)|^2 = \chi(-1)\tau(\chi)\tau(\overline{\chi}) = \mathfrak{d}.$$

For a proof of the following proposition, known as the Birch-Manin formula, the reader is referred to [17], Chap. I, §8.

Proposition 2.3.3. *The assumptions being as before, the quantity*

$$L^*(g, \chi; j) := \frac{(j-1)!\tau(\chi)}{(-2\pi i)^{j-1}\Omega_g} L(g, \chi; j) \tag{2.16}$$

(for $1 \leq j \leq k-1$) is related to modular symbol $\mathbf{I}_g$ via the formula

$$L^*(g, \chi; j) = \sum_{a \bmod \mathfrak{d}} \chi(a) \mathbf{I}_g[j, a]. \tag{2.17}$$

If we further assume that $\chi(-1) = (-1)^{j-1}\mathcal{G}(\mathfrak{c})$, then

$$L^*(g, \chi; j) \in K_g. \qquad \square$$

The chosen embedding of K_g into $\mathbb{C}_p$ will allow us, in light of the second part of the proposition, to view $L^*(g, \chi; j)$, a priori a complex number and often referred to as the *algebraic part* of $L(g, \chi; j)$, as an element of $\mathbb{C}_p$.

2.4 Hida Families

We begin by recalling some standard facts concerning the ring of Iwasawa functions.

It is a known fact that the group of p-adic units $\mathbb{Z}_p^\times$ decomposes canonically as the product of the group of *principal units* $1 + p\mathbb{Z}_p$ and the group $\boldsymbol{\mu}_{p-1}$ of $(p-1)$st roots of unity[6]:

$$\begin{aligned} \mathbb{Z}_p^\times &= (1 + p\mathbb{Z}_p) \times \boldsymbol{\mu}_{p-1} \\ t &= \langle t \rangle \cdot \omega(t), \end{aligned} \tag{2.18}$$

where $\langle \cdot \rangle$ denotes the projection to principal units and where the projection to roots of unity is given by the *Teichmüller character* $\omega(\cdot)$. For all $t \in \mathbb{Z}_p^\times$, one knows that the relations

$$\langle t \rangle = \lim_{n\to\infty} t^{1-p^n}, \quad \omega(t) \equiv t \pmod p$$

hold. Now let

$$\tilde{\boldsymbol{\Lambda}} = \mathbb{Z}_p[[\mathbb{Z}_p^\times]] := \lim_{\leftarrow} \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times], \quad \boldsymbol{\Lambda} = \mathbb{Z}_p[[(1 + p\mathbb{Z}_p)^\times]]$$

denote the usual *Iwasawa algebras*, i.e., the completed group rings of $\mathbb{Z}_p^\times$ and $(1 + p\mathbb{Z}_p)^\times$ respectively. For each element $t \in \mathbb{Z}_p^\times$, let $[t] \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ denote

[6] Here we need to assume that $p \geq 3$. When $p = 2$, the corresponding decomposition is $\mathbb{Z}_2^\times = (1 + 4\mathbb{Z}_2) \times \boldsymbol{\mu}_2$.

the corresponding element of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. For each continuous character

$$\kappa : \mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p^\times$$

we denote by the same symbol the unique continuous ring homomorphism $\kappa : \tilde{\mathbf{\Lambda}} \longrightarrow \mathbb{Z}_p$ sending $[t]$ to $\kappa(t)$ for all $t \in \mathbb{Z}_p^\times$. This induces the identification

$$\mathcal{X} := \mathrm{Hom}_{\mathrm{ct}}(\tilde{\mathbf{\Lambda}}, \mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{ct}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \cong \frac{\mathbb{Z}}{(p-1)\mathbb{Z}} \times \mathbb{Z}_p, \tag{2.19}$$

and makes it possible to regard the elements of $\tilde{\mathbf{\Lambda}}$ as functions on the *weight space* $\mathcal{X}$. We shall alternatively write $\lambda(\kappa)$ for $\kappa(\lambda)$, where $\lambda \in \tilde{\mathbf{\Lambda}}$ and $\kappa \in \mathcal{X}$. One embeds $\mathbb{Z}$ into the space $\mathcal{X}$ by the rule

$$k \mapsto \mathbf{x}_k, \qquad \mathbf{x}_k(t) := t^{k-2} \quad \text{for } t \in \mathbb{Z}_p^\times.$$

Reserving the letter κ for a generic element of $\mathcal{X}$ and following an abuse of notation, we shall write k in place of its image under the above embedding. It is noted that under such embedding, the element 2 corresponds to the augmentation map on $\tilde{\mathbf{\Lambda}}$ and $\mathbf{\Lambda}$. It is also noted that $\mathcal{X}$ with its natural topology contains $\mathbb{Z}^{\geq 2}$ as a dense subset. Hence, $U \cap \mathbb{Z}^{\geq 2}$ is dense in U, for any open subset U of $\mathcal{X}$. For any such open subset U of $\mathcal{X}$, we let $\mathcal{A}(U)$ be the space of p-adic analytic functions defined on U, and remark that $\mathcal{A}(U)$ is a $\mathbb{Z}_p$-algebra.

Definition 2.4.1. *Let M be a positive integer prime to p. A p-adic family of modular forms of* **level** *M is a formal power series*

$$\sum_{n\geq 0}\mathcal{F}_n(\kappa)q^n \in \mathcal{A}(U)[[q]],$$

where each $\mathcal{F}_n(\kappa)\in\mathcal{A}(U)$ is a p-adic analytic function on U, with the property that for all sufficiently large even integers $k\in U$, its **weight**-*k*-**specialisation**

$$\sum_n \mathcal{F}_n(k)q^n$$

obtained by packaging the values at k of all the functions $\mathcal{F}_n$ into a single power series in $q=e^{2\pi iz}$ is the Fourier expansion of a classical modular form of weight k and level $N=pM$.

For example, let us recall the definition of the Eisenstein series[7]

$$G_k(z):=\frac{\zeta(1-k)}{2}+\sum_{n\geq 1}\sigma_{k-1}(n)q^n\in M_k(\Gamma(1))$$

of even weight $k\geq 4$ and then remark that the *modified* Eisenstein series

$$G_k^*(z):=G_k(z)-p^{k-1}G_k(pz)=\frac{(1-p^{k-1})\zeta(1-k)}{2}+\sum_{n\geq 1}\sigma_{k-1}^*(d)q^n$$

of *level* p and weight k (chosen from now on for technical reasons to be congruent to 0 (mod $p-1$)) are known to live in a family (cf. [5] or [19]). This means firstly that, given any $n\geq 1$, the assignment

$$\mathcal{F}_n^*\ :\ \kappa\longmapsto\sigma_{\kappa-1}^*(n):=\sum_{\substack{d|n\\ p\nmid d}}d^{\kappa-1}$$

[7]Here $\zeta(s)$ (for $s\in\mathbb{C}$) is the Riemann zeta-function.

is a well-defined $\mathbb{Z}_p$-valued analytic function on $\mathbb{Z}_p$, and secondly that, the assignment

$$k \equiv 0 \pmod{p-1} \longmapsto \frac{(1-p^{k-1})\zeta(1-k)}{2}$$

also extends[8] to an analytic function $\dfrac{\zeta^*(1-\kappa)}{2}$ on $\mathbb{Z}_p$ (except for a simple pole at $k=0$), where $\zeta^*(s)$ (for $s \in \mathbb{Z}_p$) is the celebrated *Kubota-Leopoldt* p-adic zeta-function. Hence, the formal power series

$$1 + \sum_{n\geq 1} \frac{2\mathcal{F}_n^*(\kappa)}{\zeta^*(1-\kappa)} q^n \in \mathcal{A}(U)[[q]]$$

is a *level one* p-adic family of modular forms whose weight-k-specialisation gives the q-expansion of the classical eigenform

$$E_k^*(z) := E_k(z) - p^{k-1}E_k(pz) = 1 + \frac{2}{(1-p^{k-1})\zeta(1-k)} \sum_{n\geq 1} \sigma_{k-1}^*(n) q^n$$

of level p.

Remark 2.4.2. It is fairly easy to show that any modular form of level N lives in a p-adic family of level M. One can simply multiply the form in question by the family of Eisenstein series thus constructed.

In contrast to this remark, a much more challenging question is, given an *eigenform* $f(z)$ of level N and some weight $k \geq 2$, does there exist a p-adic family of *eigenforms* of level M whose weight-k-specialisation is $f(z)$?

[8]This latter fact is actually a consequence of the former due to a very general result of Serre in [19] concerning the Fourier coefficients of p-adic modular forms.

We should point out that the trick used in the preceding remark (i.e., multiplying a eigenform by the family of Eisenstein Series as above) will not produce a family of eigenforms, for the product of two eigenforms is not necessarily an eigenform. Nevertheless, the question stated above has been affirmatively answered by Hida [13] in the ordinary (i.e., zero slop) case and by Coleman [6] in the general (i.e., finite slop) case. In below we will be content to state Hida's work, since this thesis is only concerned with the ordinary case, namely, with the case where the p-adic valuation of the p-th Fourier coefficient of the form in question is zero.

Theorem 2.4.3. (Hida) *Every ordinary cuspidal eigenform $f(z)$ of weight $k \geq 2$ and level N lives in a p-adic family of eigenforms of level M.* $\square$

Recall that the elliptic curve E has been assumed to have multiplicative reduction at p. This implies that the p-th Fourier coefficient a_p of f_E is ± 1, and hence a p-adic unit. Therefore, it follows from Hida's theorem that one can attach to f_E a *formal* power series

$$\mathfrak{h}_E = \sum_{n=1}^{\infty} \mathcal{F}_n q^n = \sum_{n=1}^{\infty} \mathcal{F}_n(\kappa) q^n \tag{2.20}$$

with the coefficients $\mathcal{F}_n = \mathcal{F}_n(\kappa)$ in the algebra $\mathcal{A}(U)$, where U is an appropriate open neighbourhood of $2 \in \mathcal{X}$. The formal q-expansion $\mathfrak{h}_E$ gives rise to a family of cuspidal eigenforms and it also interpolate the newform f_E at $k = 2$. More precisely, we have the following.

1. For any integer $k \geq 2$ in U, the q-expansion

$$f_k := \sum_{n=1}^{\infty} \mathcal{F}_n(k) q^n, \qquad q = e^{2\pi i z}, \tag{2.21}$$

is a classical normalized cuspidal eigenform of weight k on $\Gamma_0(N)$.

2. $f_2 = f_{_E}$.

Remark 2.4.4. In Hida's theorem, we may, and do for simplicity, assume that U is contained in the residue disc of 2 modulo $p-1$. This is made possible by the observation that since the weight space $\mathcal{X}$ is identified with the direct product of $\mathbb{Z}/(p-1)\mathbb{Z}$ (equipped with the discrete topology) and $\mathbb{Z}_p$ (equipped with the p-adic topology), it could be viewed as a disjoint union of $p-1$ copies of $\mathbb{Z}_p$.

Remark 2.4.5. As in the case of the family of Eisenstein series discussed earlier, if $k > 2$, then the form

$$f_k = \sum_{n=1}^{\infty} a_n(k) q^n, \quad (a_n(k) := \mathcal{F}_n(k))$$

(while being new at all the prime divisors of $M = N/p$), is *old* at the prime p in the sense that it arises from a normalised newform

$$\hat{f}_k = \sum_{n=1}^{\infty} \hat{a}_n(k) q^n$$

of weight k on $\Gamma_0(M)$. More precisely, one has

$$f_k(z) = \hat{f}_k(z) - \beta_p(k) \hat{f}_k(pz),$$

where $\beta_p(k)$ is the *non-unit Frobenius root* appearing in the factorisation of the Euler factor at p of the L-series of $\hat{f}_k$ as

$$1 - \hat{a}_p(k) p^{-s} + p^{k-1-2s} = (1 - \alpha_p(k) p^{-s})(1 - \beta_p(k) p^{-s}).$$

It is possible to order the Frobenius unit and non-unit eigenvalues $\alpha_p(k)$ and $\beta_p(k)$ in such a way that

$$\alpha_p(k) = a_p(k), \quad \beta_p(k) = p^{k-1}a_p(k)^{-1},$$

where $a_p(k)$ is the p-th Fourier coefficient of the form f_k. With this conventions, one can also verify that if $p \not| n$, then $\hat{a}_n(k) = a_n(f_k)$. Therefore, the relationship between the L-series of f_k and $\hat{f}_k$, twisted both by χ, is

$$\begin{aligned} L(f_k, \chi; s) &= \left(1 - \chi(p)a_p(k)^{-1}p^{k-1-s}\right) L(\hat{f}_k, \chi; s) \\ &= (1 - \chi(p)\beta_p(k)p^{-s})L(\hat{f}_k, \chi; s). \end{aligned} \tag{2.22}$$

It follows from the preceding discussion that the number field $K_{\hat{f}_k}$ generated by the Fourier coefficients of $\hat{f}_k$ is contained in that of f_k; it could therefore, via the same embeddings, be viewed as a subfield of both $\mathbb{C}$ and $\mathbb{C}_p$.

To the form f_k are associated the Shimura periods $\Omega^{\pm}_{f_k}$ (as in Proposition 2.3.2) as well as the modular symbol $\mathbf{I}_{f_k}$. One may also use the same periods $\Omega^{\pm}_k$ to define the modular symbol $\mathbf{I}_{\hat{f}_k}$ attached to the form $\hat{f}_k$. In order to relax the notation, we shall write

$$\Omega^{\pm}_k := \Omega^{\pm}_{f_k} = \Omega^{\pm}_{\hat{f}_k}, \qquad \Omega_k := \Omega_{f_k} = \Omega_{\hat{f}_k}, \qquad \mathbf{I}_k := \mathbf{I}_{f_k}, \qquad \hat{\mathbf{I}}_k := \mathbf{I}_{\hat{f}_k}.$$

One uses these periods to define likewise as in (2.16) the algebraic parts of the special values of $L(f_k, \chi; s)$ and $L(\hat{f}_k, \chi; s)$. For $1 \leq j \leq k-1$, one then verifies that

$$L^*(f_k, \chi; j) = (1 - \chi(p)a_p(k)^{-1}p^{k-1-j})L^*(\hat{f}_k, \chi; j). \tag{2.23}$$

To define the algebraic part of the special values of the $\mathcal{G}$-twisted L-function $L(f_k/K, \mathcal{G}; s)$ of the form f_k over the real quadratic field K, let us first recall from Subsection 2.2.2 the relation

$$L(f_k/K, \mathcal{G}; s) = L(f_k, \chi_1; s)L(f_k, \chi_2; s). \tag{2.24}$$

For any $1 \leq j \leq k-1$, the algebraic part of $L(f_k/K, \mathcal{G}; j)$ is now defined as the quantity

$$L^*(f_k/K, \mathcal{G}; j) := \frac{\sqrt{\Delta}(j-1)!^2}{(-2\pi i)^{2j-2}\Omega_k^2} L(f_k/K, \mathcal{G}; j). \tag{2.25}$$

The algebraic part of $L(\hat{f}_k/K, \mathcal{G}; j)$ is also defined similarly.

2.5 Measure-Valued Modular Symbols

For a compact totally disconnected topological space Y we let $\mathrm{Cont}(Y)$ denote the module of $\mathbb{Z}_p$-valued continuous functions on Y and $\mathrm{Step}(Y)$ denote the submodule of locally constant functions. We equip $\mathrm{Cont}(Y)$ with the topology induced by the sup-norm and note that $\mathrm{Step}(Y)$ is a dense submodule. We define the module of $\mathbb{Z}_p$-valued *measures* on Y to be

$$\mathrm{Meas}(Y) := \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Step}(Y), \mathbb{Z}_p).$$

Every measure $\mu \in \mathrm{Meas}(Y)$ is easily seen to have a unique extension to a continuous homomorphism $\mu : \mathrm{Cont}(Y) \longrightarrow \mathbb{Z}_p$. Hence

$$\mathrm{Meas}(Y) \cong \mathrm{Cont.Hom}_{\mathbb{Z}_p}(\mathrm{Cont}(Y), \mathbb{Z}_p).$$

We endow $\mathrm{Meas}(Y)$ with the weak topology dual to $\mathrm{Cont}(Y)$. If $\mu \in \mathrm{Meas}(Y)$, $F \in \mathrm{Cont}(Y)$ and $X \subseteq Y$ is a compact-open set, we will habitually write $\displaystyle\int_Y F d\mu$ (respectively $\mu(X)$) for $\mu(F)$ (respectively for $\mu(\mathbf{1}_X)$), where $\mathbf{1}_X$ is the indicator (i.e., characteristic) function of X.

Let now[9]

$$L_* := \mathbb{Z}_p^2 = \mathbb{Z}_p \oplus \mathbb{Z}_p \tag{2.26}$$

denote the *standard* $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$. For any $\mathbb{Z}_p$-lattice L in $\mathbb{Q}_p^2$, we write L' for the set of *primitive vectors* in L, that is, the set of those vectors $v \in L$ which are not divisible by p in L:

$$L' := L \setminus pL = \{v \in L \ : \ v \notin pL\}. \tag{2.27}$$

Note, for example, that

$$L'_* = (\mathbb{Z}_p \times \mathbb{Z}_p^\times) \sqcup (\mathbb{Z}_p^\times \times p\mathbb{Z}_p), \tag{2.28}$$

which is a simple consequence of the fact that $\mathbb{Z}_p = \mathbb{Z}_p^\times \sqcup p\mathbb{Z}_p$.[10]

[9] The author apologizes for overusing the letter L for too many different purposes: various L-functions, $\mathbb{Z}_p$-sublattices of $\mathbb{Q}_p^2$,...!

[10] The symbol $\sqcup$ stands for the *disjoint union*.

We equip the space of continuous $\mathbb{C}_p$-valued functions on L'_* with the right action of $\mathbf{GL_2}(\mathbb{Z}_p)$ defined by the rule

$$(F|\gamma)(x,y) := F(ax+by, cx+dy), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and denote its continuous dual by $\mathbf{D}_*$, i.e.,

$$\mathbf{D}_* := \mathrm{Meas}(L'_*).$$

The action of the group $\mathbb{Z}_p^\times$ on L'_* given by $t(x,y) := (tx, ty)$ gives rise to a natural $\tilde{\mathbf{\Lambda}}$-module structure on $\mathbf{D}_*$ by setting

$$\int_{L'_*} F(x,y)d([t]\cdot\mu)(x,y) := \int_{L'_*} F(tx,ty)d\mu(x,y),$$

for all $t \in \mathbb{Z}_p^\times$. If X is any compact-open subset of L'_*, we adopt the common notation

$$\int_X F d\mu := \int_{L'_*} \mathbf{1}_X F d\mu.$$

Note that the group $\mathbf{GL_2}(\mathbb{Z}_p)$ also acts on $\mathbf{D}_*$ on the left by translation, so that

$$\int_X F d(\gamma\cdot\mu) = \int_{\gamma^{-1}X} (F|\gamma) d\mu.$$

Denote by $\Gamma_0(p\mathbb{Z}_p)$ the group of matrices in $\mathbf{GL_2}(\mathbb{Z}_p)$ which are upper triangular modulo p. The space $\mathbf{D}_*$ is endowed, for all $k \in \mathbb{Z}^{\geq 2}$, with a $\Gamma_0(p\mathbb{Z}_p)$-equivariant homomorphism

$$\rho_k : \mathbf{D}_* \longrightarrow \mathbf{V}_k(\mathbb{C}_p), \quad \mu \mapsto \rho_k(\mu)$$

defined by the rule

$$\rho_k(\mu)(P) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x,y) d\mu(x,y).$$

One easily notes that any mapping $\rho : A \longrightarrow B$ between abelian groups induces a mapping on modular symbols

$$\mathcal{MS}(A) \longrightarrow \mathcal{MS}(B),\ \mathfrak{m} \mapsto \rho(\mathfrak{m}),$$

by the rule

$$\rho(\mathfrak{m})\{r \to s\} := \rho(\mathfrak{m}\{r \to s\}).$$

In particular, the homomorphisms ρ_k, for all $k \geq 2$, lift to homomorphisms on modular symbols. It can also be verified that such lift will send $\Gamma_0(M)$-invariant $\mathbf{D}_*$-valued modular symbols to $\Gamma_0(N)$-invariant $\mathbf{V}_k(\mathbb{C}_p)$-valued modular symbols (cf. [12]). That is to say, each ρ_k gives rise to a mapping

$$\rho_k : \mathcal{MS}_{\Gamma_0(M)}(\mathbf{D}_*) \longrightarrow \mathcal{MS}_{\Gamma_0(N)}(\mathbf{V}_k(\mathbb{C}_p)),$$

which is, just for simplicity, denoted by the same letter.

We denote by $\mathbf{\Lambda}^\dagger \supset \mathbf{\Lambda}$ the ring of power series with coefficients in $\mathbb{C}_p$ which converge in some neighbourhood of $2 \in \mathcal{X}$, and set

$$\mathbf{D}_*^\dagger := \mathbf{D}_* \otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}^\dagger.$$

If

$$\mu = \lambda_1\mu_1 + \cdots + \lambda_r\mu_r, \qquad \text{with } \lambda_j \in \mathbf{\Lambda}^\dagger,\ \mu_j \in \mathbf{D}_*,$$

is any element of $\mathbf{D}_*^\dagger$, then there exists a neighbourhood U_μ of $2 \in \mathcal{X}$ on which all the coefficients λ_j converge. Call such a region U_μ a *neighbourhood of regularity* for μ.

Given $\kappa \in U_\mu$, a continuous function $F(x,y)$ on L'_* is said to be *homogeneous of degree* $\kappa - 2$ if $F(tx,ty) = t^{\kappa-2}F(x,y)$, for all $t \in \mathbb{Z}_p^\times$. For any $\kappa \in U_\mu$, and any homogeneous function $F(x,y)$ of degree $\kappa-2$, the function F can be integrated against μ by the rule

$$\int_X F d\mu := \lambda_1(k)\int_X F d\mu_1 + \cdots + \lambda_r(k)\int_X F d\mu_r,$$

for any compact-open $X \subset L'_*$.

The space $\mathcal{MS}_{\Gamma_0(M)}(\mathbf{D}_*)$ is equipped with a natural action of the Hecke operator U_p, given by the formula

$$\int_X F d(U_p|\mu)\{r \to s\} := \sum_{a=0}^{p-1} \int_{\frac{1}{p}\gamma_a(X)} (F|p\gamma_a^{-1}) d\mu\{\gamma_a r \to \gamma_a s\},$$

where $\gamma_a := \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$ for each $0 \le a \le p-1$.

Let $\mathcal{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbf{D}_*)$ denote the *ordinary subspace* of $\mathcal{MS}_{\Gamma_0(M)}(\mathbf{D}_*)$, and set

$$\mathcal{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbf{D}_*)^\dagger := \mathcal{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbf{D}_*) \otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}^\dagger \subset \mathcal{MS}_{\Gamma_0(M)}(\mathbf{D}_*^\dagger).$$

Given $\mu \in \mathcal{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbf{D}_*)^\dagger$, the measure $\mu\{r \to s\}$, for $r,s \in \mathbb{P}_1(\mathbb{Q})$, may be viewed as an element of $\mathbf{D}_*^\dagger$, and a common neighbourhood of regularity

U_μ for all the measures $\mu\{r \to s\}$ can be chosen. This makes it possible to define $\rho_k(\mu)$ for all $k \in U_\mu \cap \mathbb{Z}^{\geq 2}$. All this being said, we record the following important result of Greenberg and Stevens [11], [12].

Theorem 2.5.1. (Greenberg-Stevens) *There exists a neighbourhood U of $2 \in \mathcal{X}$ and a measure-valued modular symbol $\mu_* \in \mathcal{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbf{D}_*)^\dagger$ which is regular on U, and satisfies the following two conditions:*

1. *For each $k \in U \cap \mathbb{Z}^{\geq 2}$, there exists a scalar $\lambda(k) \in \mathbb{C}_p$ such that*

$$\rho_k(\mu_*) = \lambda(k)\mathbf{I}_k. \tag{2.29}$$

In other words, for any $r, s \in \mathbb{P}_1(\mathbb{Q})$ and any homogeneous polynomial $P(X,Y) \in \mathbf{P}_k(\mathbb{Q})$ of degree $k-2$, we have

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x,y) d\mu_*\{r \to s\} = \lambda(k)\mathbf{I}_k\{r \to s\}\left(P(z,1)\right). \tag{2.30}$$

2. *$\lambda(2) = 1$.* $\square$

Remark 2.5.2. According to Proposition 1.7 of [2] the neighbourhood U in the statement of theorem above may be chosen so as to satisfy

$$\lambda(k) \neq 0, \quad \text{for all } k \in U \cap \mathbb{Z}^{\geq 2}.$$

We assume, from now on, that this has been done. We should, however, remark that the assignment $k \mapsto \lambda(k)$ need not *a priori* extend to a continuous, let alon analytic, function on the space $\mathcal{X}$ (cf. [2], Remark 1.6). An understanding of this issue is not germane to the proofs of the theorems in this thesis.

It will be important to know what happens if we integrate a homogeneous polynomial of degree $k-2$ against the measure μ_* over the larger domain $L'_* = (\mathbb{Z}_p \times \mathbb{Z}_p^\times) \sqcup (\mathbb{Z}_p^\times \times p\mathbb{Z}_p)$. Such identity is furnished by the next proposition whose proof is explained in [3].

Proposition 2.5.3. *For any $r, s \in \mathbb{P}_1(\mathbb{Q})$ and for any homogeneous polynomial $P(X,Y) \in \mathbf{P}_k(\mathbb{Q})$ of degree $k-2$, we have*

$$\begin{aligned} \int_{L'_*} P(x,y)d\mu_*\{r \to s\} &= \lambda(k)(1 - a_p(k)^{-2}p^{k-2}) \times \\ &\times \hat{\mathbf{I}}_k\{r \to s\}\left(P(z,1)\right), \end{aligned} \tag{2.31}$$

where $\hat{\mathbf{I}}_k$ is the modular symbol associated to the form $\hat{f}_k$ defined in the previous section. □

Let $\mathbf{D}$ denote the $\mathbf{\Lambda}$-module of compactly supported measures on

$$\mathcal{W} := \mathbb{Q}_p^2 \setminus \{(0,0)\} = \{v \in \mathbb{Q}_p^2 \ : \ v \neq (0,0)\},$$

and put $\mathbf{D}^\dagger := \mathbf{D} \otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}^\dagger$. The space $\mathbf{D}_*$ is contained in $\mathbf{D}$ by viewing elements of $\mathbf{D}_*$ as measures on $\mathcal{W}$ with compact support in L'_*. In Proposition 1.8 of [2] (see also [4]), a family $\{\mu_L\}$ of $\mathbf{D}^\dagger$-valued modular symbols indexed by the $\mathbb{Z}_p$-lattices L in $\mathbb{Q}_p^2$ is attached to μ_* by exploiting the action of the group

$$\sum := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL_2}(\mathbb{Z}[1/p]) \ : \ M \mid c, \quad \det(\gamma) > 0 \right\}. \tag{2.32}$$

Proposition 2.5.4. *There exists a unique collection $\{\mu_L\}$ of $\mathbf{D}^\dagger$-valued modular symbols, indexed by the $\mathbb{Z}_p$-lattices $L \subset \mathbb{Q}_p^2$, with the following properties.*

1. *For all $\gamma \in \sum$, all compact-open $X \subset \mathcal{W}$ and all homogeneous functions F, we have*

$$\int_{\gamma X} (F|\gamma^{-1}) d\mu_{\gamma L}\{\gamma r \to \gamma s\} = \int_X F d\mu_L\{r \to s\}. \tag{2.33}$$

2. $\mu_{L_*} = \mu_*$. □

2.6 p-adic L-Functions

In this section we recall very briefly the definitions and some basic properties of various p-adic L-functions which play a role in this thesis.

2.6.1 The Mazur-Swinnerton-Dyer p-adic L-Function

Our goal is to associate to any ordinary[11] normalised eigenform g of weight $k = 2$ and level N a p-adic L-function. We begin with a definition.

Definition 2.6.1. *Given any $\nu \in Meas(\mathbb{Z}_p^\times)$, the one-variable p-adic L-function $L_p(\nu, s)$ attached to ν is a $\mathbb{Z}_p$-valued function defined on $\mathbb{Z}_p$ by the rule*

$$L_p(\nu, s) := \int_{\mathbb{Z}_p^\times} \langle t \rangle^{s-1} d\nu(t).$$

[11] As before, "ordinary" simply means that the p-th Fourier coefficient $a_p(g)$ of g is a p-adic unit.

The next order of business will be to attach to g an appropriate measure ν_g which in return gives us the function $L_p(g,s)$ sought for. The construction of such measure will not be given in full generality, but rather in the case where g has rational integer Fourier coefficients.

The first observation to make is that, since g is assumed to be ordinary, exactly one of the two Frobenius roots in the factorisation of the local Euler factor of $L(g,s)$ at p is a p-adic unit (See also Remark 2.4.5.) In other words, if

$$1 - a_p(g)p^{-s} + p^{1-2s} = (1 - \alpha_p(g)p^{-s})(1 - \beta_p(g)p^{-s}),$$

then, we may without loss of generality assume that

$$\alpha_p(g) \in \mathbb{Z}_p^\times \quad \text{and} \quad \beta_p(g) \notin \mathbb{Z}_p^\times.$$

This is true since

$$\alpha_p(g) + \beta_p(g) = a_p(g) \in \mathbb{Z}_p^\times \quad \text{and} \quad \alpha_p(g)\beta_p(g) = p \in p\mathbb{Z}_p.$$

Now for each $b \in \mathbb{Z}$ prime to p and for each $n > 0$, we set

$$\nu_g(b + p^n\mathbb{Z}_p) := \frac{1}{\alpha_p(g)^n}\mathbf{I}_g\{i\infty \to \frac{b}{p^n}\}(1),$$

where 1 is viewed as the constant (homogeneous) polynomial 1 of degree $2-2=0$ in two variables. For one thing, after a straightforward calculation it follows from the fact that g is an eigenform for the Atkin-Lehner operator U_p

with the eigenvalue $\alpha_p(g)$ that ν_g is a *distribution* (i.e., an additive function) on the compact-open subsets of $\mathbb{Z}_p^\times$. For another thing, due to the condition placed on $\alpha_p(g)$ and on account of Theorem 2.3.2, it follows that ν_g takes on values in $\mathbb{Z}_p$. Therefore, ν_g is a measure on $\mathbb{Z}_p^\times$. With this measure in hand we now give the following definition.

Definition 2.6.2. *The p-adic L-function associated to the form g is defined to be the p-adic function attached to the measure ν_g thus constructed. In other words, we set*

$$L_p(g,s) := L_p(\nu_g, s) = \int_{\mathbb{Z}_p^\times} \langle t \rangle^{s-1} d\nu_g(t).$$

More generally, the article [17] (see also [12]), based on ideas of [16], explains how a one variable p-adic L-function $L_p(g, \chi; s)$ can be attached to any pair consisting of an ordinary normalised eigenform $g \in S_k(N, \varepsilon)$ of weight $k \geq 2$ on $\Gamma_1(N)$ (with some *Nebentypus* character ε and with no restriction on the field K_g) and a Dirichlet character χ of conductor prime to N, viewed as a character with values in $\mathbb{C}_p^\times$. If χ is the trivial character, we stick to the simpler notation $L_p(g,s)$. In particular, since the p-th Fourier coefficient a_p of the newform f_E of Section 1.3 is ± 1, hence a p-adic unit, f_E possesses a p-adic L-function $L_p(f_E, s)$ attached to it. We recall $L(E,s) = L(f_E, s)$ and likewise set

$$L_p(E,s) := L_p(f_E, s).$$

Similar to the L-function $L(E,s)$, the p-adic L-function $L_p(E,s)$ also satisfies a functional equation (of the type (2.6)) with respect to the substitution $s \mapsto 2-s$. More precisely, if we let

$$\Lambda_p(E,s) := \langle M\rangle^{s/2} L_p(E,s),$$

then

$$\Lambda_p(E, 2-s) = -w_{_M} \Lambda_p(E,s), \tag{2.34}$$

where, as before, w_d (for $d|N$ and $(d, N/d) = 1$) is the eigenvalue of the Fricke involution W_d of (1.12) acting on $f_{_E}$. The relationship between $w_{_N}$ and $w_{_M}$ is given by a result of Atkin and Lehner as

$$w_{_N} = -a_p w_{_M}. \tag{2.35}$$

This relation is indeed equivalent to the equality $w_p = -a_p$ (cf. [12]). It follows from this relation together with the interpolation property of $L_p(E,s)$ that if E in particular has *split multiplicative reduction* at p, that is to say, if $a_p = 1$, then $L_p(E,s)$ trivially vanishes to order at least one at the central critical point $s = 1$. This property of $L_p(E,s)$ is called (after [17]) the *exceptional zero phenomenon.*

2.6.2 The Mazur-Kitagawa p-adic L-Function

The family of Mazur-Swinnerton-Dyer p-adic L-functions $L_p(f_k, s)$ associated to the weight-k-specialisations of $\mathfrak{h}_E$ can be packaged into a single two-variable p-adic L-function. Mazur and Kitagawa were the first to construct these two-variable p-adic L-functions attached to any ordinary $\mathbf{\Lambda}$-adic cusp form (cf, §3 of [12]). This $\mathbb{C}_p$-valued function $L_p(\mathfrak{h}_E;\kappa,s)$ is p-adic analytic on $U\times\mathbb{Z}_p$, and for each even integer $k\geq 2$ in U, interpolates the p-adic L-function $L_p(f_k,s)$ in the sense that

$$L_p(\mathfrak{h}_E;k,s)=\lambda(k)L_p(f_k,s),\quad (s\in\mathbb{Z}_p) \tag{2.36}$$

where $\lambda(k)$ is as in the equation (2.29).

More generally, one can attach to $\mathfrak{h}_E$ and a quadratic Dirichlet character χ a p-adic analytic function which likewise interpolates the twisted p-adic L-functions $L_p(f_k,\chi;s)$ as follows.

Let $\chi:\left(\frac{\mathbb{Z}}{\mathfrak{d}\mathbb{Z}}\right)^{\times}\longrightarrow\{\pm 1\}$ be a primitive (quadratic) Dirichlet character of conductor $\mathfrak{d}$, and as before, let $\tau(\chi)$ denote the Gauss sum attached to it.

Definition 2.6.3. *The Mazur-Kitagawa two-variable p-adic L-function associated to $\mathfrak{h}_E$ and χ is a function of $(k,s)\in U\times\mathbb{Z}_p$ defined by the rule*

$$L_p(\mathfrak{h}_E,\chi;\kappa,s):=\sum_{a=1}^{\mathfrak{d}}\chi(ap)\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}\left(x-\frac{pa}{\mathfrak{d}}y\right)^{s-1}y^{\kappa-s-1}d\mu_*\{\infty\to\frac{pa}{\mathfrak{d}}\}.$$

As alluded to in the first paragraph of this subsection, the function $L_p(\mathfrak{h}_E,\chi;\kappa,s)$ interpolates the family $\{L_p(f_k,\chi;s)\ :\ k\in\mathbb{Z}^{\geq 2}\cap U\}$. For our later applications, however, it is more convenient to state the interpolation property of $L_p(\mathfrak{h}_E,\chi;\kappa,s)$ with respect to the special values of the classical

L-functions $L(f_k, \chi; s)$. For a proof of the following theorem, the reader is referred to [2].

Theorem 2.6.4. *Suppose that $\mathfrak{d}$ is not divisible by p. Choose any $k \in U \cap \mathbb{Z}^{\geq 2}$ and suppose that $1 \leq j \leq k-1$ satisfies $\chi(-1) = (-1)^{j-1}\mathcal{G}(\mathfrak{c})$. Then, the equality*

$$L_p(\mathfrak{h}_E, \chi, k, j) = \lambda(k)(1 - \chi(p)a_p(k)^{-1}p^{j-1})L^*(f_k, \chi, j) \tag{2.37}$$

holds. □

Remark 2.6.5. In light of the relation (2.23), Theorem 2.6.4 can also be rewritten in terms of the special values of the L-series of $\hat{f}_k$ as

$$\begin{aligned} L_p(\mathfrak{h}_E, \chi, k, j) &= \lambda(k)(1 - \chi(p)a_p(k)^{-1}p^{j-1}) \times \\ &\times (1 - \chi(p)a_p(k)^{-1}p^{k-j-1})L^*(\hat{f}_k, \chi, j). \end{aligned}$$

Note in particular that, after specialising to $j = k/2$, one obtains

$$L_p(\mathfrak{h}_E, \chi, k, k/2) = \lambda(k)\left(1 - \chi(p)a_p(k)^{-1}p^{\frac{k}{2}-1}\right)^2 L^*(\hat{f}_k, \chi, k/2). \tag{2.38}$$

That the Euler factor appearing in this formula is a perfect square will play a significant role in later applications.

2.7 Optimal Embeddings of K

Given any quadratic number field K, and any elliptic curve E defined over $\mathbb{Q}$ of conductor N, one says that the pair (E, K) satisfies the *Heegner hypothesis* if all the prime divisors of N are split in K. We recall that the real quadratic field K and the elliptic curve E of Chapter 1 satisfy this condition. Notice that this assumption implies that $\gcd(N, \Delta) = 1$ and that the discriminant Δ is a square modulo $4N$. Another consequence implied by our running assumption is the equality

$$\operatorname{sign}(E, K) = 1, \tag{2.39}$$

where $\operatorname{sign}(E, K)$ denotes the sign in the functional equation satisfied by the L-function $L(E/K, s)$ of E over K (See [8] or [1] for an explanation of this relation.) In particular, the Birch and Swinnerton-Dyer conjecture predicts that the group $E(K)$ has even rank.

Definition 2.7.1. *A $\mathbb{Q}$-algebra embedding*

$$\Psi : K \longrightarrow \mathbf{M_2}(\mathbb{Q})$$

of K into the (indefinite everywhere split quaternion) algebra $\mathbf{M_2}(\mathbb{Q})$ is called an **optimal embedding** *of K (or sometimes an* **optimal embedding** *of $\mathcal{O}_K$) of level N if*

$$\Psi(K) \cap \mathbf{M_0}(N) = \Psi(\mathcal{O}_K),$$

where

$$\mathbf{M_0}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \ : \ N \mid c \right\}.$$

That such embeddings exist is a consequence of the Heegner hypothesis. In fact, one has the following lemma whose proof may be found for example in [1], p. 185.

Lemma 2.7.2. *Suppose that no prime divisor of N is ramified in K. Then the following are equivalent.*

1. *The Heegner hypothesis holds for the pair (E, K).*

2. *Optimal embeddings of K of level N exist.*

3. *There exists a* **cyclic ideal** *of $\mathcal{O}_K$ of norm N, i.e., an ideal $\mathcal{N}$ of $\mathcal{O}_K$ such that the quotient ring $\mathcal{O}_K/\mathcal{N}$ is isomorphic to $\mathbb{Z}/N\mathbb{Z}$.* □

We remark that the group $\mathbf{M_0}(N)^\times$, and hence its subgroup

$$\Gamma_0(N) = \mathbf{M_0}(N)_1^\times := \{\gamma \in \mathbf{M_0}(N) \ : \ \det\gamma = 1\},$$

acts on the set of optimal embeddings by conjugation. That is to say, if Ψ is an optimal embedding, then so is the conjugate embedding

$$\Psi^\gamma = \gamma\Psi\gamma^{-1} : K \longrightarrow \mathbf{M_2}(\mathbb{Q}), \quad \alpha \mapsto \gamma\Psi(\alpha)\gamma^{-1}, \tag{2.40}$$

for all $\gamma \in \mathbf{M_0}(N)^\times$. The set of equivalence classes of this action is in bijection with the ideal class-group of K, where the bijection goes as follows.

One attaches to an ideal class $A \in \mathcal{C}_K$ of K an optimal embedding Ψ_A by choosing a representative (i.e., a fractional ideal) $\mathfrak{a}$ in A of norm prime to N, a $\mathbb{Z}$-basis $\{e_1, e_2\}$ for $\mathfrak{a}$ such that e_1 belongs to $\mathfrak{a} \cap \mathcal{N}$, and then defining

Ψ_A as[12]

$$\Psi_A(\alpha) := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{where} \quad \begin{cases} \alpha e_1 = ae_1 + ce_2; \\ \alpha e_2 = be_1 + de_2. \end{cases}$$

The resulting embedding is independent of the choice of $\mathfrak{a}$ and its chosen basis $\{e_1, e_2\}$ up to conjugation by $\mathbf{M_0}(N)^\times$, so that the assignment

$$A \longrightarrow \Psi_A$$

sets up a bijection between ideal class-group of K and optimal embeddings of $\mathcal{O}_K$ into $\mathbf{M_0}(N)$ up to conjugation by $\mathbf{M_0}(N)^\times$.

If $\mathfrak{a}$ represents $A \in \mathcal{C}_K^+$ as a narrow ideal class, one may also insist that the basis of $\mathfrak{a}$ be *oriented*, i.e., that[13]

$$\det \begin{pmatrix} e_1 & e_1' \\ e_2 & e_2' \end{pmatrix} > 0.$$

Then Ψ_A becomes well defined up to conjugation by $\Gamma_0(N)$, and the assignment sets up a bijection between the narrow ideal class-group of K and the set of equivalent classes of the oriented optimal embeddings of $\mathcal{O}_K$ into $\mathbf{M_0}(N)$. In summary, we have the following lemma whose proof may be found either in [1], page 186 or [7], Proposition 1.4.

[12] Note that this definition generalises the assignment $A \to \gamma_A$ given in (1.2).

[13] We recall that for $\alpha \in K$, its Galois conjugate (over $\mathbb{Q}$) is denoted by α'.

Lemma 2.7.3. *As before, let h_K (respectively h_K^+) denote the class-number (respectively narrow class-number) of K. Then there are exactly h_K (respectively h_K^+) distinct optimal embeddings of K of level N, up to conjugation by $\mathbf{M_0}(N)^\times$ (respectively by $\Gamma_0(N)$).* $\square$

Thanks to the identifications discussed preceding this lemma, the set of oriented optimal embeddings of $\mathcal{O}_K$ into $\mathbf{M_0}(N)$, taken up to conjugation by $\mathbf{M_0}(N)^\times$ (respectively by $\Gamma_0(N)$) becomes a *principal homogeneous space* for the action of $\mathcal{C}_K$ (respectively of $\mathcal{C}_K^+$). Note also that the same set could also be viewed, through the Artin reciprocity map (1.1), as a principal homogeneous space for the action of the Galois group $\mathrm{Gal}(\mathcal{H}_K^+/K)$. These actions will be denoted respectively by[14]

$$(\mathfrak{p}, \Psi) \longrightarrow \Psi^{\mathfrak{p}} \qquad \text{and} \qquad (\sigma_{\mathfrak{p}}, \Psi) \longrightarrow \Psi^{\sigma_{\mathfrak{p}}},$$

and both notations will be used interchangeably.

Remark 2.7.4. If $h_K^+ = 1$, then it follows from global class field theory that there exists a p-unit element in K, i.e., an element $u_p \in \mathcal{O}_K$ whose norm $\mathrm{Norm}_{K/\mathbb{Q}}(u_p) = p$. In such circumstance, one easily shows that the image of u_p in $\mathbf{M_2}(\mathbb{Q})$ under any optimal embedding is of determinant p. We remark that in the general case, the rule of this p-unit in the computations of the next chapter is to be played by a prime ideal $\mathfrak{p}$ of K lying over p. In fact,

[14]It is really the class $[\Psi]$ represented by Ψ on which the (narrow) ideal class $[\mathfrak{p}]$ represented by $\mathfrak{p}$ acts, etc.

one can again show that the the image of $\mathfrak{p}$ is a matrix of determinant p of the type

$$\begin{pmatrix} p & \star \\ 0 & 1 \end{pmatrix}. \tag{2.41}$$

This remark makes it possible, following an abuse of notation, to write $\det \mathfrak{p} = p$. For an explicit description of such matrix representation of $\mathfrak{p}$, see [18], page 859, the relation (6.1.11).

Definition 2.7.5. *Let Ψ be an optimal embedding of K of level N. Attached to Ψ is the binary quadratic form*

$$Q_\Psi(X,Y) := cX^2 + (d-a)XY - bY^2, \tag{2.42}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \Psi(\sqrt{\Delta})$. *Since $\Delta > 0$, the equation $Q_\Psi(z,1) = 0$ has two real roots τ and τ'. In fact, these are the fixed points of the action of $\Psi(K^\times)$ on $\mathbb{P}_1(\mathbb{Q})$ by Möbius transformations. We now set*

$$\mathbf{e} := \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \quad \mathbf{e}' := \begin{pmatrix} \tau' \\ 1 \end{pmatrix}, \quad L_\Psi := \mathbb{Z}_p\mathbf{e} \oplus \mathbb{Z}_p\mathbf{e}'. \tag{2.43}$$

Remark 2.7.6. It will be important for later applications to know the variations under conjugation of the quadratic forms $Q_\Psi(X,Y)$ and the $\mathbb{Z}_p$-lattices L_Ψ associated to optimal embeddings. Such relationships are readily verified to be given by

$$Q_{\gamma\Psi\gamma^{-1}}(X,Y) = (\det\gamma)\cdot(Q_\Psi|\gamma^{-1})(X,Y), \quad L_{\gamma\Psi\gamma^{-1}} = \gamma L_\Psi. \tag{2.44}$$

We should also notice that the $\mathbb{Z}_p$-lattice L_Ψ associated to Ψ is preserved by the action of $\Psi(K)$. One may order the two eigenvectors $\mathbf{e}$ and $\mathbf{e}'$ of this action in such a way that for any $\alpha \in K$,

$$\Psi(\alpha) \cdot \mathbf{e} = \alpha \mathbf{e}.$$

We end this section with another lemma whose statement will be of some use afterwards. Before doing so, however, one more definition is required.

Definition 2.7.7. *Let L be a $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$. The generalised index of L, denoted by $|L|$, is defined as $p^{\mathrm{ord}_p(\det\gamma)}$, where γ is any element of $\mathbf{GL_2}(\mathbb{Q}_p)$ satisfying $\gamma(\mathbb{Z}_p^2) = L$.*

One easily verifies that the generalised index of any lattice L is independent of the choice of γ. It also enjoys the following property

$$|\gamma L| = p^{\mathrm{ord}_p \det\gamma} |L|,$$

valid for any lattice L and any matric $\gamma \in \mathbf{GL_2}(\mathbb{Q}_p)$. One may also expect that the generalised index of the lattice L_Ψ is related to the quadratic form $Q_{_\Psi}$. Indeed, such relation exists and is descried precisely in the last lemma of this section whose proof may be found in [2], §3.

Lemma 2.7.8. *The function ord$_p Q_{_\Psi}$ is constant on $\mathbb{Z}_p^\times \mathbf{e} \times \mathbb{Z}_p^\times \mathbf{e}'$. More precisely, for any $(x, y) \in \mathbb{Z}_p^\times \mathbf{e} \times \mathbb{Z}_p^\times \mathbf{e}'$, the identity*

$$\mathrm{ord}_p Q_\Psi(x, y) = \mathrm{ord}_p |L_\Psi| \tag{2.45}$$

holds. □

This brings us to the end of necessary backgrounds, so that we are now ready to launch the final chapter the goal of which is to prove Theorems A and B.

Chapter 3

p-adic Deformation of Shintani Cycles

This chapter includes five sections culminating in the proof of our main results. In the first section we introduce the lattice functions c_k and $\hat{c}_k$ and then establish the main identity satisfied by them. The main object of this thesis, namely the p-adic analytic function $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$, will be introduced in the second section. In the section to follow we will finish the proof of Theorem A by proving one of the salient properties of $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$. That is to say, we will prove an interpolation formula justifying the name "p-adic L-function". The objective of the next section, namely Section 3.4, will be the proof of the Factorization Formula already mentioned in Section 1.4. And finally, the proof of Theorem B will be demonstrated in the final section of the present chapter.

3.1 Lattice Functions

As already explained several times, the main goal of this thesis is to show the existence of a p-adic analytic function which interpolates a certain $\mathcal{G}$-twisted sum of Shintani cycles and whose second derivative, evaluated at $k = 2$, is intimately related to global points on the elliptic curve E. By way of providing a context for the construction of such function, and inspired by the lattice functions in §2 and §3 of [2], we introduce the following modular symbols.

Definition 3.1.1. *(1) Let (L_1, L_2) be a pair of $\mathbb{Z}_p$-lattices of $\mathbb{Q}_p^2$ where L_2 is a sublattice of L_1 with index p. For $r, s \in \mathbb{P}_1(\mathbb{Q})$, the $\mathbf{V}_k(\mathbb{C}_p)$-valued modular symbol $c_k(L_1, L_2)$ is defined by the integral formula*

$$c_k(L_1, L_2)\{r \to s\}(P) \quad := \quad \int_{L_1' \cap L_2'} P d\mu_{L_2}\{r \to s\}, \tag{3.1}$$

where $P \in \mathbf{P}_k(\mathbb{C}_p)$ is any homogeneous polynomial of degree $k - 2$.

(2) Let L be any $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$. For $r, s \in \mathbb{P}_1(\mathbb{Q})$, the $\mathbf{V}_k(\mathbb{C}_p)$-valued modular symbol $\hat{c}_k(L)$ is defined by the integral formula

$$\hat{c}_k(L)\{r \to s\}(P) \quad := \quad \int_{L'} P d\mu_L\{r \to s\}. \tag{3.2}$$

Using the invariance property (2.33) of Proposition 2.5.4, one immediately verifies that both c_k and $\hat{c}_k$ are $\sum$-invariant in the sense that for each

$\gamma \in \sum$, we have

$$\begin{aligned} c_k(\gamma L_1, \gamma L_2)\{\gamma r \to \gamma s\}(P|\gamma^{-1}) &= c_k(L_1, L_2)\{r \to s\}(P), & (3.3) \\ \hat{c}_k(\gamma L)\{\gamma r \to \gamma s\}(P|\gamma^{-1}) &= \hat{c}_k(L)\{r \to s\}(P). & (3.4) \end{aligned}$$

Another simple observation is that c_k and $\hat{c}_k$ are both homogeneous functions of degree p^{k-2}, namely,

$$c_k(pL_1, pL_2) = p^{k-2} c_k(L_1, L_2), \quad \hat{c}_k(pL) = p^{k-2}\hat{c}_k(L). \tag{3.5}$$

We now prove a very crucial identity which relates these two modular symbols.

Proposition 3.1.2. (Main Identity) *For any pair (L_1, L_2) as in Definition 3.1.1 and for any positive integer $k \geq 2$ in U, we have*

$$\begin{aligned} (1 - a_p(k)^{-1}p^{k-2})c_k(L_1, L_2) &= \hat{c}_k(L_2) - a_p(k)^{-1}\hat{c}_k(pL_1) \\ &= \hat{c}_k(L_2) - a_p(k)^{-1}p^{k-2}\hat{c}_k(L_1). \quad (3.6) \end{aligned}$$

Proof. Recall the standard lattice $L_* = \mathbb{Z}_p \oplus \mathbb{Z}_p$ of Section 2.5. Let us also introduce $L_\infty := \mathbb{Z}_p \oplus p\mathbb{Z}_p$, which is a sublattice of L_* with index p. It is a simple matter of computation to see that

$$\frac{1}{p}L'_\infty \cap L'_* = \mathbb{Z}_p \times \mathbb{Z}_p^\times, \qquad L'_* \cap L'_\infty = \mathbb{Z}_p^\times \times p\mathbb{Z}_p. \tag{3.7}$$

The usefulness of introducing these lattices arises from the fact that in the special case where

$$L_1 = \frac{1}{p}L_\infty \quad \text{and} \quad L_2 = L_*,$$

the statement of the lemma is just a manifestation of the identity

$$\mathbf{I}_k\{r \to s\}(P) = \hat{\mathbf{I}}_k\{r \to s\}(P) - a_p(k)^{-1}\hat{\mathbf{I}}_k\{\theta^{-1}r \to \theta^{-1}s\}(P|\theta); \tag{3.8}$$

an identity which is implicit in the proof of Proposition 2.3 of [3]. Here θ is an element[1] in the group $\sum$ of (2.32) satisfying

$$\theta(\frac{1}{p}L_\infty) = L_* \quad \text{and} \quad \theta(L_*) = L_\infty. \tag{3.9}$$

The second remark to follow is that the group $\sum$ acts transitively on the set of pairs of latices (L_1, L_2), where L_2 is a sublattice of L_1 with index[2] p. This fact is indeed a consequence of the Strong Approximation Theorem (cf. [8], Chapter 9, in particular Exercise 1 of that chapter.) These two facts combined will yield the general case. To see this, given an arbitrary pair (L_1, L_2) with $[L_1 : L_2] = p$, let us choose $\gamma \in \sum$ to satisfy

$$\gamma\left(\frac{1}{p}L_\infty\right) = L_1, \quad \gamma(L_*) = L_2. \tag{3.10}$$

Replacing $r,\ s$ and P respectively by $\gamma^{-1}r,\ \gamma^{-1}s$ and $P|\gamma$ in (3.8) and multiplying the factor $\lambda(k)(1-a_p(k)^{-2}p^{k-2})$ to both sides of the resulting equality, we arrive at

$$\begin{aligned}
&\lambda(k)(1-a_p(k)^{-2}p^{k-2})\mathbf{I}_k\{\gamma^{-1}r \to \gamma^{-1}s\}(P|\gamma) = \\
&\qquad = \lambda(k)(1-a_p(k)^{-2}p^{k-2})\hat{\mathbf{I}}_k\{\gamma^{-1}r \to \gamma^{-1}s\}(P|\gamma) \\
&\qquad - a_p(k)^{-1}\lambda(k)(1-a_p(k)^{-2}p^{k-2})\hat{\mathbf{I}}_k\{\theta^{-1}\gamma^{-1}r \to \theta^{-1}\gamma^{-1}s\}(P|\gamma\theta).
\end{aligned}$$

[1] Of which there are infinitely many.

[2] This set is by definition the set of unoriented edges of the *Bruhat-Tits tree* of the group $\mathbf{PGL_2}(\mathbb{Q}_p)$ (cf. [8], Chapter 5).

By applying (2.30), the last equality can be rewritten, in light of (3.7), as

$$
\begin{aligned}
(1 - a_p(k)^{-2}p^{k-2}) \int_{\frac{1}{p}L'_\infty \cap L'_*} (P|\gamma) d\mu_*\{\gamma^{-1}r \to \gamma^{-1}s\} = \\
&= \int_{L'_*} (P|\gamma) d\mu_*\{\gamma^{-1}r \to \gamma^{-1}s\} \\
&- a_p(k)^{-1} \int_{L'_*} (P|\gamma\theta) d\mu_*\{\theta^{-1}\gamma^{-1}r \to \theta^{-1}\gamma^{-1}s\} \\
&= \int_{L'_*} (P|\gamma) d\mu_*\{\gamma^{-1}r \to \gamma^{-1}s\} \\
&- a_p(k)^{-1} \int_{L'_\infty} (P|\gamma) d\mu_{L_\infty}\{\gamma^{-1}r \to \gamma^{-1}s\},
\end{aligned}
$$

where the last equality follows from the invariant property of (2.33) under θ. This last equality can be rewritten in terms of the lattice functions c_k and $\hat{c}_k$ as

$$
\begin{aligned}
(1 - a_p(k)^{-2}p^{k-2}) c_k(\tfrac{1}{p}L_\infty, L_*)\{\gamma^{-1}r \to \gamma^{-1}s\}(P|\gamma) = \\
= \hat{c}_k(L_*)\{\gamma^{-1}r \to \gamma^{-1}s\}(P|\gamma) - a_p(k)^{-1}\hat{c}_k(L_\infty)\{\gamma^{-1}r \to \gamma^{-1}s\}(P|\gamma).
\end{aligned}
$$

This will finish the proof once we apply the invariance property of c_k and $\hat{c}_k$ with respect to γ as in (3.3) and (3.4). $\square$

3.2 Yet Another p-adic L-Function

The goal of this section is to introduce the p-adic analytic function $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ as the main object of interest in our work. In order to define it, we first associate to any narrow ideal class—or equivalently, to any optimal embedding of K—a p-adic analytic function. Before doing so, however, we prove the following general lemma.

Lemma 3.2.1. *Let L be a $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$, let F be a homogeneous function, and let X be a compact-open set in $\mathcal{W} = \mathbb{Q}_p^2 \setminus \{(0,0)\}$. Also assume that $\gamma \in \sum$ preserves L, F and X, i.e.,*

$$\gamma L = L, \quad \gamma X = X, \quad F|\gamma = F.$$

Then for any $r,\ s \in \mathbb{P}_1(\mathbb{Q})$, we have

$$\int_X F d\mu_L\{r \to \gamma r\} = \int_X F d\mu_L\{s \to \gamma s\}. \tag{3.11}$$

Proof. According to the modular symbol property, we have

$$\begin{aligned}
\int_X F d\mu_L\{r \to s\} \quad + \quad & \int_X F d\mu_L\{s \to \gamma s\} = \\
= \quad & \int_X F d\mu_L\{r \to \gamma s\} \\
= \quad & \int_X F d\mu_L\{r \to \gamma r\} + \int_X F d\mu_L\{\gamma r \to \gamma s\} \\
= \quad & \int_X F d\mu_L\{r \to \gamma r\} + \int_{\gamma^{-1}X} (F|\gamma) d\mu_{\gamma^{-1}L}\{r \to s\} \\
= \quad & \int_X F d\mu_L\{r \to \gamma r\} + \int_X F d\mu_L\{r \to s\},
\end{aligned}$$

where the penultimate equality follows from (2.33), whereas the last equality follows from the trivial action of γ on L, X and F. A simple cancelation now completes the proof. □

For the following definition we recall that ϵ_K is the fundamental unit of K of positive norm. And now:

Definition 3.2.2. *Let $[\Psi]$ be a class of optimal embeddings of K of level N and set $\gamma_\Psi := \Psi(\epsilon_K)$. Also choose an arbitrary base point $z_0 \in \mathbb{P}_1(\mathbb{Q})$ on the* *extended upper half-plane. To this data we associate the function $\mathcal{L}_p(\Psi,\kappa)$, as a function of $\kappa \in U$, given by the rule*[3]

$$\mathcal{L}_p(\Psi,\kappa) := \int_{\mathbb{Z}_p^\times \mathbf{e} \times \mathbb{Z}_p^\times \mathbf{e}'} \langle Q_{\Psi}(x,y)\rangle^{\frac{\kappa-2}{2}} d\mu_{\Psi}\{z_0 \to z_{\Psi}\}, \tag{3.12}$$

where, in order to ease the notation, we have written μ_{Ψ} (respectively z_{Ψ}) in place of the measure associated to L_Ψ (respectively in place of $\gamma_{\Psi} z_0$).

Remark 3.2.3. One sees easily that Proposition 2.5.4 together with the relations (2.44) imply that the integral appearing in the definition of $\mathcal{L}_p(\Psi,\kappa)$ is unchanged if Ψ is replaced by $\gamma\Psi\gamma^{-1}$. That the definition of $\mathcal{L}_p(\Psi,\kappa)$ is also independent of the choice of the base point z_0 follows from the last lemma.

Remark 3.2.4. It follows from Lemma 2.7.8 that the special value of $\mathcal{L}_p(\Psi,k)$ at a positive even integer $k \in U$ is given by[4]

$$\mathcal{L}_p(\Psi,k) = |L_\Psi|^{-\frac{k-2}{2}} \int_{\mathbb{Z}_p^\times \mathbf{e} \times \mathbb{Z}_p^\times \mathbf{e}'} Q_{\Psi}(x,y)^{\frac{k-2}{2}} d\mu_{\Psi}\{z_0 \to z_{\Psi}\}.$$

Our next objective is to prove a formula expressing $\mathcal{L}_p(\Psi,k)$ in terms of the lattice function $\hat{c}_k$. Before doing so, however, let us first introduce some useful notation which will lighten the exposition. We fix once and for all a prime ideal $\mathfrak{p}$ of K lying above p. Remark 2.7.4 then implies that[5]

$$\mathfrak{p}^j L_\Psi = \mathfrak{p}^j\left(\mathbb{Z}_p\mathbf{e} \oplus \mathbb{Z}_p\mathbf{e}'\right) = p^j\mathbb{Z}_p\mathbf{e} \oplus \mathbb{Z}_p\mathbf{e}', \tag{3.13}$$

[3] Recall that for $t \in \mathbb{Z}_p^\times$, $\langle t\rangle = t\omega(t)^{-1}$, where ω is the Teichmüller character.

[4] Recall that $|L|$ is the generalised index of a lattice L defined as in Definition 2.7.7.

[5] The same relations are also explained in detail (using an adèlic approach) in Lemma 3.10 of [2].

where the action of $\mathfrak{p}$ on the lattice L_Ψ is given by the multiplication on the left of the 2×2 matrix (2.41). Therefore, for each j, the lattice $\mathfrak{p}^j L_{_\Psi}$ contains (respectively is contained in) the lattice $\mathfrak{p}^{j+1} L_{_\Psi}$ (respectively the lattice $\mathfrak{p}^{j-1} L_{_\Psi}$) with index p. For any $j \in \mathbb{Z}$ and any $k \in U \cap \mathbb{Z}^{\geq 2}$, we set

$$\Gamma_k^{(j)}[\Psi] := |\mathfrak{p}^j L_{_\Psi}|^{-\frac{k-2}{2}} \hat{c}_k(\mathfrak{p}^j L_{_\Psi})\{z_0 \to z_{_\Psi}\}(Q_{_\Psi}^{\frac{k-2}{2}}).$$

Whenever convenient, we suppress Ψ from the notation $\Gamma_k^{(j)}[\Psi]$ and simply write $\Gamma_k^{(j)}$, if this results in no ambiguity.

Remark 3.2.5. If the lattice L_Ψ happens to be L_*, then one computes

$$\begin{aligned}
\Gamma_k^{(j)} &= |\mathfrak{p}^j L_*|^{-\frac{k-2}{2}} \hat{c}_k(\mathfrak{p}^j L_*)\{z_0 \to \Psi(\epsilon_{_K}) z_0\} \left(Q_{_\Psi}^{\frac{k-2}{2}}\right) \\
&= (\det \mathfrak{p}^{-j})^{\frac{k-2}{2}} \int_{\mathfrak{p}^j L'_*} Q_{_\Psi}^{\frac{k-2}{2}} \, d\mu_{_{\mathfrak{p}^j L_*}}\{z_0 \to \Psi(\epsilon_{_K}) z_0\} \\
&= \int_{L'_*} \left(Q_{_\Psi}|\mathfrak{p}^j\right)^{\frac{k-2}{2}} d\mu_*\{z_0 \to \Psi^{\mathfrak{p}^j}(\epsilon_{_K}) z_0\} \\
&= \lambda(k)(1 - a_p(k)^{-2} p^{k-2}) \hat{\mathbf{I}}_k\{z_0 \to \Psi^{\mathfrak{p}^j}(\epsilon_{_K}) z_0\} \left(Q_{_{\Psi^{\mathfrak{p}^j}}}(z,1)^{\frac{k-2}{2}}\right),
\end{aligned} \tag{3.14}$$

where the penultimate equality follows from (2.33) together with the action of $\mathfrak{p}^j$ on Ψ. This important calculation will be exploited crucially in the proof of Theorem A (see the proof of Proposition 3.3.1.)

Before stating the next lemma we remark that if t denotes the order[6] of $\mathfrak{p}$ in the group $\mathcal{C}_K^+$, since $\mathfrak{p}^t$ is the trivial class, $\Gamma_k^{(j)}$ as a function of $j \in \mathbb{Z}$

[6] I.e., the order of the ideal class containing $\mathfrak{p}$.

is t-periodic. That is to say, for all $k \in U \cap \mathbb{Z}^{\geq 2}$, we have

$$\Gamma_k^{(j+t)} = \Gamma_k^{(j)}. \tag{3.15}$$

We now assert that:

Lemma 3.2.6. *For any integer $k > 2$ in U, we have*[7]

$$\begin{aligned} \mathcal{L}_p(\Psi, k) &= \frac{1}{1 - a_p(k)^{-2} p^{k-2}} \times \\ &\times \left[(1 + a_p(k)^{-2} p^{k-2}) \Gamma_k^{(0)} - a_p(k)^{-1} p^{\frac{k-2}{2}} (\Gamma_k^{(-1)} + \Gamma_k^{(1)}) \right]. \end{aligned} \tag{3.16}$$

Proof. We should first remark that since the p-th Fourier coefficient $a_p(k)$ of the eigenform f_k is a p-adic unit (cf. Remark 2.4.5), the denominator of the fraction above never vanishes. We now proceed the proof. On account of the relation (3.13), one readily verifies, for any $j \in \mathbb{Z}$, that

$$\mathfrak{p}^j L'_{\Psi} = (p^j \mathbb{Z}_p \mathbf{e} \times \mathbb{Z}_p^\times \mathbf{e}') \sqcup (p^j \mathbb{Z}_p^\times \mathbf{e} \times p\mathbb{Z}_p \mathbf{e}').$$

This in return implies that

$$\mathfrak{p}^j L'_{\Psi} \cap \mathfrak{p}^{j+1} L'_{\Psi} = p^{j+1} \mathbb{Z}_p \mathbf{e} \times \mathbb{Z}_p^\times \mathbf{e}'.$$

It is now fairly easy, by setting $j = 0$ and $j = 1$ in the last equality, to see that the domain of integration in Definition 3.2.2 can be expressed as

$$\mathbb{Z}_p^\times \mathbf{e} \times \mathbb{Z}_p^\times \mathbf{e}' = \left(\mathfrak{p}^{-1} L'_{\Psi} \cap L'_{\Psi} \right) - \left(L'_{\Psi} \cap \mathfrak{p} L'_{\Psi} \right).$$

[7] For "why not $k \geq 2$?", please see Remark 3.2.8.

Hence,

$$\begin{aligned}\mathcal{L}_p(\Psi,k) &= |L_{\Psi}|^{-\frac{k-2}{2}}\int_{\mathbb{Z}_p^\times \mathbf{e}\times\mathbb{Z}_p^\times \mathbf{e}'} Q_{\Psi}^{\frac{k-2}{2}}\,d\mu_{\Psi}\{z_0\to z_{\Psi}\}\\ &= |L_{\Psi}|^{-\frac{k-2}{2}}\left(\int_{\mathfrak{p}^{-1}L'_{\Psi}\cap L'_{\Psi}} Q_{\Psi}^{\frac{k-2}{2}}\,d\mu_{\Psi}\{z_0\to z_{\Psi}\}\right.\\ &- \left.\int_{L'_{\Psi}\cap\mathfrak{p}L'_{\Psi}} Q_{\Psi}^{\frac{k-2}{2}}\,d\mu_{\Psi}\{z_0\to z_{\Psi}\}\right).\end{aligned}$$

Now we use the modular symbols c_k and $\hat{c}_k$ to rewrite the above equality as

$$\begin{aligned}\mathcal{L}_p(\Psi,k) &= |L_{\Psi}|^{-\frac{k-2}{2}}\Bigg[c_k(\mathfrak{p}^{-1}L_{\Psi},L_{\Psi})\{z_0\to z_{\Psi}\}(Q_{\Psi}^{\frac{k-2}{2}})\\ &- a_p(k)^{-1}c_k(L_{\Psi},\mathfrak{p}L_{\Psi})\{z_0\to z_{\Psi}\}(Q_{\Psi}^{\frac{k-2}{2}})\Bigg]\\ &= \frac{|L_{\Psi}|^{-\frac{k-2}{2}}}{1-a_p(k)^{-2}p^{k-2}}\Bigg[\Big(\hat{c}_k(L_{\Psi})-a_p(k)^{-1}p^{k-2}\hat{c}_k(\mathfrak{p}^{-1}L_{\Psi})\Big)\\ &- a_p(k)^{-1}\Big(\hat{c}_k(\mathfrak{p}L_{\Psi})-a_p(k)^{-1}p^{k-2}\hat{c}_k(L_{\Psi})\Big)\Bigg]\{z_0\to z_{\Psi}\}(Q_{\Psi}^{\frac{k-2}{2}})\\ &= \frac{|L_{\Psi}|^{-\frac{k-2}{2}}}{1-a_p(k)^{-2}p^{k-2}}\Bigg[(1+a_p(k)^{-2}p^{k-2})\hat{c}_k(L_{\Psi})-\\ &- a_p(k)^{-1}p^{\frac{k-2}{2}}\Big(p^{\frac{k-2}{2}}\hat{c}_k(\mathfrak{p}^{-1}L_{\Psi})+p^{-\frac{k-2}{2}}\hat{c}_k(\mathfrak{p}L_{\Psi})\Big)\Bigg]\{z_0\to z_{\Psi}\}(Q_{\Psi}^{\frac{k-2}{2}}).\end{aligned}$$

Finally once we observe that (see also the paragraph preceding Lemma 2.7.8)

$$|\mathfrak{p}^jL_{\Psi}| = p^j|L_{\Psi}|,\qquad \text{for all } j\in\mathbb{Z},$$

we may rewrite $\mathcal{L}_p(\Psi,k)$ in terms of $\Gamma_k^{(j)}$'s as

$$\begin{aligned}\mathcal{L}_p(\Psi,k) &= \frac{1}{1-a_p(k)^{-2}p^{k-2}}\times\\ &\times\Big[(1+a_p(k)^{-2}p^{k-2})\Gamma_k^{(0)}-a_p(k)^{-1}p^{\frac{k-2}{2}}(\Gamma_k^{(-1)}+\Gamma_k^{(1)})\Big].\end{aligned}$$

The proof is complete. □

The next proposition will pave the way for the proof of the interpolation formula of Proposition 3.3.1 culminating in the proof of Theorem A.

Proposition 3.2.7. *For any $j \in \mathbb{Z}$, write Ψ_j for the action of the j-th power of $\mathfrak{p}$ on Ψ. Then, for all $k \in \mathbb{Z}^{>2} \cap U$, we have*[8]

$$\begin{aligned}
\sum_{j=0}^{t-1} \mathcal{G}(\mathfrak{p}^j)\mathcal{L}_p(\Psi_j, k) &= \frac{\left(1 - \chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}\right)^2}{1 - a_p(k)^{-2}p^{k-2}} \sum_{j=0}^{t-1} \chi_1(p)^j \Gamma_k^{(j)} \\
&= \frac{1 - \chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}}{1 + \chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}} \sum_{j=0}^{t-1} \chi_1(p)^j \Gamma_k^{(j)}, \qquad (3.17)
\end{aligned}$$

where as before t is the order of $\mathfrak{p}$ in the group $\mathcal{C}_K^+$.

Proof. Firstly, notice that the second equality above is a simple consequence of the fact that since χ_1 is a quadratic character, $\chi_1(p)^2 = 1$, and hence

$$1 - a_p(k)^{-2}p^{k-2} = \left(1 - \chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}\right)\left(1 + \chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}\right).$$

Secondly, we argue that since p splits completely in K as $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}'$, as one of the instances of (2.1), we have

$$\mathcal{G}(\mathfrak{p}) = \chi_1(p).$$

Also as noted earlier, $\Gamma_k^{(j)} = \Gamma_k^{(j+t)}$. These two facts combined with Lemma

[8] In this proposition, Γ_k^j is still a shorthand for $\Gamma_k^j[\Psi]$.

3.2.6 then allow us to write the left hand-side as

$$\begin{aligned}\sum_{j=0}^{t-1}\mathcal{G}(\mathfrak{p}^j)\mathcal{L}_p(\Psi_j,k) &= \frac{1}{1-a_p(k)^{-2}p^{k-2}}\Bigg[(1+a_p(k)^{-2}p^{k-2})\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)} \\ &- a_p(k)^{-1}p^{\frac{k-2}{2}}\sum_{j=0}^{t-1}\chi_1(p)^j\Big(\Gamma_k^{(j-1)}+\Gamma_k^{(j+1)}\Big)\Bigg] \\ &= \frac{1}{1-a_p(k)^{-2}p^{k-2}}\Bigg[(1+a_p(k)^{-2}p^{k-2})\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)} - \\ &- \chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}\Big(\sum_{j=0}^{t-1}\chi_1(p)^{j-1}\Gamma_k^{(j-1)} + \\ &+ \sum_{j=0}^{t-1}\chi_1(p)^{j-1}\Gamma_k^{(j+1)}\Big)\Bigg].\end{aligned}$$

Now we note that $\chi_1(p)^2=1$, and that $\chi_1(p)^t=\mathcal{G}(\mathfrak{p})^t=\mathcal{G}(\mathfrak{p}^t)=1$. Therefore we may write

$$\sum_{j=0}^{t-1}\chi_1(p)^{j-1}\Gamma_k^{(j-1)}=\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)},$$

and

$$\sum_{j=0}^{t-1}\chi_1(p)^{j-1}\Gamma_k^{(j+1)}=\sum_{j=0}^{t-1}\chi_1(p)^{j+1}\Gamma_k^{(j+1)}=\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)}.$$

Putting all this together yields

$$\begin{aligned}\sum_{j=0}^{t-1}\mathcal{G}(\mathfrak{p}^j)\mathcal{L}_p(\Psi_j,k) &= \frac{1}{1-a_p(k)^{-2}p^{k-2}}\Bigg[(1+a_p(k)^{-2}p^{k-2})\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)} \\ &- \chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}\Big(\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)}+\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)}\Big)\Bigg] \\ &= \frac{\Big(1+a_p(k)^{-2}p^{k-2}-2\chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}\Big)}{1-a_p(k)^{-2}p^{k-2}}\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)} \\ &= \frac{\Big(1-\chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}\Big)^2}{1-a_p(k)^{-2}p^{k-2}}\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)}.\end{aligned}$$

The proof is complete. $\square$

Remark 3.2.8. Note that we excluded $k = 2$ in the statement of Lemma 3.2.6 as well as in the first equality of the statement of the last proposition. However, in the equality (3.17) the variable k is allowed to assume the value 2, since the denominator of the right hand-side in (3.17) does not vanish for $k = 2$. So, the restriction $k > 2$ was temporary and may be removed from now on.

We are now ready to define the p-adic analytic function $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ as the main object of interest in this thesis.

Definition 3.2.9. *Fix once and for all an optimal embedding Ψ, and without loss of generality, assume that its associated lattice is the standard lattice L_* of (2.26). To the data consisting of the elliptic curve E, the real quadratic field K and the genus character $\mathcal{G}$ we attach the p-adic analytic function $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ of the variable $\kappa \in U$ defined by*

$$\mathcal{L}_p(E/K, \mathcal{G}; \kappa) := \sum_{\mathfrak{a} \in \mathcal{C}_K^+} \mathcal{G}(\mathfrak{a}) \mathcal{L}_p(\Psi^{\mathfrak{a}}, \kappa), \tag{3.18}$$

where as before $\mathcal{C}_K^+$ is the narrow ideal class-group of K.

3.3 Proof of Theorem A

As reflected in the terminology, $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ enjoys the following interpolation property. We remark that the proposition to follow will also finish the proof of Theorem A, as stated in Section 1.3.

Proposition 3.3.1. (Interpolation Formula) *For every $k \in \mathbb{Z}^{\geq 2} \cap U$, we have*

$$\begin{aligned}
\mathcal{L}_p(E/K, \mathcal{G}; k) &= \lambda(k)\left(1 - \chi_1(p) a_p(k)^{-1} p^{\frac{k-2}{2}}\right)^2 \\
&\times \left(\sum_{\mathfrak{a} \in \mathcal{C}_K^+} \mathcal{G}(\mathfrak{a}) \mathbf{C}(\hat{f}_k, \Psi^{\mathfrak{a}}(\epsilon_K))\right) \\
&= \lambda(k)\left(1 - \chi_1(p) a_p(k)^{-1} p^{\frac{k-2}{2}}\right)^2 \\
&\times \left(\sum_{\mathfrak{a} \in \mathcal{C}_K^+} \mathcal{G}(\mathfrak{a}) \frac{2\pi i}{\Omega_k} \int_{z_0}^{\Psi^{\mathfrak{a}}(\epsilon_K) z_0} \hat{f}_k(z) Q_{\Psi^{\mathfrak{a}}}(z,1)^{\frac{k-2}{2}} dz\right).
\end{aligned}$$

Proof. Let $\mathcal{T}$ denote a set of representatives for the quotient group $\mathcal{C}_K^+/\langle \mathfrak{p} \rangle$. Therefore, any element $\mathfrak{a}$ of $\mathcal{C}_K^+$ has a unique representation of the form $\boldsymbol{\delta}\mathfrak{p}^j$, as $\boldsymbol{\delta}$ runs through $\mathcal{T}$ and $0 \leq j \leq t-1$. Equivalently, for any $\boldsymbol{\delta} \in \mathcal{T}$, if we write $\Psi_j^{\boldsymbol{\delta}}$ for the optimal embedding obtained by acting the j-th power of $\mathfrak{p}$ on $\Psi^{\boldsymbol{\delta}}$, then every equivalence class of optimal embeddings of K can be uniquely written as $[\Psi_j^{\boldsymbol{\delta}}]$, where $0 \leq j \leq t-1$, and $\boldsymbol{\delta} \in \mathcal{T}$. Therefore, we have

$$\begin{aligned}
\mathcal{L}_p(E/K, \mathcal{G}; k) &= \sum_{\mathfrak{a} \in \mathcal{C}_K^+} \mathcal{G}(\mathfrak{a}) \mathcal{L}_p(\Psi^{\mathfrak{a}}, k) \\
&= \sum_{\boldsymbol{\delta} \in \mathcal{T}} \mathcal{G}(\boldsymbol{\delta}) \sum_{j=0}^{t-1} \mathcal{G}(\mathfrak{p}^j) \mathcal{L}_p(\Psi_j^{\boldsymbol{\delta}}, k) \\
&= \frac{1 - \chi_1(p) a_p(k)^{-1} p^{\frac{k-2}{2}}}{1 + \chi_1(p) a_p(k)^{-1} p^{\frac{k-2}{2}}} \sum_{\boldsymbol{\delta} \in \mathcal{T}} \mathcal{G}(\boldsymbol{\delta}) \sum_{j=0}^{t-1} \chi_1(p)^j \Gamma_k^{(j)}[\Psi^{\boldsymbol{\delta}}].
\end{aligned}$$

The last remark to follow is that, without loss of generality, we may assume that the $\mathbb{Z}_p$-lattice associated to the optimal embedding $\Psi^{\boldsymbol{\delta}}$ is the standard lattice L_*. With this convention, and by invoking Remark 3.2.5, we deduce that

$$
\begin{aligned}
\mathcal{L}_p(E/K,\mathcal{G};k) &= \frac{1-\chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}}{1+\chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}}\sum_{\boldsymbol{\delta}\in\mathcal{T}}\mathcal{G}(\boldsymbol{\delta})\sum_{j=0}^{t-1}\chi_1(p)^j\Gamma_k^{(j)}[\Psi^{\boldsymbol{\delta}}] \\
&= \frac{1-\chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}}{1+\chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}}\lambda(k)\left(1-a_p(k)^{-2}p^{k-2}\right)\times \\
&\times \sum_{\boldsymbol{\delta}\in\mathcal{T}}\mathcal{G}(\boldsymbol{\delta})\left(\sum_{j=0}^{t-1}\chi_1(p)^j\hat{\mathbf{I}}_k\{z_0\to\Psi^{\boldsymbol{\delta}\mathfrak{p}^j}(\epsilon_K)z_0\}\left(Q_{\Psi^{\boldsymbol{\delta}\mathfrak{p}^j}}(z,1)^{\frac{k-2}{2}}\right)\right). \\
&= \lambda(k)\left(1-\chi_1(p)a_p(k)^{-1}p^{\frac{k-2}{2}}\right)^2\times \\
&\times \sum_{\mathfrak{a}\in\mathcal{C}_K^+}\mathcal{G}(\mathfrak{a})\hat{\mathbf{I}}_k\{z_0\to\Psi^{\mathfrak{a}}(\epsilon_K)z_0\}\left(Q_{\Psi^{\mathfrak{a}}}(z,1)^{\frac{k-2}{2}}\right).
\end{aligned}
$$

But this last equality is equivalent to the statement of the proposition, and we are done. □

The next proposition justifies the designation of the term "p-adic L-function", since it will give an interpolation formula for the central critical values of the complex L-functions $L(\hat{f}_k/K,\mathcal{G};s)$. However, it is not $\mathcal{L}_p(E/K,\mathcal{G};\kappa)$, but rather its square

$$L_p(E/K,\mathcal{G};\kappa) := \mathcal{L}_p(E/K,\mathcal{G};\kappa)^2, \tag{3.19}$$

which makes an appearance in the interpolation formula. The same phenomenon will be observed in the Factorisation Formula of Proposition 3.4.1. We cannot say much about the factorisation of $\mathcal{L}_p(E/K,\mathcal{G};\kappa)$, yet we will be able to decompose its square into the product of two restricted Mazur-Kitagawa L-functions.

Proposition 3.3.2. *For any $k \in U \cap \mathbb{Z}^{\geq 2}$, we have*

$$\begin{aligned} L_p(E/K, \mathcal{G}; k) &= \lambda(k)^2 \left(1 - \chi_1(p) a_p(k)^{-1} p^{\frac{k-2}{2}}\right)^4 \times \\ &\times \frac{\Delta^{\frac{k-1}{2}} (\frac{k}{2} - 1)!^2}{(2\pi i)^{k-2} \Omega_k^2} L(\hat{f}_k/K, \mathcal{G}; k/2). \end{aligned} \tag{3.20}$$

Proof. The result is a direct consequence of the last proposition together with the following statement

$$\left(\sum_{\mathfrak{a} \in \mathcal{C}_K^+} \mathcal{G}(\mathfrak{a}) \int_{z_0}^{\Psi^{\mathfrak{a}}(\epsilon_K) z_0} \hat{f}_k(z) Q_{\Psi^{\mathfrak{a}}}(z,1)^{\frac{k-2}{2}} dz \right)^2 = \frac{\Delta^{\frac{k-1}{2}} (\frac{k}{2} - 1)!^2}{(2\pi i)^k} L(\hat{f}_k/K, \mathcal{G}; k/2).$$

For a proof of this formula the reader is referred to the page 862 of [18]. □

Remark 3.3.3. By setting

$$L^*(\hat{f}_k/K, \mathcal{G}; k/2) := \frac{\sqrt{\Delta} (\frac{k}{2} - 1)!^2}{(2\pi i)^{k-2} \Omega_k^2} L(\hat{f}_k/K, \mathcal{G}; k/2),$$

the statement of the proposition above can be rewritten as

$$\begin{aligned} L_p(E/K, \mathcal{G}; k) &= \lambda(k)^2 \left(1 - \chi_1(p) a_p(k)^{-1} p^{\frac{k-2}{2}}\right)^4 \Delta^{\frac{k-2}{2}} L^*(\hat{f}_k/K, \mathcal{G}; k/2) \\ &= \lambda(k)^2 \left(1 - \chi_1(p) a_p(k)^{-1} p^{\frac{k-2}{2}}\right)^2 \Delta^{\frac{k-2}{2}} L^*(f_k/K, \mathcal{G}; k/2), \end{aligned} \tag{3.21}$$

valid for all $k \in \mathbb{Z}^{\geq 2} \cap U$. Notice that the second equality follows from (2.22), (2.24) and (2.25).

3.4 Factorization of $L_p(E/K, \mathcal{G}; \kappa)$

Finally, a crucial ingredient needed in the proof of Theorem B is the following formula which furnishes a factorization for $L_p(E/K, \mathcal{G}; \kappa)$ in terms of restrictions to the central critical line $\kappa = 2s$ of two Mazur-Kitagawa p-adic L-functions.

Proposition 3.4.1. *For all $\kappa \in U$,*

$$L_p(E/K, \mathcal{G}; \kappa) = \Delta^{\frac{\kappa-2}{2}} L_p(\mathfrak{h}_E, \chi_1; \kappa, \kappa/2) L_p(\mathfrak{h}_E, \chi_2; \kappa, \kappa/2).$$

Proof. Since the two sides are continuous functions of $\kappa \in U$, and since $U \cap \mathbb{Z}^{\geq 2}$ is dense in U, it will suffice to prove the statement for $k \in U \cap \mathbb{Z}^{\geq 2}$. Having made such observation, just by comparing the relations (2.16) and (2.25) with $\hat{f}_k$ in place of g and f_k respectively, and in light of (2.24), we deduce that

$$L^*(\hat{f}_k/K, \mathcal{G}; k/2) = L^*(\hat{f}_k, \chi_1; k/2) L^*(\hat{f}_k, \chi_2; k/2). \tag{3.22}$$

On the other hand, the formula (2.38) with $\chi = \chi_1$ and $\chi = \chi_2$ respectively reads

$$L_p(\mathfrak{h}_E, \chi_1; k, k/2) = \lambda(k) \left(1 - \chi_1(p) a_p(k)^{-1} p^{\frac{k-2}{2}}\right)^2 L^*(\hat{f}_k, \chi_1; k/2), \tag{3.23}$$

and

$$L_p(\mathfrak{h}_E, \chi_2; k, k/2) = \lambda(k) \left(1 - \chi_2(p) a_p(k)^{-1} p^{\frac{k-2}{2}}\right)^2 L^*(\hat{f}_k, \chi_2; k/2). \tag{3.24}$$

Since p is assumed to be split in K, we have $\chi_K(p) = 1$. This in turn implies that $\chi_1(p) = \chi_2(p)$. The desired result is now an immediate consequence of (3.21), (3.22), (3.23) and (3.24). $\square$

3.5 Proof of Theorem B

In this section we shall prove Theorem B as stated in Chapter 1. In the proof, as we will see shortly, there is crucial role played by the main result of [2]. For that reason and also for the convenient of the reader, let us state this result first.

Theorem 3.5.1. (Bertolini-Darmon) *Suppose that the elliptic curve $E/\mathbb{Q}$ of conductor N has at least two distinct primes of semi-stable reduction, one of which being p. Let χ be a Dirichlet character of conductor $\mathfrak{d}$ prime to N, which satisfies $\chi(-N) = w_{N}$, and $\chi(p) = a_p$. Then*

1. *The p-adic L-function $L_p(\mathfrak{h}_E, \chi, \kappa, \kappa/2)$ vanishes to order at least 2 at $k = 2$.*

2. *There exits a global point $\mathbf{P}_\chi \in (E(\mathbb{Q}(\sqrt{\mathfrak{d}_1})) \otimes \mathbb{Q})^\chi$, and $t \in \mathbb{Q}^\times$, such that*
$$\frac{d^2}{d\kappa^2} L_p(\mathfrak{h}_E, \chi, \kappa, \kappa/2)\Big|_{k=2} = t \log_E^2(\mathbf{P}_\chi).$$

3. *The point $\mathbf{P}_\chi$ is of infinite order if and only if $L'(E/\mathbb{Q}, \chi, 1) \neq 0$.*

4. *The image of t in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is equal to that of $L^*(f_E, \psi, 1)$, where ψ is any quadratic Dirichlet character satisfying*

   (a) *$\psi(\ell) = \chi(\ell)$ for all primes ℓ dividing $M = N/p$;*

   (b) *$\psi(p) = -\chi(p)$;*

   (c) *$L(f_E, \psi, 1) \neq 0$.* □

Remark 3.5.2. The main result of [15], due to R. Murty and K. Murty, guarantees the existence of infinity many characters ψ satisfying the conditions listed above.

We have gathered, by now, all the necessary materials needed to prove the main contribution of this thesis, namely Theorem B. Before we proceed with the proof and just for convenience, however, let us first recall its statement.

Theorem 3.5.3. *As in Theorem A assume that all the primes $\ell | N$ are split in K. Suppose further that, apart from p, the elliptic curve E has at least one other prime of semistable reduction, that $\chi_1(-N) = w_N$ and that $\chi_1(p) = a_p$. Then:*

1. *$\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ vanishes to order at least two at $k = 2$;*
2. *There exist global points $P_1 \in (E(\mathbb{Q}(\sqrt{\mathfrak{d}_1})) \otimes \mathbb{Q})^{\chi_1}$ and $P_2 \in (E(\mathbb{Q}(\sqrt{\mathfrak{d}_2})) \otimes \mathbb{Q})^{\chi_2}$ and a rational number $t \in \mathbb{Q}^\times$ such that*
$$\frac{d^2}{d\kappa^2} \mathcal{L}_p(E/K, \mathcal{G}; \kappa)\Big|_{k=2} = t \log_E(P_1) \log_E(P_2);$$
3. *The two points P_1 and P_2 are simultaneously of infinite order if and only if*
$$L''(E/K, \mathcal{G}; 1) \neq 0.$$

Proof. (1) We first remark that since all the prime divisors of N are split in K and since K is a real quadratic field,

$$\chi_K(-N) = \chi_K(p) = 1.$$

We also recall that $\chi_1\chi_2 = \chi_K$. It follows from this and from the running assumptions $\chi_1(-N) = w_N$ and $\chi_1(p) = a_p$ that

$$\chi_1(-N) = \chi_2(-N) = w_N, \quad \chi_1(p) = \chi_2(p) = a_p.$$

So, the characters χ_1 and χ_2 both satisfy the two conditions of Theorem 3.5.1. In other words, the two L-functions

$$L_p(E, \chi_1; s) \quad \text{and} \quad L_p(E, \chi_2; s)$$

have each an exceptional zero at $s = 1$ in the sense of [2] (see also the closing paragraph of Subsection 2.6.1), and therefore the two restricted to the line $s = \kappa/2$ Mazur-Kitagawa p-adic L-functions

$$L_p(\mathfrak{h}_E, \chi_1; \kappa, \kappa/2) \quad \text{and} \quad L_p(\mathfrak{h}_E, \chi_2; \kappa, \kappa/2)$$

vanish to order at least two at $k = 2$. It follows from this and from the factorisation formula of Proposition 3.4.1 that the function $L_p(E/K, \mathcal{G}; \kappa)$ vanishes to order at least four at $k = 2$. On account of (3.19), part (*1*) follows.

(2) Part 2 of Theorem 3.5.1 also guarantees the existence of two global points

$$P_1 \in (E(\mathbb{Q}(\sqrt{\mathfrak{d}_1})) \otimes \mathbb{Q})^{\chi_1}$$

and

$$P_2 \in (E(\mathbb{Q}(\sqrt{\mathfrak{d}_2})) \otimes \mathbb{Q})^{\chi_2},$$

and two rational numbers $t_1, t_2 \in \mathbb{Q}^\times$ satisfying

$$\frac{d^2}{d\kappa^2} L_p(\mathfrak{h}_E, \chi_1; \kappa, \kappa/2)\Big|_{k=2} = t_1 \log_E^2(P_1)$$

and

$$\frac{d^2}{d\kappa^2} L_p(\mathfrak{h}_E, \chi_2; \kappa, \kappa/2)\Big|_{k=2} = t_2 \log_E^2(P_2).$$

On the other hand, as we just proved, $L_p(E/K, \mathcal{G}; \kappa)$ vanishes to order at least four at $k = 2$, or what amounts to the same thing, $\mathcal{L}_p(E/K, \mathcal{G}; \kappa)$ vanishes to order at least two at $k = 2$. It follows from this and the vanishing properties of the two Mazur-Kitagawa p-adic L-functions at $k = 2$ mentioned above that

$$\begin{aligned}
\left(\frac{d^2}{d\kappa^2}\mathcal{L}_p(E/K, \mathcal{G}; \kappa)\Big|_{k=2}\right)^2 &= \frac{1}{6}\frac{d^4}{d\kappa^4} L_p(E/K, \mathcal{G}; \kappa)\Big|_{k=2} \\
&= \left(\frac{d^2}{d\kappa^2} L_p(\mathfrak{h}_E, \chi_1; \kappa, \kappa/2)\Big|_{k=2}\right) \\
&\times \left(\frac{d^2}{d\kappa^2} L_p(\mathfrak{h}_E, \chi_2; \kappa, \kappa/2)\Big|_{k=2}\right) \\
&= t_1 t_2 \log_E^2(P_1) \log_E^2(P_2). \qquad (3.25)
\end{aligned}$$

Next we remark that according to part 4 of Theorem 3.5.1,

$$t_1 \equiv^\times t_2 \equiv^\times L^*(f_E, \psi;\ 1) \pmod{(\mathbb{Q}^\times)^2}, \qquad (3.26)$$

where ψ can be any quadratic Dirichlet character satisfying

- $\psi(l) = 1$ for all l dividing $M = N/p$;
- $\psi(p) = -1$;
- $L(f_E, \psi; 1) \neq 0$.

It is readily seen from (3.26) that $t_1 t_2 \in (\mathbb{Q}^\times)^2$. Combining this fact with (3.25) establishes the proof of part (*2*).

(3) The conditions placed on the character χ_j $(j = 1, 2)$ imply that the sign in the functional equation satisfied by $L(E, \chi_j, s)$ is minus one (see the closing paragraph in Subsection 2.2.1), and therefore we have

$$L(E, \chi_1; 1) = 0 \qquad \text{and} \qquad L(E, \chi_2; 1) = 0.$$

On the other hand, by invoking Theorem 3.5.1 one more time we see that the point P_j $(j = 1, 2)$ is of infinite order if and only if

$$L'(E, \chi_j; 1) \neq 0.$$

However, these last two non-vanishing conditions, namely,

$$L'(E, \chi_1; 1) \neq 0 \qquad \text{together with} \qquad L'(E, \chi_2; 1) \neq 0,$$

in light of the relation

$$L(E/K, \mathcal{G}; s) = L(E, \chi_1; s) L(E, \chi_2; s),$$

(see (2.11)) are equivalent to

$$L''(E/K, \mathcal{G}; 1) \neq 0.$$

This completes the proof. $\square$

Chapter 4

Epilogue: Future Plans

We intend to extend the results of this work in two directions. On the one hand, we wish to generalize the concept of Stark-Heegner points (cf. for instance Chapter 9 of [8]) to a setting where there are several inert primes, and on the other hand, we aim at extending the main result of [3] to Shimura curve parametrization situation. It is a known fact that elliptic curves over $\mathbb{Q}$ are equipped with a systematic collection of Heegner points arising from the theory of complex multiplication and defined over abelian extensions of those *imaginary* quadratic number fields fields $F \subset \mathbb{C}$ in which all the prime divisors of the conductor N of E are split. If F is replaced by a *real* quadratic field K, however, such construction is no longer available to systematically produce rational and algebraic points on E, for one needs in the first place the field F to intersect the Poincaré upper half-plane. There are hopes, nevertheless, that by modifying the Heegner hypothesis and by utilizing p-adic tools, one still would be able (at least conjecturally, loc cit,) to manufacture a host of algebraic points on E defined over abelian extensions of K. So, suppose that the conductor of E is of the form $N = pM$, where $p \nmid M$, and that K satisfies the following *modified* Heegner condition:

1. All the primes dividing M are split in K;

2. The prime p is inert in K.

The already quoted reference [8] describes a conjectural recipe for constructing certain canonical points in $E(\mathcal{H}_K)$, where $\mathcal{H}_K$ is the Hilbert class-field of K. The idea behind Darmon's construction is to attach p-adic periods to $f_{_E}$ in a way which formally suggests viewing $f_{_E}$ as a "mock Hilbert modular form" on the quotient space $\Gamma\backslash(\boldsymbol{\mathcal{H}}_p \times \boldsymbol{\mathcal{H}})$, where $\boldsymbol{\mathcal{H}}_p = \mathbb{P}_1(\mathbb{C}_p)\backslash\mathbb{P}_1(\mathbb{Q}_p)$ is the p-adic *counterpart* of $\boldsymbol{\mathcal{H}}$, and where $\Gamma \subset \mathbf{SL}_2(\mathbb{Z}[1/p])$ is the subgroup of matrices which are upper triangular modulo M. The main ingredient that enters into the definition of such p-adic periods is the theory of modular symbols associated to $f_{_E}$, which as seen before, states that the period integrals

$$\mathbf{I}_{f_E}\{r \to s\} := \frac{1}{\Omega^+}\mathrm{Re}\left(\int_r^s 2\pi i f_{_E}(z)dz\right), \quad (r, s \in \mathbb{P}_1(\mathbb{Q}))$$

take integer values, for a suitable choice of a real period $\Omega^+ \in \mathbb{R}$. Two issues arise immediately. One is what if there are more inert primes in K.

The other issue is that if the form $f_{_E}$ arises from a Shimura curve parametrization (associated to a division quaternion algebra) rather than a modular parametrization, such modular symbols cease to be in our disposal, for no parabolic matrix would arise from such quaternion algebras, and therefore, the corresponding Shimura curves have no cusps. In order to handle both issues at the same time, suppose that N admits a factorization as a product of three relatively prime integers

$$N = pN^+N^-,$$

where N^- is the square-free product of an *even* number of primes, and let B be "the" indefinite quaternion algebra ramified exactly at primes $l \mid N^-$. That such—unique up to isomorphism—quaternion algebra exists is a consequence of global class field theory. Now let R be an Eichler $\mathbb{Z}[1/p]$-order of level N^+ in B, and also let $\Gamma := R_1^\times$ be the group of reduced norm one elements in R. One can diagonally embed Γ into the direct product $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{SL}_2(\mathbb{Q}_p)$, and then make it act on the space $\boldsymbol{\mathcal{H}}_p \times \boldsymbol{\mathcal{H}}$ and then try to carry over the constructions of the "one inert prime" setting to the "several inert primes" situation.

We should point out that very recently Matthew Greenberg in [10] has succeeded in carrying out this program by taking a "cohomological" approach. We believe that there exists a second approach to this problem which utilises Hida families of quaternionic automorphic forms. The evidence that this program might be fulfilled is that in an alternative approach to Stark-Heegner points Bertolini and Darmon [3] have recently succeeded in a certain sense to circumvent modular symbols by proving a formula in which p-adic logarithm[1] of Stark-Heegner points are expressed in terms of period integrals attached to the Hida family $\{f_k\}$ (ibid., in particular Theorem 2.5 and Corollary 2.6 therein.) The advantage of this second approach is that it allows one (we hope!) to generalize the work done in this thesis as well as the work of Bertolini and Darmon in [3]. We believe that this approach can be extended to the setting of Shimura curve parametrisations since there is a similar theory of Hida families interpolating quaternionic automorphic forms, and since the special-value-formula of Popa, a key tool used by Bertoloni and Darmon in [3] and also used in this thesis, has been already proved in sufficient generality.[2]

[1] This might be viewed as a disadvantage(!), comparing to the "cohomological—integration on $\boldsymbol{\mathcal{H}}_p \times \boldsymbol{\mathcal{H}}$—style" taken by Greenberg in [10] which works directly with Stark-Heegner points rather than their formal group logarithms.

[2] The author has been informed of this issue by Matthew Greenberg.

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