CM cycles on varieties fibered over Shimura curves, and p-adic L-functions

by

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Abstract

Let f be a modular form of weight $k \geq 4$ on a Shimura curve, let K be a quadratic imaginary field, and fix a rational prime p which is inert in K and divides the level of f. The goal of this thesis is to construct and study a collection of algebraic cycles on an appropriate Chow motive which encode data about the anticyclotomic p-adic L-function $L_p(f, K, s)$ attached to f and K introduced by Bertolini-Darmon-Iovita-Spieß in [BDIS02]. In our setting, this function of a p-adic variable s vanishes in the critical range $s = 1, \dots, k-1$, and we study its derivative. After constructing this motive and the corresponding cycles, we compute their image under a p-adic analogue of the Griffiths-Weil Abel-Jacobi map, and show how this recovers the derivatives of the p-adic L-function at all the points in the critical range.

Our main result can be viewed as a generalization of the result obtained by Iovita-Spieß in [IS03], which gives a similar formula for the "central" value $s = k/2$. It can also be seen as an extension of the construction of Bertolini-Darmon-Prasanna appearing in [BDP09] to the Shimura curve setting.

Abrégé

Soit f une forme modulaire de poids $k \geq 4$ sur une courbe de Shimura, soit K un corps quadratique imaginaire, et soit p un premier fixé qu'on suppose inerte dans K. Le but de cette thèse est de construire une collection de cycles algébriques sur un motif de Chow approprié, et de démontrer qu'ils sont liés à la fonction-L p-adique anti-cyclotomique $L_p(f, K, s)$ attachée à f et K introduite par Bertolini-Darmon-Iovita-Spieß dans [BDIS02]. Cette fonction d'une variable p-adique s s'annule dans l'intervalle critique $s = 1, \ldots, k-1$, et nous nous intéressons à sa dérivée. Après avoir construit le motif et les cycles correspondants, nous calculons leur image par un analogue p-adique de l'application d'Abel-Jacobi, et nous retrouvons la dérivée de $L_p(f, K, s)$ dans l'intervalle critique.

Notre résultat principal est une généralisation du théorème obtenu par Iovita-Spieß dans [IS03], qui donne une formule du même genre pour la valeur centrale $s = k/2$. Cette thèse étend également les constructions introduites par Bertolini-Darmon-Prasanna dans [BDP09] au cadre des courbes de Shimura.

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Dedication

To Quim, pure potential.

Contents

Introduction

A classical problem

Consider an elliptic curve E defined over the field $\mathbb{Q}$ of rational numbers. The Mordell-Weil theorem asserts that the group $E(\mathbb{Q})$ of rational points on E is finitely-generated. While it is easy to compute its torsion, the rank of $E(\mathbb{Q})$, called the *arithmetic rank*, is an invariant that is far from being understood. Another invariant attached to E is its Hasse-Weil L function, $L(E/\mathbb{Q}, s)$. Thanks to results of Wiles [Wil95], this is an entire complex function, satisfying a functional equation relating its behavior at s and $2-s$. One may consider the order of vanishing of $L(E/\mathbb{Q}, s)$ at $s = 1$, which is called the *analytic rank* of E. The famous conjecture of Birch and Swinnerton-Dyer [BD65], which we will refer to as the BSD conjecture, predicts that the arithmetic and analytic ranks coincide. The conjecture is not only more precise, since it also predicts the first nonzero coefficient of $L(E/\mathbb{Q}, s)$ at $s = 1$, but it is actually much more general, and can be formulated for abelian varieties over arbitrary number fields. However, very little is known about the BSD conjecture, even in the simplest cases.

One of the breakthroughs in the field is due to Gross and Zagier [GZ86], who proved a formula relating the central critical value of $L(E/\mathbb{Q}, s)$ to the Néron-Tate height of a Heegner point on a modular curve. A few years later Kolyvagin introduced the technique of Euler systems which, together with the formula of Gross and Zagier, proved the BSD conjecture when the analytic rank is at most one.

Given a normalized eigenform f of arbitrary weight $n+2$, one can construct an L-function $L(f/\mathbb{Q}, s)$ in such a way that for weight 2 the function $L(f/\mathbb{Q}, s)$ coincides with $L(E_f/\mathbb{Q}, s)$, where E_f is obtained from f using a construction of Eichler and Shimura. Regarding this generalization, Zhang proved in [Zha97] a formula similar to that given in [GZ86] for modular forms of arbitrary weight. This formula relates the derivative of $L(f/\mathbb{Q}, s)$ to the Néron-Tate height of certain Heegner cycles, a class of cycles in the Kuga-Sato variety supported on CM-divisors, which were already studied in [GZ86].

p-adic L-functions

One essential and striking feature of the BSD conjecture is that it relates complex-analytic and arithmetic data. The analytic world being *a priori* so far from the rational world makes this conjecture unreachable to us using current ideas. Since p-adic analysis is closely related to arithmetic, a more tractable approach is provided by a p-adic theory of L-functions replacing its complex counterpart. This theory is not settled yet, and many different proposals have appeared in the last decades.

Fix a quadratic imaginary field K, and let f be a modular form defined over $\mathbb{Q}$. The goal of the different theories of p-adic L-functions is to produce rigid-analytic functions attached to f that interpolate the Rankin-Selberg L-function $L(f/K, s)$ in different ways. The theory has so far developed in two directions, which correspond to the two independent $\mathbb{Z}_p$-extensions of the field K: the cyclotomic and anti-cyclotomic extension.

The cyclotomic p-adic L-function

The first approach to such p-adic analogues appeared in [MD74], where Mazur and Swinnerton-Dyer constructed a p-adic L-function associated to a modular form f of arbitrary weight $n+2$ using the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Mazur, Tate and

Teitelbaum formulated in [MTT86] a conjectural formula that related the order of vanishing of this p-adic L function to that $L(f, s)$, and in [GS93], Greenberg and Stevens proved that formula in the case of weight 2. Also, Perrin-Riou [Rio92] obtained a Gross-Zagier type formula for the central value using p-adic heights.

Nekovář [Nek95] extended the result of Perrin-Riou to higher weights, by using the definition of p-adic height that he had already introduced in his earlier paper [Nek93]. Combining this result with his previous work on Euler systems [Nek92], he obtained a result of Kolyvagin-type for the cyclotomic p-adic L-function.

The anti-cyclotomic p-adic L-function

Bertolini and Darmon, in a series of papers [BD96], [BD98] and [BD99], constructed another p-adic L-function which depends instead on the *anti-cyclotomic* $\mathbb{Z}_p$-extension of a fixed quadratic imaginary field K. One important feature of this construction is that it is purely p-adic, unlike its cyclotomic counterpart. Bertolini and Darmon formulated the analogous conjectures to those of Teitelbaum [Tei90], and proved them in the case of weight 2.

After these papers, Iovita and Spieß entered the project initiated by Bertolini and Darmon. Their goal was to generalize the previous constructions and results to higher weights and to make them more conceptual. They considered certain Selmer groups, in the spirit of the conjectures of Bloch and Kato [BK90] as generalized by Fontaine and Perrin-Riou in [FR94]. In [BDIS02], taking ideas from the work of Schneider in [Sch84], the four authors construct the anti-cyclotomic p-adic L-function attached to a rigid modular form f and a quadratic imaginary field K, and obtain a formula which computes the derivative of this p-adic L-function at the central point in terms of an integral on the p-adic upper half plane, using the integration theory introduced by Coleman in [Col85]. Assume for simplicity that the ideal class number of K is 1. Using their techniques one can easily show that, when p is inert in K, the anti-cyclotomic p-adic L-function vanishes at all the critical values. Moreover, one computes a formula for the derivative at all the values in the critical range: if f is a modular form of even weight $n+2$, and we denote by $L_p(f, K, s)$ the anti-cyclotomic p-adic L-function attached to f and K, then for all $0 \leq j \leq n$, one has:

$$L_p'(f, K, j+1) = \int_{\overline{z}_0}^{z_0} f(z)(z - z_0)^j (z - \overline{z}_0)^{n-j} dz, \tag{1}$$

where, $z_0, \overline{z}_0 \in \mathcal{H}_p(K)$ are certain conjugate Heegner points on the p-adic upper-half plane. In 2003, Iovita and Spieß [IS03] interpreted the quantity appearing in the right hand side of formula (1), in the case of $j = \frac{n}{2}$, as the image of a Heegner cycle under a p-adic analogue to the Abel-Jacobi map. This thesis gives a similar geometric interpretation of the quantity appearing in the right hand side of the previous formula, for all values of j.

Contributions

Let $M_{n+2}(X)$ denote the space of modular forms on a Shimura curve X, of weight $n+2 \geq 4$. The case of weight 2 is excluded for technical reasons, and because it has already been studied by other authors. Let K be a quadratic imaginary field in which p is inert, and fix an elliptic curve E with complex multiplication. In this setting, we construct a Chow motive $\mathcal{D}_n$ over X, and a family of algebraic cycles Δ_φ supported in the fibers over CM-points of X, indexed by isogenies $\varphi\colon E \to E'$, of elliptic curves with complex multiplication. The motive $\mathcal{D}_n$ is obtained from a self-product of a certain number of abelian surfaces, together with a self-product of the elliptic curve E. The cycles Δ_φ are essentially the graph of φ, and are expected to carry more information than the classical Heegner cycles.

One can define a map analogous to the classical Abel-Jacobi map for curves, but for varieties defined over p-adic fields. This map, denoted $\mathrm{AJ}_{K,p}$, assigns to a null-homologous algebraic cycle an element in the dual of the de Rham realization of the motive $\mathcal{D}_n$. This motive has precisely been constructed so that this realization is

$$M_{n+2}(X) \otimes_{\mathbb{Q}_p} \operatorname{Sym}^n H^1_{\mathrm{dR}}(E/K).$$

One can choose generators ω and η for the group $H^1_{\mathrm{dR}}(E/K)$, and it thus makes sense to evaluate $\mathrm{AJ}_{K,p}(\Delta_\varphi)$ on an element of the form $f \wedge \omega^j \eta^{n-j}$. By explicitly computing this map, and combining the result with the formula (1), we obtain the following result (see Corollary 6.10 for a more precise and general statement):

Theorem. *There exist explicit isogenies φ and $\overline{\varphi}$ as above and a constant $\Omega \in K$ such that, for all $0 \leq j \leq n$:*

$$\mathrm{AJ}_{K,p}(\Delta_\varphi - \Delta_{\overline{\varphi}})(f \wedge \omega^j \eta^{n-j}) = \Omega^{j-n} L_p'(f, K, j+1).$$

This result is to be regarded as a p-adic Gross-Zagier type formula for the anticyclotomic p-adic L-function. Note however that instead of heights it involves the p-adic Abel-Jacobi map. It can also be seen as a generalization of the main result of Iovita and Spieß in [IS03] to all values in the critical range.

Structure

This document consists of this introduction and seven chapters. The first four chapters introduce background material. Therefore no claim of originality is made, although some details have been added in certain places.

Chapter 1 introduces background on rigid analytic geometry. In particular, the theory of Coleman integration is developed at the appropriate level of generality.

Chapter 2 introduces the main geometric object of study in this work, namely Shimura curves. Both their moduli interpretation as both as a p-adic uniformization result due to Čerednik-Drinfel'd are needed later to state the problem and to obtain the main result.

Chapter 3 deals with the several definitions of the higher Abel-Jacobi maps that will be needed in the sequel, and describes formulas for natural pairings on the cohomology of open Shimura curves with coefficients.

Chapter 4 follows [BDIS02] in introducing the anti-cyclotomic p-adic L-function, which is the object of the main application of our results.

In the second part there is an exposition of the results obtained so far.

Chapter 5 describes a motive $\mathcal{D}_n$ which is inspired by the constructions appearing in [IS03] and in [BDP09]. We study its p-adic étale realization and its de Rham realization.

Chapter 6 begins by giving a formula for the values of the derivative of the anti-cyclotomic p-adic L-function. The goal of this project is to give a geometric interpretation of these values. It continues by defining a family of cycles on the motive $\mathcal{D}_n$ studied in the previous chapter. The remainder of the chapter computes the image under the p-adic Abel-Jacobi map of certain linear combinations of these cycles, which are shown to coincide with the values of the derivative found before.

Chapter 7 points out different directions towards which the future research on this subject could be directed.

Notation

We denote by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the ring of integers and the fields of the rational, real and complex numbers, respectively. For each prime p, we denote by $\mathbb{Q}_p$ the field obtained by completing $\mathbb{Q}$ with respect to the non-archimedean valuation $|\cdot|_p$ corresponding to p, which is normalized so that it satisfies

$$|p|_p{:=}p^{-1}.$$

If q is a prime power, we denote by $\mathbb{F}_q$ the finite field with q elements.

In general, if K is any field, we denote by K^{sep} the separable closure of K, which coincides with the algebraic closure K^{alg} of K if K is perfect. We write K^{ur} for the maximal unramified extension of K. We also write G_K for the fundamental group of K, also known as the absolute Galois group:

$$G_K = \mathrm{Gal}(K^{\text{sep}}/K).$$

We denote by $\mathbb{C}_p$ the ring obtained by completing the algebraic closure of $\mathbb{Q}_p$ with respect to the unique extension of the valuation $|\cdot|_p$. We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$, and isomorphisms $\mathbb{C} \cong \mathbb{C}_p$ for each prime p.

The notation $A{:=}B$ means that A is defined to be B, and $A{=:}B$ means that B is defined to be A.

By $\mathbb{P}^1(\mathbb{C}_p)$ we mean the set of lines of $\mathbb{C}_p^2$. Fixing a coordinate function z, we think of $\mathbb{P}^1(\mathbb{C}_p)$ as the union $\mathbb{A}^1(\mathbb{C}_p) \cup \{\infty\}$. There is a norm

$$|\cdot|\colon \mathbb{C}_p \to \mathbb{Q},$$

which we normalize so that $|p| = p^{-1}$. This gives an extended norm on $\mathbb{P}^1(\mathbb{C}_p)$, where we declare $|\infty|:=\infty$.

For two integers r and s, their greatest common divisor is written (r, s).

Given a ring A, an A-algebra B and an A-module M, we write M_B for the tensor product $M \otimes_A B$. Also, if G is a group acting A-linearly on M, we denote by M^G the submodule consisting of those elements fixed by the action of G.

Part I

Background

Chapter 1

Rigid geometry

In this chapter we recall some notions from rigid-analytic geometry that are needed in the sequel. After introducing the basic notions in Section 1.1 we introduce in Section 1.2 the only rigid spaces that will be considered in this work, which are the p-adic upper-half plane and its quotients by certain arithmetic subgroups of GL_2. In Section 1.3 we present the theory of Coleman integration, following roughly [Col82].

1.1 Basics

1.1.1 Affinoids and rigid-analytic spaces

We only introduce the minimum amount of definitions to construct the objects involved in this work. For more details the reader is invited to look at the standard references [BGR84] or [FVDP04].

Definition 1.1. A *closed disk* in $\mathbb{P}^1(\mathbb{C}_p)$ is a set either of the form

$$B[a,r]:=\{z \in \mathbb{P}^1(\mathbb{C}_p) \mid |z-a| \leq r\},$$

for $r \in |\overline{\mathbb{Q}}_p^\times|$ and $a \in \mathbb{C}_p$, or of the form

$$B'[a,r]:=\{z \in \mathbb{P}^1(\mathbb{C}_p) \mid |z-a| \geq r\}.$$

For convenience, we set $\mathbb{B}^1 := B[0,1]$ to be the closed unit disk. If $K/\mathbb{Q}_p$ is a field extension, and $a \in K$ and $r \in |K^\times|$, we say that the disk $B[a,r]$ is *K-rational*. The *open disks* $B(a,r)$ and $B'(a,r)$ are defined by changing the inequalities by strict inequalities in the previous definitions.

A *connected affinoid* is the complement of a non-empty finite union of open disks, and it is said to be K-rational if it is fixed by any automorphism fixing K. An *affinoid* is a finite union of connected affinoids. The complement of the union of two disjoint open disks is called a *closed annulus*, which is a particular type of connected affinoid. A *wide open annulus* is a set of the form $A(a,r,R) := \{z \in \mathbb{P}^1(\mathbb{C}_p) \mid r < |z-a| < R\}$, with $a \in \mathbb{A}^1(\mathbb{C}_p)$ and $r, R \in |\mathbb{C}_p| \cup \infty$.

Remarks 1.2.

1. A connected affinoid can be written as the intersection of finitely many closed disks.

2. Let B_1, B_2 be two open disks. If $B_1 \cup B_2 \neq \mathbb{P}^1(\mathbb{C}_p)$, then $B_1 \cap B_2$ is either empty or an open disk. If $B_1 \cup B_2 = \mathbb{P}^1(\mathbb{C}_p)$, then $B_1 \cap B_2$ is a wide open annulus.

3. Let F_1, F_2 be two connected affinoids, with $F_1 \cap F_2 \neq \emptyset$ and $F_1 \cup F_2 \neq \mathbb{P}^1(\mathbb{C}_p)$. Then both $F_1 \cap F_2$ and $F_1 \cup F_2$ are connected affinoids.

4. Any affinoid F can be written uniquely (up to reordering) as a finite union of connected affinoids, which are called the connected components of F.

5. A subset of $\mathbb{P}^1(\mathbb{C}_p)$ is a connected affinoid if and only if it is conformal via a linear fractional transformation to a set of the form

$$\mathbb{B}^1 \setminus \bigcup_{a \in S} B(a, r_a),$$

where S is a finite subset of $\mathbb{B}^1$ and $r_a \in |\mathbb{C}_p|$.

We want to do function theory on subsets of $\mathbb{P}^1(\mathbb{C}_p)$. The topology induced on $\mathbb{P}^1(\mathbb{C}_p)$ by the norm $|\cdot|$ is not good enough: for example, $\mathbb{P}^1(\mathbb{C}_p)$ is totally disconnected, and so its space of locally constant functions is infinite-dimensional. To remedy this problem one endows $\mathbb{P}^1(\mathbb{C}_p)$ with a *Grothendieck topology*, called a G-topology:

Definition 1.3. Let X be a set. A *G-topology* T on X is the datum of:

1. A family $\mathcal{F}$ of subsets of X, called the *admissible sets*, or T-opens, such that $\emptyset$ and X belong to $\mathcal{F}$ and such that it is closed under finite intersections.

2. For each $U \in \mathcal{F}$, a set $\mathrm{Cov}(U)$ of set-theoretic coverings of U by elements of $\mathcal{F}$, called *admissible coverings*, or T-coverings, satisfying:

   (a) $\{U\} \in \mathrm{Cov}(U)$,

   (b) For $V, U \in \mathcal{F}$ satisfying $V \subset U$ and $\mathcal{U} \in \mathrm{Cov}(U)$, then the covering $\mathcal{U} \cap V := \{U' \cap V \mid U' \in \mathcal{U}\}$ belongs to $\mathrm{Cov}(V)$, and

   (c) For $U \in \mathcal{F}$, let $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ be any covering. For each $i \in I$, let $\mathcal{U}_i \in \mathrm{Cov}(U_i)$. Then $\cup_{i\in I}\mathcal{U}_i := \{U' \mid U' \in \mathcal{U}_i \text{ for some } i \in I\}$ is an element of $\mathrm{Cov}(U)$.

One defines pre-sheaves, sheaves and Čech complexes for a G-topology in a natural way. For more details, see [FVDP04, Section 2.4].

Definition 1.4. We define the *weak G-topology* on $\mathbb{P}^1(\mathbb{C}_p)$ as follows:

1. The admissible opens are $\emptyset, \mathbb{P}^1(\mathbb{C}_p)$ and all the affinoids.

2. A covering $\{U_i\}$ of an admissible U is admissible if all the U_i are admissible and if U is already the union of finitely many of the U_i's.

Sometimes it is best to change a given G-topology by accepting more sets as admissible, without changing the sheaf theory on it. There is a notion of a G-topology T' being *slightly finer* than another G-topology T, which makes this concept precise. One defines the *strong G-topology* to be the finest topology which is slightly finer than the weak G-topology. In our situation this is easily described:

Proposition 1.5. *The strong G-topology on $\mathbb{P}^1(\mathbb{C}_p)$ is described as:*

1. *Every open subset U of $\mathbb{P}^1(\mathbb{C}_p)$ is admissible.*

2. *An open covering $\{U_i\}_{i\in I}$ of an open $U \subseteq \mathbb{P}^1(\mathbb{C}_p)$ is admissible if for every affinoid $F \subseteq U$ there is a finite subset $J \subseteq I$ and affinoids $F_j \subseteq U_j$ for all $j \in J$, such that $F \subseteq U_{j\in J} F_j$.*

1.1.2 Some sheaves on $\mathbb{P}^1$

Definition 1.6. The sheaf $\mathcal{O}$ of *analytic functions* on $\mathbb{P}^1(\mathbb{C}_p)$ is defined as follows:

1. $\mathcal{O}(\emptyset) = 0$ and $\mathcal{O}(\mathbb{P}^1(\mathbb{C}_p)) = \mathbb{C}_p$.

2. For $U \subset \mathbb{P}^1(\mathbb{C}_p)$ an affinoid, $\mathcal{O}(U)$ is the completion of $\mathrm{Rat}(U)$ with respect to the sup-norm, where $\mathrm{Rat}(U)$ is the $\mathbb{C}_p$-algebra of rational functions with poles outside U.

Proposition 1.7 ([FVDP04, Theorem 2.5.1])**.** *$\mathcal{O}$ is an acyclic sheaf on $\mathbb{P}^1(\mathbb{C}_p)$ for the weak G-topology.*

Definition 1.8. The sheaf $\mathcal{M}$ of *meromorphic functions* on $\mathbb{P}^1(\mathbb{C}_p)$ is defined by:

1. $\mathcal{M}(\emptyset) = 0$ and $\mathcal{M}(\mathbb{P}^1(\mathbb{C}_p)) = \mathbb{C}_p(z)$ (rational functions on $\mathbb{P}^1(\mathbb{C}_p)$).

2. $\mathcal{M}(U) = \mathrm{Frac}(\mathcal{O}(U))$ for any admissible $U \subset \mathbb{P}^1(\mathbb{C}_p)$.

One can verify that $\mathcal{M}$ is also an acyclic sheaf. See [FVDP04, Proposition 2.4.6].

Definition 1.9. The sheaf $\mathcal{L}$ of *locally-analytic functions* on $\mathbb{P}^1(\mathbb{C}_p)$ is defined by:

1. $\mathcal{L}(\emptyset) = 0$.

2. For each any admissible $U \subset \mathbb{P}^1(\mathbb{C}_p)$, the space $\mathcal{L}(U)$ is the space of functions $f\colon U \to \mathbb{C}_p$ such that, for each point $x \in U$, there is a power series $F_x(z)$ converging in a neighborhood $V_x \subset U$ of x (for the topology induced from that of $\mathbb{C}_p$), and such that $f|_U = F_x$.

1.1.3 Rigid analytic spaces

We want to define the notion of a rigid space. Let $\mathcal{G}$ be a non-empty collection of affinoids of $\mathbb{P}^1(\mathbb{C}_p)$, satisfying:

1. If the union of $X_1, X_2 \in \mathcal{G}$ is not $\mathbb{P}^1(\mathbb{C}_p)$, then $X_1 \cup X_2 \in \mathcal{G}$,
2. If $X_1 \subseteq X_2$ are two affinoids and $X_2 \in \mathcal{G}$, then $X_1 \in \mathcal{G}$.

To such a collection one associates data $\Omega = \Omega(\mathcal{G})$, called the *rigid space* associated to $\mathcal{G}$, as follows:

1. A topological space Ω: as a set, it is the direct limit (increasing union) of all the $X \in \mathcal{G}$. A subset $U \subseteq \Omega$ is declared open if $U \cap X$ is open in X, for all $X \in \mathcal{G}$.
2. A G-topology on Ω: the admissible sets are $\mathcal{F}{:=}\mathcal{G} \cup \{\emptyset, \Omega\}$, and a covering $\{U_i\}_{i\in I}$ of $U \in \mathcal{F}$ is admissible if all $U_i \in \mathcal{F}$ and every affinoid $F \subseteq U$ is contained in a finite union $U_{j\in J} U_j$,
3. A structure sheaf $\mathcal{O}$, defined by: for $X \in \mathcal{G}$, define $\mathcal{O}(X)$ as the algebra of regular functions on X. Also, $\mathcal{O}(\emptyset) = 0$, and $\mathcal{O}(\mathcal{G}){:=}\varprojlim_{X\in\mathcal{G}} \mathcal{O}(X)$.

In the next section we define a certain analytic subspace of $\mathbb{P}^1(\mathbb{C}_p)$, the p-adic upper-half plane $\mathcal{H}_p$. It is defined by giving it as a subset of $\mathbb{P}^1(\mathbb{C}_p)$, and giving a collection of affinoids $\mathcal{G}$ as above, so that $\mathcal{H}_p = \Omega(\mathcal{G})$.

1.2 The p-adic upper-half plane and its quotients

In this section we define the p-adic upper-half plane over $\mathbb{Q}_p$. The construction for general p-adic fields is given in [DT08, Chapter 3]. In order to describe its rigid-analytic structure, it is convenient to introduce a combinatorial device called the Bruhat-Tits tree. We follow the exposition of [Dar04].

1.2.1 The Bruhat-Tits tree

In this subsection we define the *Bruhat-Tits tree* $\mathcal{T}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$. It is a graph which has as set of vertices $\mathfrak{V}(\mathcal{T})$ the similarity classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. Two vertices are connected by an edge whenever they have representative lattices Λ_1 and Λ_2 satisfying

$$p\Lambda_2 \subsetneq \Lambda_1 \subsetneq \Lambda_2.$$

The set of edges of $\mathcal{T}$ will be denoted $\mathfrak{E}(\mathcal{T})$. Note that the above symmetrical relation makes $\mathcal{T}$ an unoriented graph. In fact, $\mathcal{T}$ is a $(p+1)$-regular tree; that is, each vertex has exactly $(p+1)$ neighbors. There is a natural action of $\mathrm{PGL}_2(\mathbb{Q}_p)$ on $\mathcal{T}$ by acting on the lattices, and this action respects the edges, yielding an action of $\mathrm{PGL}_2(\mathbb{Q}_p)$ on $\mathcal{T}$ by (continuous) graph automorphisms.

Fix an ordering of the edges of $\mathcal{T}$, and denote by $\vec{\mathfrak{E}}(\mathcal{T})$ the set of ordered edges. If the edge e connects the vertices v_1 and v_2, we write $v_1 = o(e)$ and $v_2 = t(e)$. We also write $\bar{e}$ for the opposite edge, which has $o(\bar{e}) = v_2$ and $t(\bar{e}) = v_1$.

The Bruhat-Tits tree $\mathcal{T}$ has a distinguished vertex, written v_0, which corresponds to the homothety class of the standard lattice $\mathbb{Z}_p^2$ inside $\mathbb{Q}_p^2$. The edges e with $o(e) = v_0$ correspond to the $(p+1)$ sublattices of index p in $\mathbb{Z}_p^2$, which in turn are in bijection with the points of $\mathbb{P}^1(\mathbb{F}_p)$. These edges are called $e_0, e_1, \dots e_{p-1}, e_\infty \in \vec{\mathfrak{E}}(\mathcal{T})$. We will abuse language and think of $\mathcal{T}$ as a contractible topological space. Given an edge $e \in \mathfrak{E}(\mathcal{T})$, we write $[e] \subset \mathcal{T}$ for the closed edge (which contains the two vertices that e connects), and $]e[$ for the open edge.

1.2.2 The p-adic upper-half plane as a rigid-analytic space

The p-adic upper-half plane $\mathcal{H}_p$ can be defined as a formal scheme over $\mathbb{Z}_p$, but we are only interested in the rigid-analytic space associated to its generic fiber: the set of the $\mathbb{C}_p$-valued points of $\mathcal{H}_p$ is $\mathcal{H}_p(\mathbb{C}_p){:=}\mathbb{P}^1(\mathbb{C}_p)\setminus\mathbb{P}^1(\mathbb{Q}_p)$, and it has the structure of a rigid-analytic space, which we describe below. Note that $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathcal{H}_p(\mathbb{C}_p)$

by fractional linear transformations: for $\tau \in \mathcal{H}_p(\mathbb{C}_p)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$,

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + b}.$$

We describe a covering by basic affinoids and annuli, using the Bruhat-Tits tree defined above. Let

$$\mathrm{red}\colon \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\overline{\mathbb{F}}_p)$$

denote the natural map given by reduction modulo $\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$, the maximal ideal of the ring of integers of $\mathbb{C}_p$.

Definition 1.10. Let $\tilde{x}$ be a point in $\mathbb{P}^1(\overline{\mathbb{F}}_p)$. The *residue class* of $\tilde{x}$ is the subset of $\mathbb{P}^1(\mathbb{C}_p)$:

$$R = R_{\tilde{x}} := \{x \in \mathbb{P}^1(\mathbb{C}_p) \mid \mathrm{red}(x) = \tilde{x}\}.$$

Define the set A_0 to be $\mathrm{red}^{-1}(\mathbb{P}^1(\overline{\mathbb{F}}_p) \setminus \mathbb{P}^1(\mathbb{F}_p))$. Concretely:

$$A_0 = \left\{ \tau \in \mathcal{H}_p(\mathbb{C}_p) \middle| \begin{array}{c} |\tau - t| \geq 1, \quad \forall\, 0 \leq t \leq p-1 \\ |\tau| \leq 1 \end{array} \right\}.$$

This is the prototypical example of a *standard affinoid*. Define also a collection of annuli

$$W_t := \left\{ \tau \mid \frac{1}{p} < |\tau - t| < 1 \right\}, \quad 0 \leq t \leq p-1,$$

as well as

$$W_\infty := \{\tau \mid 1 < |\tau| < p\}.$$

Note that A_0 and the annuli W_t and W_∞ are mutually disjoint. The goal is to construct a "reduction map" $r\colon \mathcal{H}_p(\mathbb{C}_p) \to \mathcal{T}$. We first define it on the set

$$X_1 := A_0 \cup (\cup_{0 \leq t \leq p-1} W_t) \cup W_\infty, \tag{1.1}$$

by setting

$$r(\tau) := \begin{cases} v_0 & \text{if } \tau \in A_0 \\ e_t & \text{if } \tau \in W_t. \end{cases}$$

The map r is extended by requiring it to be compatible with the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on both $\mathcal{H}_p$ and $\mathcal{T}$: for all $\tau \in \mathcal{H}_p(\mathbb{C}_p)$ and for all $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$, we require $r(\gamma\tau) = \gamma r(\tau)$.

For each vertex $v \in \mathfrak{V}(\mathcal{T})$, let $\mathcal{A}_v{:=}r^{-1}(\{v\})$. For each edge $e \in \mathfrak{E}(\mathcal{T})$, write $\mathcal{A}_{[e]}{:=}r^{-1}([e])$ and $\mathcal{A}_{]e[}{:=}r^{-1}(]e[)$. Then the collection $\mathcal{G}{:=}\{\mathcal{A}_{[e]}\}_{e\in\mathfrak{E}(\mathcal{T})}$ gives a covering of $\mathcal{H}_p(\mathbb{C}_p)$ by standard affinoids, and their intersections are:

$$\mathcal{A}_{[e]} \cap \mathcal{A}_{[e']} = \begin{cases} \emptyset & \text{if } [e] \cap [e'] = \emptyset, \\ \mathcal{A}_v & \text{if } [e] \cap [e'] = \{v\}. \end{cases}$$

As an example, note that $\mathcal{A}_{v_0} = A_0$ and that for $0 \le t \le p-1$, $\mathcal{A}_{[e_t]}$ is the union of two translates of A_0 glued along W_t. This covering gives the rigid-analytic space structure to $\mathcal{H}_p{:=}\Omega(\mathcal{G})$.

In general, if X is any topological space and $\mathcal{U} = \{U_i\}_{i\in I}$ is a covering of X, one can construct an abstract simplicial complex $\mathcal{N}$ from $\mathcal{U}$, called the *nerve of the covering* $\mathcal{U}$, as follows:

1. The empty set belongs to $\mathcal{N}$, and
2. A finite set $J \subseteq I$ belongs to $\mathcal{N}$ if and only if

$$\bigcup_{j\in J} U_j \neq \emptyset.$$

Note that the covering that we have constructed has the tree $\mathcal{T}$ as its nerve: there are no triple intersections and no loops.

1.2.3 The boundary of $\mathcal{H}_p$

The boundary of $\mathcal{H}_p$ is the set $\mathbb{P}^1(\mathbb{Q}_p)$, which has been removed from $\mathbb{P}^1(\mathbb{C}_p)$ in order to obtain $\mathcal{H}_p(\mathbb{C}_p)$. If $\{x_n\}_{n\geq 1}$ is a sequence of points in $\mathcal{H}_p(\mathbb{C}_p)$ approaching $x \in \mathbb{P}^1(\mathbb{Q}_p)$, then the sequence $\{r(x_n)\}_{n\geq 1}$ gives a path on $\mathcal{T}$ which contains a subsequence with no backtracking. This motivates the following definition:

Definition 1.11. An *end* of $\mathcal{T}$ is an equivalence class of sequences $\{e_i\}_{i\geq 1}$ of edges $e_i \in \vec{\mathfrak{E}}(\mathcal{T})$, such that $t(e_i) = o(e_{i+1})$, and such that $t(e_{i+1}) \neq o(e_i)$. Two such sequences are identified if a shift of one is the same as the other, for large enough i. Write $\mathfrak{E}_\infty(\mathcal{T})$ for the space of ends.

Choose once and for all an edge $e_0 \in \vec{\mathfrak{E}}(\mathcal{T})$ such that its stabilizer inside $\mathrm{PGL}_2(\mathbb{Q}_p)$ is the image of the image of the unit group in the Eichler order

$$R{:=}\left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}_p) \mid c \equiv 0 \pmod{p}\right\}.$$

The following lemma is immediate.

Lemma 1.12. *The map $\beta \mapsto \beta \cdot e_0$ identifies $\mathrm{PGL}_2(\mathbb{Q}_p)/\operatorname{stab}(e_0)$ with $\vec{\mathfrak{E}}(\mathcal{T})$. The inverse map will be written $e \mapsto \beta_e$.*

Lemma 1.13. *The map*

$$N\colon \{e_i\}_i \mapsto \lim_i \beta_{e_i}(\infty)$$

identifies $\mathfrak{E}_\infty(\mathcal{T})$ with $\mathbb{P}^1(\mathbb{Q}_p)$.

Proof. Let $(x : y)$ be the coordinates of a point $P \in \P^1(\mathbb{Q}_p)$. Consider the lattice $L_0 = \mathbb{Z}_p \oplus \mathbb{Z}_p$, and consider $l{:=}(-y, x)$ as an point in the lattice L_0. The sequence:

$$\{[L_0], [l\mathbb{Z} + pL_0], [l\mathbb{Z} + p^2 L_0], \ldots\}$$

represents an end which maps to (x, y) under the map N, showing surjectivity of N.

Conversely, given a sequence of lattices Λ_i representing an end

$$\{[\Lambda_0], [\Lambda_1], \ldots\},$$

and supposing that $\Lambda_0 = L_0$, note that the intersection of all the lattices Λ_i is a one-dimensional subspace of $\mathbb{Q}_p^2$, thus giving a point $l \in \mathbb{P}^1(\mathbb{Q}_p)$. Using that the lattices Λ_i represent an end, we can show that they are of the form

$$\Lambda_i = \mathbb{Z} l_i \oplus p^i L_0,$$

with the sequence $\{l_i\}$ converging to l, and therefore N is injective. $\square$

In this way, the p-adic topology on $\mathbb{P}^1(\mathbb{Q}_p)$ induces a topology on $\mathfrak{E}_\infty(\mathcal{T})$. For an edge $e \in \vec{\mathfrak{E}}(\mathcal{T})$, write $U(e)$ for the compact open subset of $\mathfrak{E}_\infty(\mathcal{T})$ consisting of those ends having a representative which contains e. This is a basis for the topology, and we can compactify $\mathcal{T}$ by adding to it its ends. Calling this completed tree $\mathcal{T}^*$, we can extend r to a map $r\colon \mathbb{P}^1(\mathbb{C}_p) \to \mathcal{T}^*$.

1.2.4 Quotients of $\mathcal{H}_p$ by arithmetic subgroups

In [GvdP83] one can find the general theory of Schottky groups, which are those groups that lead to manageable quotients of $\mathcal{H}_p$. We will restrict our attention to a very special class of those groups, which are related to the p-adic uniformization of Shimura curves.

Let B be a definite rational quaternion algebra of discriminant N^- coprime to p. Fix an Eichler $\mathbb{Z}[\frac{1}{p}]$-order R of level N^+ in B, and fix an isomorphism $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$. Let Γ be the group of elements of reduced norm 1 in R.

Lemma 1.14. *The group* Γ *is a discrete cocompact subgroup of* $\mathrm{SL}_2(\mathbb{Q}_p)$.

Proof. See [Shi94, Proposition 9.3]. □

Suppose for simplicity that Γ contains no elliptic points, and consider the topological quotient $\pi\colon \mathcal{H}_p \to X_\Gamma{:=}\Gamma\backslash\mathcal{H}_p$. Since Γ is discrete, the space X_Γ can be given a structure of rigid-analytic space in a way so that π is a morphism of rigid-analytic spaces. An admissible covering is indexed by the quotient graph $\Gamma\backslash\mathcal{T}$, in the same way that was done for $\mathcal{H}_p$. In this way one obtains a complete curve called a *Mumford-Shottky curve*. In Chapter 2 we will see how these curves are related to Shimura curves.

In Chapter 3 we describe certain pairings on the cohomology of a Shimura curve. In order to get explicit formulas, we will need the notion of a good fundamental domain. By a *half-tree* in $\mathcal{T}$ we mean a connected component of $\mathcal{T}\backslash]e[$, for some edge $e \in \mathfrak{E}(\mathcal{T})$.

Definition 1.15 ([dS89, Section 2.5])**.** Let F be a bounded connected subset of $\mathcal{T}$. We say that F is a *good fundamental domain* for $\Gamma\backslash\mathcal{T}$ if:

1. F is the complement of $2g$ pairwise disjoint half-trees in $\mathcal{T}$, denoted

$$B_1, \dots, B_g, C_1, \dots, C_g.$$

   Denote by $b_1, \dots, b_g, c_1, \dots, c_g$ the $2g$ free edges of F, oriented such that $o(b_i) \in B_i$ and $t(c_i) \in C_i$.

2. The group Γ is generated by $\gamma_1, \dots, \gamma_g$, where γ_i maps B_i isomorphically onto $\mathcal{T} \setminus (C_i \cup \{c_i\})$. In particular, $\gamma_i(b_i) = c_i$.

If F is a good fundamental domain for $\Gamma\backslash\mathcal{T}$, write $\mathcal{F} := \mathrm{red}^{-1}(F)$, which gives a fundamental domain for $\Gamma\backslash\mathcal{H}_p$.

1.3 p-adic integration

The theory of p-adic integration was constructed initially by Coleman in [Col89], [Col85] and [Col82], and further developed by Coleman-Iovita in [CI03], and by de Shalit [dS89], among others. We will borrow very little from this vast theory, in order to cover only those concepts that we need in our project.

Concretely, we will construct an integration theory on rigid spaces which admit an admissible covering by a special type of open subsets of $\mathbb{P}^1(\mathbb{C}_p)$. The p-adic upper-half plane $\mathcal{H}_p$ admits such a covering, and hence we will obtain a theory of integration on $\mathcal{H}_p$ and on Mumford-Schottky curves.

We will also be interested in the integration of general vector bundles over the curves above. It turns out, however, that the bundles that we will encounter always have a basis of horizontal sections, and therefore one can integrate component-wise, thus reducing to integration with trivial coefficients.

1.3.1 Notation

Definition 1.16. A *wide open* is a set of the form

$$U{:=}\{z \in \mathbb{P}^1(\mathbb{C}_p) \mid |f(z)| < e_f, f \in S\},$$

where S a finite set of rational functions over $\mathbb{C}_p$ containing at least one non-constant function, and $e_f \in \{1, \infty\}$.

Example 1.17. The following are all examples of wide open sets:

1. The open balls $B(a, r)$, with $a \in \mathbb{A}^1(\mathbb{C}_p)$ and $r \in |\mathbb{C}_p| \cup \{\infty\}$.

2. The open annuli $\mathcal{A}_{]e[}$, for edges $e \in \mathfrak{E}(\mathcal{T})$.

3. The set X_1 defined in subsection 1.2.2.

If $X \subseteq \mathbb{P}^1(\mathbb{C}_p)$ is an affinoid and $U \supset X$ is a wide open such that the complement $U \setminus X$ is a disjoint union of annuli, we say that U is a *wide open neighborhood* of X.

Definition 1.18. A *basic wide open* is a set C of the form:

$$C = \mathbb{A}^1(\mathbb{C}_p) \setminus \bigcup_{a \in S \cup \{\infty\}} B[a, r_a],$$

where S is a finite subset of $\mathbb{B}^1$ no two elements of which are contained in the same residue class, and for each $a \in S \cup \{\infty\}$ the radius $r_a \in |\mathbb{C}_p|$ satisfies $|r_a| < 1$.

A basic wide open C is the disjoint union of a connected affinoid X, and of $|S| + 1$ wide open annuli V_a:

$$X = \mathbb{B}^1 \setminus \bigcup_{a \in S} B(a, 1),$$
$$V_a = A(a, r_a, 1), \quad a \in S \cup \{\infty\}.$$

Remark 1.19. When $\#S \geq 2$, X is characterized as the smallest connected affinoid contained in C whose complement is a finite disjoint union of annuli.

Example 1.20. The set X_1 appearing in Equation (1.1) is an example of a basic wide open, with $X(X_1) = A_0$.

1.3.2 The logarithm and integration on wide open annuli

A locally analytic homomorphism $l\colon \mathbb{C}_p^\times \to \mathbb{C}_p^+$ such that $\frac{d}{dz}l(1) = 1$ is called a *branch of the logarithm*. It can be easily shown that $l(z)$ is analytic on $B(x, |x|)$, for all $x \in \mathbb{C}_p^\times$. Let C be a basic wide open. For $z \in C$ and S as in Definition 1.16, set:

$$r(z) := \min\{|z-d| \mid d \in S\},$$

and define the thickened diagonal of C to be:

$$D = D(C) := \{(x,y) \in C \times C \mid |x-y| < r(x)\}.$$

For $0 < r < 1$, define the wide open neighborhoods of $X(C)$:

$$U_r := B(0, r^{-1}) \setminus \bigcup_{a \in S} B[a, r],$$

so that for r sufficiently close to 1 we have $U_r \subseteq C$.

For V an open of $\mathbb{P}^1(\mathbb{C}_p)$, we set

$$\Omega(V) := \mathcal{O}(V)dz,$$
$$\Omega_{\mathcal{L}}(V) := \mathcal{L}(V)dz,$$

where we recall that $\mathcal{L}$ is the sheaf of locally-analytic functions as in Definition 1.9. There are canonical derivations making the following diagram commutative:

$$\begin{tikzcd} \mathcal{L}(V) \arrow[r, "d"] & \Omega_{\mathcal{L}}(V) \\ \mathcal{O}(V) \arrow[r, "d"] \arrow[u, hook] & \Omega(V) \arrow[u, hook] \end{tikzcd}$$

We set $H^i(V)$ to be the cohomology of the complex

$$0 \to \mathcal{O}(V) \xrightarrow{d} \Omega^1(V) \to 0.$$

Let V be an annulus about $a \in \mathbb{A}^1(\mathbb{C}_p)$, and let $\omega \in \Omega(V)$ be an analytic differential on V.

Definition 1.21. The *annular residue* of ω at V, written $\operatorname{res}_V \omega$, is the coefficient of $\frac{dz}{z-a}$ in the Laurent expansion of ω on V.

The annular residue is well defined because of the following:

Lemma 1.22. *The annular residue is characterized by being the unique map*

$$\operatorname{res}_V \colon \Omega(V) \to \mathbb{C}_p$$

satisfying:

1. $\operatorname{res}_V(df) = 0$ *for all* $f \in \mathcal{O}(V)$*, and*

2. $\operatorname{res}_V\left(\frac{dz}{z-a}\right) = 1$.

It is easy to integrate analytic differentials on wide open annuli, as long as they have no residue:

Lemma 1.23. $\omega \in d\mathcal{O}(V)$ *if and only if* $\operatorname{res}_V(\omega) = 0$.

Sketch of proof. For $f \in \mathcal{O}(V)$, write:

$$f(z) = \sum_{n \geq n_0} a_n (z-a)^n.$$

Then df has an expansion:

$$\sum_{n \geq n_0} n a_n (z-a)^{n-1} dz.$$

The coefficient of -1 corresponds to $n = 0$ and is therefore 0.

Conversely, if ω has no residue, then one can integrate term by term to get an element f such that $df = \omega$. One needs only to check that the denominators introduced by this process do not change the convergence properties. $\square$

Lemma 1.24. *Let V be an annulus about a, and let $g \in \mathcal{O}(V)^\times$. Define $n \in \mathbb{C}_p$ by:*

$$n := \operatorname{res}_V \frac{dg}{g}.$$

Then $n \in \mathbb{Z}$ and g can be written as $g = c(z-a)^n(1+h)$, where $c \in \mathbb{C}_p$, $h \in \mathcal{O}(V)$, and $|h(z)| < 1$ for all $z \in V$.

There is also a local residue, for each point $a \in \mathbb{P}^1(\mathbb{C}_p)$: if ω is a meromorphic differential and t is a local parameter around a (e.g. $t = z - a$), then around a we can express $\omega = \sum_{n\in\mathbb{Z}} a_n t^n dt$, and we set

$$\operatorname{res}_a \omega := a_{-1}.$$

The two residue maps are related:

Proposition 1.25 ([FVDP04, Lemma 2.3.2])**.** *Let $V = A(a, r, R)$ be an annulus, let $\omega = f dz$ be a meromorphic differential with f a rational function on $\mathbb{P}^1(\mathbb{C}_p)$, and assume that ω has no poles on the sets $\{z \mid |z - a| = r\}$ or $\{z \mid |z - a| = R\}$. Then:*

$$\operatorname{res}_V \omega = \sum_{x\in B(a,r)} \operatorname{res}_x \omega.$$

We want to state a p-adic analogue to the residue theorem in complex analysis. First, let D be an open disk.

Definition 1.26. Let $q \in \mathbb{P}^1(\mathbb{C}_p)$ be a point which is not in D. The *boundary of D with respect to q*, which we write as $\partial_q D$, is defined as follows: choose a point $b \in D$, and consider a fractional linear transformation $z \mapsto t(z)$ satisfying $t(q) = \infty$, $t(b) = 0$, and

$$D = \{x \in \mathbb{P}^1(\mathbb{C}_p) \mid |t(x)| < 1\}.$$

Then define:

$$\partial_q D := \{x \in \mathbb{P}^1(\mathbb{C}_p) \mid |t(x)| = 1\}.$$

Remark 1.27. The boundary of D with respect to q does not depend on the choice of neither t nor b.

Let F be a connected affinoid in $\mathbb{P}^1(\mathbb{C}_p)$ which is the complement of a finite number of disjoint open disks $D_1, \ldots, D_s$. Fix $q \in F$, and let $\partial D_i := \partial_q D_i$, which we also suppose pairwise disjoint. Define the *interior of F with respect to q* to be:

$$F^\circ := F \setminus \bigcup_i \partial D_i.$$

The following is the p-adic residue formula:

Proposition 1.28. *1. Let ω be a meromorphic differential form on F such that its restriction to each ∂D_i has no poles. Then:*

$$\sum_{a\in F^\circ} \mathrm{res}_a(\omega) + \sum_i \mathrm{res}_{\partial D_i}(\omega) = 0.$$

2. Let f be a meromorphic function on F such that its restriction to each ∂D_i has neither poles nor zeros. Then:

$$\sum_{a\in F^\circ} \mathrm{ord}_a(f) + \sum_i \mathrm{ord}_{\partial D_i}(f) = 0$$

Proof. See [FVDP04, Theorem 2.3.3]. □

For every open subset $V \subseteq \mathbb{P}^1(\mathbb{C}_p)$ we define

$$\mathcal{O}_{\mathrm{Log}}(V) := \mathcal{O}(V)\left[\mathrm{Log}(f) \mid f \in \mathcal{O}(V)^\times\right].$$

Define also

$$\Omega^1_{\mathrm{Log}}(V) := \mathcal{O}_{\mathrm{Log}}(V) \otimes_{\mathcal{O}(V)} \Omega^1(V).$$

Lemma 1.24 implies:

Corollary 1.29. *If V is an annulus about a, then*

$$\mathcal{O}_{\mathrm{Log}}(V) = \mathcal{O}(V)[\mathrm{Log}(z-a)].$$

Wide open annuli also satisfy a uniqueness principle:

Proposition 1.30. *Let V be a wide open annulus, and let $f \in \mathcal{O}_{\mathrm{Log}}(V)$. If f vanishes on a non-empty open subset of V, then f vanishes identically on V.*

This proposition and the previous results make it possible to compute the cohomology of a wide open annulus. By $H^i_{\mathrm{Log}}(V)$ we denote the ith cohomology of the complex:

$$0 \to \mathcal{O}_{\mathrm{Log}}(V) \xrightarrow{d} \Omega^1_{\mathrm{Log}}(V) \to 0.$$

Lemma 1.31. *Let V be a wide open annulus. Then:*

1. $H^0_{\mathrm{Log}}(V) = \mathbb{C}_p$, *and*

2. $H^1_{\mathrm{Log}}(V) = 0$.

Remark 1.32. We underline what the previous lemma says. Firstly, the second statement says that, given $\omega \in \Omega^1_{\mathrm{Log}}(V)$, there exists $f \in \mathcal{O}_{\mathrm{Log}}(V)$ such that $df = \omega$. The first statement says that if f and f' belong to $\mathcal{O}_{\mathrm{Log}}(V)$ and satisfy $df = df'$, then $f' = f + c$, for some constant $c \in \mathbb{C}_p$. Therefore there is an integration theory on V, as long as we allow logarithms.

1.3.3 The Frobenius and the Dwork principle

Let X be an affinoid, and let $q = p^n$. Consider $\mathcal{O}(X)$, its ring of rigid-analytic functions. It has a maximal ideal

$$\mathcal{O}(X)^\circ := \{f \in \mathcal{O}(X) \mid |f| < 1\},$$

and therefore we can consider $\tilde{\mathcal{O}}(X)$ to be the corresponding reduction, which is an $\overline{\mathbb{F}}_p$-algebra. We say that X has *good reduction* if $\tilde{\mathcal{O}}(X)$ is regular.

Definition 1.33. A morphism $\phi\colon X \to X$ is called a $\mathbb{F}_q$-*Frobenius morphism* if the reduction of its pullback $\tilde{\phi}^*\colon \tilde{\mathcal{O}}(X) \to \tilde{\mathcal{O}}(X)$ is of the form

$$f \mapsto F^{-1}(f^q),$$

where F is an extension of the absolute Frobenius automorphism of $\overline{\mathbb{F}}_q/\mathbb{F}_q$ to $\tilde{\mathcal{O}}(X)$.

Let $Y \subseteq \mathbb{P}^1(\mathbb{C}_p)$ be a rigid space, and let $X \subseteq Y$ be an affinoid of Y. A wide open $U \subseteq Y$ which contains X is called a *wide open neighborhood* of X.

Definition 1.34. A pair (U, ϕ) is a $\mathbb{F}_q$-Frobenius neighborhood of X in Y if $U \subseteq Y$ is a wide open neighborhood of X, and $\phi\colon U \to Y$ is a morphism whose restriction to X is an $\mathbb{F}_q$-Frobenius morphism of X.

Fact. *If X has good reduction, then X has a Frobenius neighborhood.*

If X is an affinoid inside $\mathbb{P}^1(\mathbb{C}_p)$ and (U,ϕ) is a $\mathbb{F}_q$-Frobenius neighborhood of X, we define recursively

$$U_1{:=}U, \quad U_i{:=}\{x \in U_{i-1} \mid \phi(x) \in U_{i-1}\}.$$

Then (U_n, ϕ^n) is an $\mathbb{F}_{q^n}$-Frobenius neighborhood of X.

The logarithm introduced above has a rigidity property with respect to Frobenius that will be later exploited to integrate on more general spaces. This is explained in the following key lemma due to Coleman.

Lemma 1.35 ([Col82, Lemma 2.5]). *Let S be a finite subset of $\mathbb{A}^1(\mathbb{C}_p)$, and let X be a subaffinoid of $\mathbb{A}^1(\mathbb{C}_p) \setminus S$ with good reduction. Let (U,ϕ) be a $\mathbb{F}_q$-Frobenius neighborhood of X contained in $\mathbb{A}^1(\mathbb{C}_p) \setminus S$. Then there exists an $\mathbb{F}_{q^n}$-Frobenius neighborhood (V,ϕ^n), with $V \subseteq U_n$ for some $n \in \mathbb{Z}_{\geq 0}$, such that:*

1. *For each connected component B of $\mathbb{A}^1(\mathbb{C}_p) \setminus X$, $\phi(B \cap V) \subseteq B$, and*

2. *For all $a \in S$,*

$$(\phi^*)^n \mathrm{Log}(z-a) - q^n \mathrm{Log}(z-a) \in \mathcal{O}(V).$$

Let X be an affinoid with good reduction, and let ϕ be a Frobenius morphism of X. The following result is due to Dwork.

Proposition 1.36 ([Kat]). *For each residue class R of X, there exists some $n \in \mathbb{Z}_{\geq 1}$ and some $\varepsilon \in R$ such that:*

$$\lim_{m\to\infty} \phi^{nm}(x) = \varepsilon,$$

for all $x \in R$. The point ε is called a Teichmüller point.

Proof. There exists $n \in \mathbb{Z}_{\geq 1}$ such that $\phi^n(R) \subseteq R$. Let $x \in R$ be any point. The sequence $\{\phi^{nm}(x)\}_{m\geq 1}$ is seen to be convergent, and the limit point ε, which has to exist, will satisfy $\phi^n(\varepsilon) = \varepsilon$. One then sees that as m increases, $\phi^{nm}(x)$ becomes arbitrarily close to $\phi^{nm}(y)$ for $x, y \in R$, and therefore such an ε is unique. □

1.3.4 Integration on basic wide opens

In this subsection we define logarithmic F-crystals. Fix a branch of the logarithm $\mathrm{Log}(z)$. Let $U \subseteq \mathbb{A}^1(\mathbb{C}_p)$ be an arbitrary open, and let $M \subseteq \mathcal{L}(U)$ be an $\mathcal{O}(U)$-module. If $f\colon V \to U$ is any morphism of rigid spaces, we write $M(V)$ for the pullback f^*M, and similarly write $\Omega_M(V)$ for $f^*\Omega_M$. That is:

$$M(V) = M \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \subseteq \mathcal{L}(V),$$

and similarly for $\Omega_M(V)$. Also, if $dM \subseteq \Omega_M(U)$, then we set:

$$H^1(M(V)) := \Omega_M(V)/dM(V).$$

Let C be a basic wide open.

Definition 1.37 ([Col82, Section IV])**.** A *logarithmic F-crystal* on C is an $\mathcal{O}(C)$-submodule $M \subseteq \mathcal{L}(C)$ containing $\mathcal{O}(C)$, and such that:

1. $M(X)$ is analytic in each residue class of X and for all $a \in S$, $M(V_a) \subseteq A_{\mathrm{Log}}(V_a)$.
2. $dM \subseteq \Omega_M(C)$.
3. $p_1^*M = p_2^*M$, where $p_i\colon D(C) \to C$ are the two canonical projections.
4. $M(U_r)$ satisfies the uniqueness principle for any $0 \leq r < 1$ such that $U_r \subseteq C$.
5. The natural map $H^1(M) \xrightarrow{\sim} H^1(M(U_r))$ is an isomorphism for all $0 \leq r < 1$ such that $U_r \subseteq C$.
6. There is a Frobenius neighborhood (U, ϕ) of X in C such that
   (a) $\phi^*(M) \subseteq M(U)$, and
   (b) There exists $b \in \mathbb{C}_p$ which is not a root of unity, and such that for all $\omega \in \Omega_M(C)$,
   $$\phi^*\omega - b\omega \in dM(U).$$

Let M be a logarithmic F-crystal on a basic wide open space C.

Lemma 1.38. *There exists a Frobenius neighborhood (W,ϕ) of X in C, and $n \geq 0$, such that:*

1. *The pair (W,ϕ) satisfies the condition (6) for M, with some power of b in place of b,*
2. *$\phi(W \cap V_a) \subseteq V_a$ for each $a \in S$, and*
3. *$\mathrm{Log}\left(\frac{\phi(z)-a}{(z-a)^q}\right) \in \mathcal{O}(W)$ for each $a \in S$.*

Proof. See [Col82, Lemma 4.2]. □

Lemma 1.39. *Let $\omega \in \Omega_M(C)$. There exists a locally-analytic function $F_\omega \in \mathcal{L}(C)$, unique up to an additive constant, which satisfies:*

1. *$dF_\omega = \omega$,*
2. *There is a wide open neighborhood V of C such that $\phi^* F_\omega - bF_\omega \in M(V)$, for some $b \in \mathbb{C}_p$ which is not a root of unity, and*
3. *The restriction of F_ω to the underlying affinoid X is analytic in each residue class of X, and the restriction to V_a is in $\mathcal{O}_{\mathrm{Log}}(V_a)$ for all $a \in S$.*

Given M, define an $\mathcal{O}(C)$-module M' as follows:

$$M' := M + \sum_{\omega \in \Omega_M(C)} F_\omega \mathcal{O}(C).$$

Theorem 1.40. *The $\mathcal{O}(C)$-module M' is the unique minimal logarithmic F-crystal on C which contains M and such that $dM' \supseteq \Omega_M(C)$.*

Remark 1.41. Logarithmic F-crystals on C satisfy the uniqueness principle on C.

It is easy to see that, for C a basic wide open, the ring of rigid-analytic functions $\mathcal{O}(C)$ is a logarithmic F-crystal. This allows to define $A^1(C) := \mathcal{O}(C)'$, and we obtain:

Theorem 1.42 (Coleman)**.** *Let $\omega \in \Omega^1(C)$. There exists a unique (up to constants) function $F_\omega \in A^1(C)$, such that $dF_\omega = \omega$.*

Proof. See [Col82, Theorem 5.1]. □

Let Y be a rigid-analytic space which can be covered by a family $\mathcal{C}$ of basic wide opens, which intersect at basic wide opens, and such that the nerve of the covering is simply connected. Let $\mathcal{A}^1$ be the sheaf of $\mathcal{O}_Y$-modules defined by $\mathcal{A}^1(U){:=}A^1(U)$ for each $U \in \mathcal{C}$.

Corollary 1.43. *There is a short exact sequence:*

$$0 \to \mathbb{C}_p \to H^0(Y, \mathcal{A}^1) \xrightarrow{d} H^0(Y, \mathcal{A}^1) \otimes_{\mathcal{O}_Y(Y)} \Omega^1(Y) \to 0$$

Proof. Let $\omega \in H^0(Y, \mathcal{A}^1) \otimes_{\mathcal{O}_Y(Y)} \Omega^1(Y)$. Let $\mathcal{N} = (\mathfrak{V}(\mathcal{N}), \mathfrak{E}(\mathcal{N}))$ be the nerve of the covering. To each $v \in \mathfrak{V}(\mathcal{N})$, there is a corresponding open $U \in \mathcal{C}$, and a local primitive F'_v of $\omega|_V$, which is defined up to a constant. Therefore the map:

$$e \mapsto F'_{o(e)} - F'_{t(e)} \in \mathbb{C}_p$$

is a 1-cocycle. Since $\mathcal{N}$ is simply connected, this is a coboundary. So there exists a 0-cocycle $v \mapsto c_v \in \mathbb{C}_p$ such that:

$$F'_{o(e)} - F'_{t(e)} = c_{t(e)} - c_{o(e)}.$$

Define F by $F_v{:=}F'_v + c_v$. The previous equation proves that this glues, and gives a global primitive to ω.

Lastly, the fact that $\ker d = \mathbb{C}_p$ is equivalent to the fact that F is uniquely defined up to constants. □

Chapter 2

Shimura curves

In this chapter we introduce the different ways in which Shimura curves appear in this work. We start by introducing basic definitions of quaternion algebras in Section 2.1. In Section 2.2 we define Shimura curves as the solution to certain moduli problems. We continue in Section 2.3 to describe certain special points which play a very important role in our project. In Section 2.4 we describe a uniformization result of Čerednik and Drinfel'd which is a p-adic analogue to the complex uniformization of modular curves. In Section 2.5 we explain what are the spaces of modular forms on a Shimura curve, and we end in Section 2.6 by relating these to classical modular forms.

2.1 Quaternion algebras

This section based in [Bes95] and it is just meant to introduce the required notation.

2.1.1 First definitions

Definition 2.1. An associative algebra D over a field F is:

- *simple algebra* if it does not have any nontrivial double-sided ideals.

- *semisimple algebra* if it the direct sum of simple algebras.
- a *division algebra* if every nonzero element has an inverse.
- *central algebra* if the center of D is F.

Definition 2.2. A *quaternion algebra* over a field F is a 4-dimensional central simple F-algebra B. If F equals $\mathbb{Q}$, we call B a *rational quaternion algebra*.

Example 2.3. The F-algebra of two-by-two matrices with entries in F, which we write $M_2(F)$, is a quaternion algebra. This is the only example of a quaternion algebra over F which is not a division algebra.

A quaternion algebra B comes equipped with an anti-involution $x \mapsto \overline{x}$, called the *canonical anti-involution* of B. For all $a, b \in F$ and all $x, y \in B$, it satisfies:

$$\overline{ax + by} = a\overline{x} + b\overline{y}, \quad \overline{\overline{x}} = x, \quad \overline{xy} = \overline{y}\,\overline{x}.$$

The existence of the canonical anti-involution allows for the following:

Definition 2.4. The *reduced trace* of B is the additive homomorphism $\operatorname{trd}\colon B \to F$ which maps x to $x + \overline{x}$. The *reduced norm* of B is the multiplicative homomorphism $\operatorname{nrd}\colon B \to F$ which maps x to $x\overline{x}$. If $B \cong M_2(F)$, then nrd and trd are the usual determinant and trace of matrices, respectively.

Remark 2.5. Note that neither the reduced norm nor the reduced trace are algebra homomorphisms. They are just group homomorphisms, where the group structure is either the underlying additive structure (for the trace) or the multiplicative structure (for the norm).

Remark 2.6. In general if B is any finite-dimensional F algebra, given $b \in B$ one can consider the F-linear map m_b given by $y \mapsto yb$. This gives an element of $\operatorname{End}_{F-\mathrm{vs}}(B)$. Since B is a finite-dimensional F-vector space, the choice of a basis of B yields an identification $\operatorname{End}_{F-\mathrm{vs}}(B) \cong \operatorname{End}(F^4) \cong M_4(F)$. Hence one may define $t(b) := \operatorname{tr}(m_b)$ (here by tr we mean the trace of an endomorphism), and

$n(b) := \det(m_b)$. They satisfy $t(b) = 2\,\mathrm{trd}(b)$ and $n(b) = \mathrm{nrd}(b)^2$, respectively. Hence the distinction of calling trd and nrd the *reduced* trace and norm.

If B is a quaternion algebra over a field F and $\mathrm{char}(F) \neq 2$, then one can find an F-basis $\{1, i, j, k\}$ for B, satisfying the relations:

$$ij = -ji = k, \quad i^2 = r, \quad j^2 = s,$$

for some elements $r, s \in F$. Such a quaternion algebra is written $\left(\frac{r,s}{F}\right)$. A similar description can be given in characteristic 2, but we will not need it in the sequel.

Example 2.7 (Hamilton quaternions)**.** The rational quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ is known as the algebra of (rational) *Hamilton quaternions*. We will denote it by $\mathbb{H}$.

A field extension F'/F is said to *split* a quaternion algebra B if

$$B \otimes_F F' \cong M_2(F').$$

An algebraically closed extension always splits B, so if $\overline{F}$ is an algebraic closure of F one can compute the trace and norm of B by restricting the trace and determinant via the inclusion $B \hookrightarrow B \otimes \overline{F} \cong M_2(\overline{F})$.

Lemma 2.8. *A quadratic extension F'/F splits B if and only if F' embeds in B as an F-algebra.*

The main involution of B restricts to the nontrivial Galois automorphism on any quadratic extension embedded in B. In particular, if B has a basis $\{1, i, j, k\}$ as before, then

$$\overline{(x + yi + zj + wk)} = x - yi - zj - wk.$$

Theorem 2.9 (Skolem-Noether)**.** *Any two embeddings of an F-algebra F' into the quaternion algebra B are conjugate to one another. Also, any F-automorphism of B is inner. That is, it is of the form $x \mapsto y^{-1}xy$ for some invertible element $y \in B$.*

2.1.2 Local theory

Given a B a rational quaternion algebra, and p a prime of $\mathbb{Q}$ (which can be the infinite prime, also), let $B_p{:=}B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ (where $\mathbb{Q}_\infty = \mathbb{R}$). Then B_p is a quaternion algebra over $\mathbb{Q}_p$.

Lemma 2.10. *There are exactly two isomorphism classes of quaternion algebras over $\mathbb{Q}_p$, given by $M_2(\mathbb{Q}_p)$ and by $\mathbb{H}_p$, where for $p \neq \infty$ a model for $\mathbb{H}_p$ is given as*

$$\left\{ \left(\begin{smallmatrix} a & b \\ pb^\sigma & a^\sigma \end{smallmatrix} \right), a, b \in L_p \right\} \subseteq M_2(L_p).$$

Here $L_p = \mathbb{Q}_{p^2}$ is the unique unramified quadratic extension of $\mathbb{Q}_p$, and σ is the nontrivial automorphism of L_p over $\mathbb{Q}_p$. Finally, $\mathbb{H}_\infty = \mathbb{H} \otimes_{\mathbb{Q}} \mathbb{R}$ is the classical algebra of Hamilton quaternions.

Definition 2.11. The quaternion algebra B is called *ramified* at p if $B_p \cong \mathbb{H}_p$, and it is said to be *split* at p otherwise (that is, if $B_p \cong M_2(\mathbb{Q}_p)$). If B is ramified at ∞ one says that B is *definite*. If B is split at ∞ one says that B is *indefinite*.

Theorem 2.12 (Eichler)**.** *A rational quaternion algebra is determined up to isomorphism by its set of ramifying primes, including the infinite prime, which is always a finite set of* even *cardinality. Moreover, given any set of even cardinality of places of $\mathbb{Q}$, there exists a (necessarily unique) quaternion algebra B which is ramified precisely at those places.*

Definition 2.13. The *discriminant of a rational quaternion algebra* is the product of the finite ramifying primes.

Remark 2.14. The discriminant of a rational quaternion algebra B is always a square-free positive integer, which is divisible by an odd (resp. even) number of primes if B is definite (resp. indefinite)

The following theorem gives a criterion for a quadratic number field to split a rational quaternion algebra.

Theorem 2.15 (Hasse-Brauer-Noether-Albert)**.** *Let B be a rational quaternion algebra and let F be a quadratic number field. There is an embedding of F into B if and only each prime which divides the discriminant of B is either inert or ramified in F.*

2.1.3 Ideals and orders

Definition 2.16. An *ideal in a rational quaternion algebra B* is a free $\mathbb{Z}$-lattice of rank 4 in B. An *order* is an ideal which is a subring of B. A *maximal order* is an order which is not strictly contained in any other order.

Definition 2.17. The *right order* (resp *left order*) of an ideal I of B is the order $R(I) = \{x \in B \mid Ix \subseteq I\}$ (resp. $L(I) = \{x \in B \mid xI \subseteq I\}$).

Definition 2.18. Let I be an ideal in a rational quaternion algebra B, and choose a basis $\{e_i\}$ for it:

$$I = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4.$$

The *discriminant* of I is the rational number:

$$\mathrm{disc}(I) := \sqrt{\det(\mathrm{trd}(e_i e_j'))}.$$

Lemma 2.19 ([Vig80, Cor III.5.3])**.** *An order $\mathcal{R}$ in a rational quaternion algebra B is* maximal *if and only if*

$$\mathrm{disc}(\mathcal{R}) = \mathrm{disc}(B).$$

Definition 2.20. An *Eichler order* in a rational quaternion algebra is an order $\mathcal{R}$ which is the intersection of two maximal orders.

Proposition 2.21. *If $\mathcal{R}$ is Eichler order in a rational quaternion algebra B of discriminant D, then the local order $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is conjugate to one of the following orders: if $2 \neq p \mid D$,*

$$\mathcal{R}_p = \left\{ \left(\begin{smallmatrix} a & b \\ pb^\sigma & a^\sigma \end{smallmatrix} \right) \mid a, b \in \mathcal{O}_{L_p} \right\} \subseteq M_2(\mathcal{O}_{L_p}),$$

where $\mathcal{O}_{L_p}$ is the ring of integers of L_p, and v a non-residue modulo p. If $p \nmid D$,

$$\mathcal{R}_p = \left\{ \left(\begin{smallmatrix} a & b \\ cp^n & d \end{smallmatrix} \right) \mid a, b, c, d \in \mathbb{Z}_p \right\} \subseteq M_2(\mathbb{Q}_p),$$

for some non-negative integer n. Finally,

$$\mathcal{R}_2 = \{ \left(\begin{smallmatrix} a & b \\ 2b^\sigma & a^\sigma \end{smallmatrix} \right) \mid a, b \in \mathcal{O}_{L_2} \} \subseteq M_2(\mathcal{O}_{L_2}),$$

where L_2 may be taken to be $\mathbb{Q}(\sqrt{5})$ and $\mathcal{O}_{L_2} = \mathbb{Z}_2 \oplus \mathbb{Z}_2(1+\sqrt{5})/2$.

Definition 2.22. The *level of an Eichler order* if the product of all prime powers p^n appearing in the previous proposition. It is always an integer prime to the discriminant of B.

Proposition 2.23. *The discriminant of an Eichler order of level N in a rational quaternion algebra of discriminant D is ND.*

Proposition 2.24. *Let $\mathcal{B}$ be an indefinite rational quaternion algebra (i.e. unramified at ∞). Then every left ideal of an Eichler order in $\mathcal{B}$ is principal.*

In the rest of this work the quaternion algebras that will appear will be denoted by either B if they are definite, or by $\mathcal{B}$ if they are indefinite. This convention seems to be in accordance with a large number of those authors using the p-adic uniformization of Shimura curves.

2.2 Shimura curves as moduli spaces

A good exposition of the theory of Shimura curves and their p-adic uniformization can be found in [BC91]. Here we just recall the basic facts.

Fix an integer N which can be factored as $N = pN^-N^+$, where p a prime which will remain fixed, N^- is a positive squarefree integer with an odd number of prime divisors none of which equals p, and N^+ is a positive integer relatively prime to pN^-. Let $\mathcal{B}$ be the indefinite rational quaternion algebra of discriminant pN^-. Fix a maximal order $\mathcal{R}^{\max}$ in $\mathcal{B}$, and an Eichler order $\mathcal{R}$ of level N^+ contained in $\mathcal{R}^{\max}$.

Definition 2.25. Let S be a $\mathbb{Q}$-scheme. An *abelian surface with quaternionic multiplication* (by $\mathcal{R}^{\max}$) and level N^+-structure over S is a triple (A, i, G) where

1. A is a (principally polarized) abelian scheme over S of relative dimension 2;
2. $i\colon \mathcal{R}^{\max} \hookrightarrow \mathrm{End}_S(A)$ is an inclusion defining an action of $\mathcal{R}^{\max}$ on A;
3. G is a subgroup scheme of A which is locally isomorphic to $\mathbb{Z}/N^+\mathbb{Z}$ and is stable and locally cyclic under the action of $\mathcal{R}$.

When no confusion may arise, such a triple will be called an *abelian surface with QM*.

Definition 2.26. The *Shimura curve* $X{:=}X_{N^+,pN^-}/\mathbb{Q}$ is the *coarse* moduli scheme representing the moduli problem over $\mathbb{Q}$:

$$S \mapsto \{ \text{ isomorphism classes of abelian surfaces with QM over } S \ \}.$$

Proposition 2.27 (Drinfel'd)**.** *The Shimura curve X_{N^+,pN^-} is a smooth, projective and geometrically connected curve over $\mathbb{Q}$.*

Proof. See [BC91, Chapter III]. □

It is simpler from a technical point of view to work with a Shimura curve which is a fine moduli space. For that, we need to rigidify the moduli problem, as follows.

Definition 2.28. Let $M \geq 3$ be an integer relatively prime to N. Let S be a $\mathbb{Q}$-scheme. An *abelian surface with QM and full level M-structure* (QM by $\mathcal{R}^{\max}$ and level N^+-structure is understood) is a quadruple $(A, i, G, \overline{\nu})$ where (A, i, G) is as before, and $\overline{\nu}\colon (\mathcal{R}^{\max}/M\mathcal{R}^{\max})_S \to A[M]$ is a $\mathcal{R}^{\max}$-equivariant isomorphism from the constant group scheme $(\mathcal{R}^{\max}/M\mathcal{R}^{\max})_S$ to the group scheme of M-division points of A.

Definition 2.29. The Shimura curve $X_M = X_{N^+,pN^-,M}$ is defined to be the fine moduli scheme classifying the abelian surfaces with QM and full level M-structure.

Remark 2.30. The curve X_M is still smooth and projective over $\mathbb{Q}$. However, it is not geometrically-connected. In fact, as we will see below, it is the disjoint union of $\#(\mathbb{Z}/M\mathbb{Z})^\times$ components.

Forgetting the level M-structure yields a Galois covering $q\colon X_M \to X$, with Galois group

$$(\mathcal{R}^{\max}/M\mathcal{R}^{\max})^\times/\{\pm 1\} \cong \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})/\{\pm 1\}.$$

2.3 Heegner points on a Shimura curve

Let F be a field of characteristic zero.

Definition 2.31. An abelian surface A defined over F (with $i\colon \mathcal{R}^{\max} \hookrightarrow \mathrm{End}_F(A)$ and level-N structure) is said to have *complex multiplication* (CM) if $\mathrm{End}_{\mathcal{R}^{\max}}(A)$, the ring of endomorphism of A which commute with the action of $\mathcal{R}^{\max}$, strictly contains $\mathbb{Z}$. In that case, $\mathcal{O} := \mathrm{End}_{\mathcal{R}^{\max}}(A)$ is an order in an imaginary quadratic number field K, and one says that A has CM by $\mathcal{O}$.

Definition 2.32. An point on the Shimura curve X_M is called a *Heegner point* if it can be represented by a quadruple $(A, i, G, \overline{\nu})$ such that A has complex-multiplication by $\mathcal{O}$ and G is $\mathcal{O}$-stable. If we drop the condition of G being $\mathcal{O}$-stable, then we call it a *CM point*.

Remark 2.33. Suppose that A has QM by $\mathcal{R}^{\max}$ and CM by $\mathcal{O}_K$. Then $\mathcal{O}_K$ splits $\mathcal{R}^{\max}$, and therefore:

$$\mathrm{End}_F(A) \cong \mathcal{O}_K \otimes \mathcal{R}^{\max} \cong M_2(\mathcal{O}_K).$$

By $\mathrm{End}_F(A)$ we mean the endomorphisms of A as an algebraic variety over F. Fixing an isomorphism $\mathrm{End}(A) \cong M_2(\mathcal{O}_K)$ yields an isomorphism $A \cong E \times E$, where E is an elliptic curve defined over H, the Hilbert class field of F, with $\mathrm{End}_H(E) \cong \mathcal{O}_K$. Explicitly, one can obtain each of the two copies of E by applying to A the endomorphism corresponding to the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, E is an elliptic curve with complex multiplication.

2.4 The p-adic uniformization of Shimura curves

We will use a uniformization result due to Čerednik and Drinfel'd, which gives an explicit realization of the Shimura curves X and X_M as quotients of the p-adic upper-half plane. Let B be the *definite* rational quaternion algebra of discriminant N^-, and let R be an Eichler $\mathbb{Z}[\frac{1}{p}]$-order of level N^+ in B. Define the group

$$\Gamma := \{x \in R^\times \mid \mathrm{nrd}(x) = 1\}.$$

Fix an isomorphism

$$\iota_p \colon B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} M_2(\mathbb{Q}_p).$$

Proposition 2.34. *The isomorphism ι_p identifies the group Γ with a discrete co-compact subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$.*

Proof. See [Shi94, Proposition 9.3]. □

The previous proposition makes it possible to consider the quotient $X_\Gamma := \Gamma \backslash \mathcal{H}_p$. The celebrated result of Čerednik-Drinfel'd gives a deep relationship of the Shimura curves X and X_M defined above, with X_Γ.

Theorem 2.35 (Čerednik-Drinfel'd)**.** *There is an isomorphism of rigid-analytic varieties:*

$$(X_{\mathbb{Q}_p^{ur}})^{an} \cong X_\Gamma := \Gamma \backslash \mathcal{H}_p.$$

Moreover, for any integer $M \geq 3$, let Γ_M be the subgroup of units of reduced norm congruent to 1 modulo M. There is an isomorphism of rigid-analytic varieties:

$$(X_M)^{an}_{\mathbb{Q}_p^{ur}} \cong \Gamma \backslash \left(\mathcal{H}_p \times (R/MR)^\times\right) \cong \coprod_{(\mathbb{Z}/M\mathbb{Z})^\times} \Gamma_M \backslash \mathcal{H}_p,$$

which exhibits X_M as a disjoint union of Mumford curves, and hence it is semistable.

Proof. Although the result is original of Čerednik and Drinfel'd, a more detailed exposition of the proof can be found in [BC91, Chap. III, 5.3.1]. $\square$

2.5 Modular forms on a Shimura curve

Let $n \geq 0$ be an even integer. In later chapters we will exclude $n = 0$ for technical reasons, but in this and the following section we can afford to be more general.

We want to explain the different ways of identifying modular forms with sections of certain sheaves associated to the Shimura curve $X{:=}X_{N^+,pN^-}$ as defined in Definition 2.26.

Definition 2.36. Let K be a field of characteristic 0. A *modular form of weight $n + 2$ on X defined over K* is a global section of the sheaf $\Omega_{X_K/K}^{\otimes 1+n/2}$ on X_K. We denote by $M_{n+2}(X, K)$ the space of such modular forms.

Let K be either $\mathbb{Q}_p^{\mathrm{ur}}$ or any complete field contained in $\mathbf{C}_p$ which contains $\mathbb{Q}_{p^2}$. Using the result of Čerednik-Drinfel'd stated in Theorem 2.35 we can give a more concrete description of $M_{n+2}(X, K)$.

Definition 2.37. A p-adic modular form of weight $n + 2$ for Γ is a rigid analytic function $f\colon \mathcal{H}_p(\mathbb{C}_p) \to \mathbb{C}_p$, defined over K, such that

$$f(\gamma z) = (cz + d)^{n+2} f(z) \qquad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

Denote the space of such p-adic modular forms by $M_{n+2}(\Gamma) = M_{n+2}(\Gamma, K)$.

Proposition 2.38. *There is a canonical isomorphism*

$$M_{n+2}(\Gamma, K) \xrightarrow{\sim} M_{n+2}(X, K),$$

which maps f to $\omega_f{:=}f(z)dz^{\otimes 1+n/2}$.

2.6 The Jacquet-Langlands correspondence

In order to justify our interest in modular forms over Shimura curves, we would like to relate them to more familiar objects. Let $\mathbb{T}$ be the abstract Hecke algebra generated by the Hecke operators T_ℓ for $\ell \nmid N$ and U_ℓ for $\ell \mid N$. The Hecke algebra $\mathbb{T}$ acts naturally on the space $M_{n+2}(X, K)$, on which also act the Atkin-Lehner involutions.

Theorem 2.39 (Jacquet-Langlands)**.** *Let K be a field. There is a canonical (up to scaling) isomorphism*

$$M_{n+2}(X, K) \xrightarrow{\sim} S_{n+2}(\Gamma_0(N), K)^{pN^-\text{-}new},$$

which is compatible with the action of $\mathbb{T}$ and the Atkin-Lehner involutions on each of the spaces.

Therefore to a classical modular pN^--new eigenform f_∞ on the modular curve $X_0(N)$, there is associated an eigenform f on the Shimura curve X. In Chapter 4 we will see the construction of a p-adic L-function attached to f which interpolates special values of the classical L-function associated to f_∞.

Chapter 3

Cohomology

In this chapter we present and develop the cohomological tools needed for the constructions and computations in this thesis. In Section 3.1 we describe harmonic cocycles. Section 3.2 introduces the category of filtered Frobenius monodromy modules, a linear algebra category whose objects encode all necessary information from suitable p-adic representations. In Section 3.3 we describe the comparison isomorphisms of Fontaine. A computationally important result is explained in Section 3.4. In Sections 3.5 and 3.6 we follow [IS03] in introducing convergent filtered F-isocrystals on a formal curve. These are sheaves equipped with various extra structures (a connection, filtration, Frobenius action); they are allowable coefficients for the de Rham cohomology to have the structure of a filtered Frobenius monodromy module. In Section 3.7 this structure is described as in [CI03]. The previous constructions are specialized to the situation that applies to this work in Section 3.8. Finally, in Section 3.9 we define the p-adic Abel-Jacobi map, following mainly [IS03].

3.1 Harmonic cocycles

Let Γ be a cocompact subgroup of $\mathrm{PSL}_2(\mathbb{Q}_p)$, and let M be a $\mathbb{C}_p[\Gamma]$-module. By $\mathcal{T} = (\mathfrak{V}(\mathcal{T}), \mathfrak{E}(\mathcal{T}))$ we denote, as in Subsection 1.2.1, the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

Definition 3.1. An M-valued 0-cocycle (resp. 1-cocycle) on $\mathcal{T}$ is an M-valued function c on $\mathfrak{V}(\mathcal{T})$ (resp. on $\mathfrak{E}(\mathcal{T})$, such that $c(\overline{e}) = -c(e)$). The $\mathbb{C}_p$-vector space of M-valued 0-cocycles (resp. 1-cocyles) is written $C^0(M)$ (resp. $C^1(M)$).

Definition 3.2. An M-valued 0-cocycle c is called *harmonic* if it satisfies, for all $v \in \mathfrak{V}(\mathcal{T})$,

$$\sum_{e \in \mathfrak{E}(\mathcal{T}), o(e)=v} c(o(e)) - c(t(e)) = 0.$$

The $\mathbb{C}_p$-vector space of M-valued harmonic 0-cocycles is written $C^0_{\mathrm{har}}(M)$.

Definition 3.3. An M-valued 1-cocycle c is called *harmonic* if it satisfies

$$\sum_{o(e)=v} c(e) = 0,$$

for all $v \in \mathfrak{V}(\mathcal{T})$.

The $\mathbb{C}_p$-vector space of M-valued harmonic 1-cocycles is written $C^1_{\mathrm{har}}(M)$.

The group Γ acts on $C^i_{\mathrm{har}}(M)$ on the left, by

$$\gamma \cdot c := \gamma \circ c \circ \gamma^{-1}, \quad \text{for } \gamma \in \Gamma \text{ and } c \in C^i_{\mathrm{har}}(M).$$

Let $\mathcal{P}_n$ be the $n+1$-dimensional $\mathbb{Q}_p$-vector space of polynomials of degree at most n with coefficients in $\mathbb{Q}_p$. The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathcal{P}_n$ on the right, by

$$P(x) \cdot \beta := (cx+d)^n P\left(\frac{ax+b}{cx+d}\right), \quad \text{for } \beta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

In this way its $\mathbb{Q}_p$-linear dual $V_n := \mathcal{P}_n^\vee$ is endowed with a left action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Definition 3.4. A *harmonic cocycle of weight* $n+2$ *on* $\mathcal{T}$ is a V_n-valued harmonic cocycle.

Define now $\mathcal{U}$ as the subspace of $M_2(\mathbb{Q}_p)$ given by matrices of trace 0. They have a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ given by

$$u \cdot \beta := \overline{\beta} u \beta,$$

where $\overline{\beta}$ is the matrix such that $\overline{\beta}\beta = \det(\beta)$.

There is a map $\Phi\colon \mathcal{U} \to \mathcal{P}_2$ intertwining the action $\mathrm{GL}_2(\mathbb{Q}_p)$, given by

$$u \mapsto P_u(x) := \operatorname{tr}\left(u \left(\begin{smallmatrix} x & -x^2 \\ 1 & -x \end{smallmatrix}\right)\right) = \operatorname{tr}\left(u \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -x \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & -x \end{smallmatrix}\right) u \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right). \tag{3.1}$$

Lemma 3.5. *The map Φ induces an isomorphism of right $\mathrm{GL}_2(\mathbb{Q}_p)$-modules.*

Remark 3.6. Let $u = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathcal{U}$. Suppose that u is invertible. Then u acts on $\mathcal{H}_p$, and a point $x \in \mathcal{H}_p$ is fixed by u if it satisfies:

$$\frac{ax + b}{cx - a} = x,$$

that is if x is a root of the polynomial:

$$-cx^2 + 2ax + b = P_u(x).$$

This is why the map $u \mapsto P_u(x)$ is introduced.

On $\mathcal{U}$ there is a pairing defined by

$$\langle u, v\rangle := -\operatorname{tr}(u\overline{v}).$$

This induces a pairing on $\mathcal{P}_2$ by transport of structure, and on the dual V_2 of $\mathcal{P}_2$ by canonically identifying $\mathcal{P}_2$ with V_2 using the pairing itself. Unwinding the definitions, we can prove the following formula:

Lemma 3.7. *Take as basis for V_2 the linear forms $\{\omega_i\}_{0\leq i\leq 2}$, dual to the basis $\{1, x, x^2\}$ of $\mathcal{P}_2$. Then the pairing $\langle\cdot,\cdot\rangle$ on V_2 is given by:*

$$\langle a\omega_0 + b\omega_1 + c\omega_2, a'\omega_0 + b'\omega_1 + c'\omega_2\rangle = 2bb' - a'c - ac'.$$

Let now $n = 2m$ be an even integer. The pairing $\langle\cdot,\cdot\rangle$ induces a perfect symmetric pairing on $\operatorname{Sym}^m V_2 = V_n$ given by the formula:

$$\langle v_1 \cdots v_m, v_1' \cdots v_m'\rangle := \frac{1}{m!} \sum_{\sigma\in\mathfrak{S}_n} \langle v_1, v'_{\sigma(1)}\rangle \cdots \langle v_m, v'_{\sigma(m)}\rangle.$$

3.2 Filtered (ϕ, N)-modules

Let K be a field of characteristic 0, which is complete with respect to a discrete valuation and has perfect residue field κ of characteristic $p > 0$. Let $K_0 \subseteq K$ be the maximal unramified subfield of K. Concretely, K_0 is the fraction field of the ring of Witt vectors of κ. Let $\sigma\colon K_0 \to K_0$ be the absolute Frobenius automorphism.

Definition 3.8. A *filtered Frobenius monodromy module* over K (also called a *filtered* (ϕ, N)*-module*) is a quadruple $(D, \mathrm{Fil}^\bullet, \phi, N)$, where

1. D is a finite dimensional K_0-vector space,
2. $\mathrm{Fil}^\bullet = \mathrm{Fil}^\bullet_D$ is an exhaustive and separated decreasing filtration on the vector space $D_K := D \otimes_{K_0} K$ over K, called the *Hodge filtration*,
3. $\phi = \phi_D\colon D \to D$ is a σ-linear automorphism, called the *Frobenius on* D, and
4. $N = N_D\colon D \to D$ is a K-linear endomorphism, called the *monodromy operator*, satisfying $N\phi = p\phi N$.

Sometimes we write D to refer to the tuple $(D, \mathrm{Fil}^\bullet_D, \phi_D, N_D)$. The category of filtered (ϕ, N)-modules over K is denoted by $MF_K^{(\phi,N)}$.

Forgetting the monodromy action or, equivalently, setting $N = 0$, gives a full subcategory of $\mathrm{MF}_K^{(\phi,N)}$, called the category of *filtered F-isocrystals over* K. If in addition we also forget the filtration, the full subcategory thus obtained is called the category of *isocrystals over* K_0.

Remark 3.9. The particle "iso" is used to remind ourselves that they do not have an integral structure on them. If one wants to work with "crystals", then one has to consider certain types of $\mathcal{O}_K$-modules instead.

Example 3.10. The field K_0 itself has a structure of filtered (ϕ, N)-module over K: the underlying K_0-vector space is K_0 itself, the Frobenius is $\phi_{K_0} = \sigma$, the monodromy N is 0, and for $K = K_0 \otimes_{K_0} K$ the filtration is given by:

$$\mathrm{Fil}^i = \begin{cases} K & \text{if } i \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.11. Let $\lambda = r/s$ be a rational number, where r and s are chosen so that $(r, s) = 1$ and $s > 0$. Let $K_0[\phi]$ be the *twisted* polynomial ring with coefficients in K_0, satisfying $\phi c = \sigma(c)\phi$, for $c \in K_0$. The quotient:

$$M_\lambda = K_0[\phi]/(K_0[\phi](\phi^s - p^r))$$

has a natural Frobenius action given by left-multiplication by ϕ. This is a fundamental example of an isocrystal.

Theorem 3.12 (Dieudonné-Manin)**.** *Assume that K_0 equals $W(k)[1/p]$, with k an algebraically-closed field. Then the category of isocrystals over K_0 is semisimple (that is, all objects are finite direct sums of simple objects, and all short exact sequences split). Its simple objects are the isocrystals M_λ of Example 3.11, for $\lambda \in \mathbb{Q}$.*

Proposition 3.13. *The category $MF_K^{(\phi,N)}$ is an additive tensor category admitting kernels and cokernels.*

Sketch of proof. Let $f \colon D \to D'$ be a morphism. Then $\ker f$ is defined as follows: As a K_0-vector space, it is just the kernel of the underlying K_0-linear map $f \colon D \to D'$. If $x \in \ker f$, then $f(\phi_D(x)) = \phi_{D'}(f(x)) = \phi_{D'}(0) = 0$, so that ϕ_D acts on $\ker f$. Similarly for the monodromy N_D. Finally, one can define $F^i_{\ker f} \ker f := \ker f \cap F^i_D D$. This gives $\ker f$ the structure of a filtered (ϕ, N)-module. The cokernel is defined similarly, where the filtration on it is the quotient filtration.

If D and D' are filtered (ϕ, N)-modules, we construct the object $D \otimes D'$ as follows. As a K_0-vector space, it is $D \otimes_{K_0} D'$. Its base extension to K is $D_K \otimes_K D'_K$, and the filtration is given by the tensor product filtration:

$$F^i_{D\otimes D'}(D_K \otimes_K D'_K) := \sum_{a+b=i} F^a_D D_K \otimes F^b_{D'} D'_K.$$

The Frobenius is given by tensoring the corresponding actions on each factor, and the monodromy is defined by:

$$N_{D\otimes D'}{:=}1_D \otimes N_{D'} + N_D \otimes 1_{D'}.$$

Finally, a sequence is exact if the underlying sequence of K_0-vector spaces is exact. For more details refer to [BC]. $\square$

Remark 3.14. The notion of short exact sequence coincides with the following: given a sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in $\mathrm{MF}_K^{(\phi,N)}$, it is exact if:

1. $\ker f = 0$,
2. $\operatorname{coker} g = 0$, and
3. f induces an isomorphism $A \xrightarrow{\sim} \ker g$, or $\operatorname{coker} f \cong C$.

Definition 3.15. If $D = (D, F_D^\bullet, \phi_D, N_D)$ is a filtered (ϕ, N)-module, and $j \in \mathbb{Z}$, we define another filtered (ϕ, N)-module $D[j]$, the jth *Tate twist* of D, as $D[j] = (D, F_D^{\bullet-j}, p^j\phi_D, N_D)$. Here we mean:

$$F^i(D[j]_K) = F^{i-j}(D_K), \qquad \text{for all } i \in \mathbb{Z}.$$

Remark 3.16. In [BC], the Tate-twist is defined using a different convention. Their jth Tate twist coincides with our $-j$th Tate twist. In their notation:

$$D[-j] = D\langle j\rangle.$$

This is done because we will be using the covariant Fontaine functors, whereas [BC] uses their contravariant counterparts.

3.3 The ring B_{st}, semistability, and comparison isomorphisms

We first recall some key facts about the ring B_{st} of Fontaine. The original construction can be found in [Fon94]. There is a very detailed exposition of this material in the paper O. Brinon and B. Conrad [BC].

The ring B_{st} is a topological K_0-algebra, with the following extra structure:

1. A continuous action of G_K such that $B_{\mathrm{st}}^{G_K} = K_0$.
2. A G_K-equivariant embedding $K_0^{\mathrm{ur}} \hookrightarrow B_{\mathrm{st}}$.
3. A σ-semilinear continuous automorphism $\phi\colon B_{\mathrm{st}} \to B_{\mathrm{st}}$ which commutes with the G_K-action.
4. An exhaustive and separated decreasing filtration Fil^i of the extension of scalars $(B_{\mathrm{st}})_K$, stable under the G_K-action.
5. A K_0-linear operator $N\colon B_{\mathrm{st}} \to B_{\mathrm{st}}$ satisfying $N\phi = p\phi N$.

We do not need to know too much about the construction of the ring B_{st} for now. For us, its importance lies in the functors that are constructed using B_{st}, which establish important equivalences of categories. Consider the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of p-adic representations of G_K, whose objects are finite-dimensional $\mathbb{Q}_p$-vector-spaces with a continuous linear G_K-action. It is an abelian tensor category, with twists given by tensoring with powers of the Tate representation $\mathbb{Q}_p(1){:=}(\varprojlim_n \boldsymbol{\mu}_{p^n})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

Given a p-adic representation V of G_K, consider:

$$\mathrm{D}_{\mathrm{st}}(V){:=}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_K}.$$

This is a K_0-vector space, and the actions of ϕ and N on B_{st} give actions on $\mathrm{D}_{\mathrm{st}}(V)$ as well. The filtration on B_{st} induces also a filtration on $\mathrm{D}_{\mathrm{st}}(V)_K = \mathrm{D}_{\mathrm{st}}(V) \otimes_{K_0} K$.

One checks that it has finite K_0-dimension, and so $\mathrm{D}_{\mathrm{st}}(V)$ becomes a filtered (ϕ, N)-module. Moreover, for each $j \in \mathbb{Z}$,

$$\mathrm{D}_{\mathrm{st}}(V(j)) = \mathrm{D}_{\mathrm{st}}(V)[-j].$$

Conversely, given a filtered (ϕ, N)-module D, one defines:

$$\mathrm{V}_{\mathrm{st}}(D) := \mathrm{Hom}_{\mathrm{MF}_K(\phi,N)}(K, D \otimes_{K_0} B_{\mathrm{st}}) \cong \mathrm{Fil}^0 \left(D \otimes_{K_0} B_{\mathrm{st}}\right)^{\phi=\mathrm{Id}, N=0}.$$

The assignments D_{st} and V_{st} are functorial, as it is easily checked. This provides functors:

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \underset{\mathrm{V}_{\mathrm{st}}}{\overset{\mathrm{D}_{\mathrm{st}}}{\rightleftarrows}} \mathrm{MF}_K(\phi, N),$$

and we are interested in how close these are to providing equivalences of categories. In order to obtain such an equivalence, we need to restrict both categories.

Definition 3.17. A p-adic representation V of G_K is *semistable* if the canonical injective map:

$$\alpha\colon \mathrm{D}_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}} = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{G_K} \otimes_{K_0} B_{\mathrm{st}} \hookrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}) \otimes_{K_0} B_{\mathrm{st}} \overset{\mathrm{Id}\otimes m}{\longrightarrow} V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$$

is surjective. The category of semistable representations, denoted $\mathrm{Rep}_{\mathrm{st}}(G_K)$ is the full subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of semistable objects.

Remark 3.18. Let X/K be a proper variety with a semi-stable model. Consider the étale cohomology groups:

$$H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p) := \left(\varprojlim_n H^i_{\mathrm{et}}(\overline{X}, \mathbb{Z}/p^n\mathbb{Z})\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

These vector spaces are naturally finite-dimensional continuous G_K-representations. Results of Fontaine-Messing, Hyodo-Kato, Faltings and Tsuji imply that these representations are semistable. They constitute in fact the main source of semistable representations.

Definition 3.19. A filtered (ϕ, N)-module D is *admissible* if it is isomorphic to $\mathrm{D}_{\mathrm{st}}(V)$ for some semistable representation V of G_K. The full subcategory of admissible filtered (ϕ, N)-modules is denoted $\mathrm{MF}^{\mathrm{ad}}_K(\phi, N)$.

The property of a filtered (ϕ, N)-module being admissible can also be phrased in terms of intrinsic properties of D itself. In this setting this property was originally called weak-admissibility, but was proven to be equivalent to admissibility by Fontaine and Colmez in [CF00]. They also prove:

Theorem 3.20 ([CF00, Theorem A]). *The functors* D_{st} *and* V_{st} *give an equivalence of categories between* $\mathrm{Rep}_{st}(G_K)$ *and* $\mathrm{MF}_K^{ad}(\phi, N)$*, which is compatible with exact sequences, tensor products and duality.*

The main use that we have for this fact is the following:

Corollary 3.21. *Let* V, W *be two objects in* $\mathrm{Rep}_{st}(G_K)$*. The functors* D_{st} *and* V_{st} *induce a canonical group isomorphism*

$$\mathrm{Ext}^1_{\mathrm{Rep}_{st}(G_K)}(V, W) \cong \mathrm{Ext}^1_{\mathrm{MF}_K^{ad}(\phi,N)}\left(\mathrm{D}_{st}(V), \mathrm{D}_{st}(W)\right),$$

where $\mathrm{Ext}^1_{\mathcal{C}}$ *denotes the extension-group bifunctor in the category* $\mathcal{C}$*.*

3.4 Extensions of filtered (ϕ, N)-modules

Let D be a filtered (ϕ, N)-module over K_0, where $K_0 = W(k)[1/p]$ with k algebraically closed. Given a rational number $\lambda = r/s$, where $r, s \in \mathbb{Z}$ are such that $(r, s) = 1$ and $s > 0$, define D_λ to be the largest subspace of D which has an $\mathcal{O}_{K_0}$-stable lattice M satisfying $\phi^s(M) = p^r M$. The subspace D_λ is called the *isotypical component* of D of slope λ. The *slopes* of D are the rational numbers λ such that $D_\lambda \neq 0$, and D is called *isotypical* of slope λ_0 if $D = D_{\lambda_0}$. As a corollary to Theorem 3.12 we obtain a decomposition of isocrystals (that is, after forgetting the filtration and monodromy):

$$D = \bigoplus_{\lambda \in \mathbb{Q}} D_\lambda.$$

Note also that $N(D_\lambda) \subseteq D_{\lambda-1}$ for all $\lambda \in \mathbb{Q}$. The following result appears in [IS03, Lemma 2.1], although its proof is mostly omitted. We present here a fully detailed proof.

Lemma 3.22. *Let D be a filtered (ϕ, N)-module, n an integer, and assume that N induces an isomorphism between the isotypical components D_{n+1} and D_n. Then there is a canonical isomorphism*

$$\mathrm{Ext}^1_{\mathrm{MF}_K^{(\phi,N)}}(K[n+1], D) \cong D/\operatorname{Fil}^{n+1} D,$$

that maps the class of an extension

$$0 \to D \xrightarrow{\iota} E \xrightarrow{\pi} K[n+1] \to 0$$

to $(s_1(1) - s_2(1)) + \operatorname{Fil}^{n+1} D_K$, where:

1. *$s_1 \colon K[n+1] \to E$ is a splitting of π which is compatible with the Frobenius and monodromy operator, but not necessarily with filtrations, and*
2. *$s_2 \colon K[n+1] \to E$ is splitting of π compatible with the filtrations, but not necessarily with the Frobenius and monodromy operators.*

Remark 3.23. The fact that the splittings s_1 and s_2 exist is part of the statement of the lemma.

Proof. First, note that by applying the snake lemma to the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Fil}^{n+1} D_K \arrow[r] \arrow[d, hook] & \operatorname{Fil}^{n+1} E_K \arrow[r] \arrow[d, hook] & \operatorname{Fil}^{n+1} K \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D_K \arrow[r] & E_K \arrow[r] & K \arrow[r] & 0,
\end{tikzcd}$$

we get an isomorphism $D_K/\operatorname{Fil}^{n+1} D_K \cong E_K/\operatorname{Fil}^{n+1} E_K$, and hence we just need to find an element in $E_K/\operatorname{Fil}^{n+1} E_K$. Explicitly, once we get $s_1(1) \in E_K$, we can consider $s_1(1) - s_2(1)$, where s_2 is a splitting of the extension which is compatible with the filtrations. Such a splitting s_2 exists because the category of K-vector spaces is semisimple. Since $\pi(s_1(1) - s_2(1)) = 0$, we can view $s_1(1) - s_2(1)$ as an element of D_K (via ι), thus making the isomorphism explicit.

The filtered (ϕ, N)-module $K[n+1]$ is pure of slope $n+1$, and the hypothesis on the monodromy action N on D gives a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & D_{n+1} \arrow[r] \arrow[d, "N", "\cong"'] & E_{n+1} \arrow[r] \arrow[d, "N"] & K_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_n \arrow[r] & E_n \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

An application of the snake lemma and the fact that the left vertical arrow is an isomorphism yields another isomorphism

$$\pi_| \colon \ker\left(E_{n+1} \xrightarrow{N} E_n\right) \xrightarrow{\sim} K_0,$$

and we define $s_1 \colon K \to E_K$ as its inverse. Then s_1 is compatible with the action of ϕ and N, by construction.

We check that the assignment of $s_1(1) + \operatorname{Fil}^{n+1} E_K$ to an extension $0 \to D \to E \to K[n+1] \to 0$ is well-defined: if the extension is trivial, then s_1 can be chosen to be compatible with Fil, and we then get

$$s_1(1) \in s_1(\operatorname{Fil}^n K[n+1]) \subseteq \operatorname{Fil}^{n+1} E_K.$$

Conversely, given $d + \operatorname{Fil}^{n+1} D_K \in D_K / \operatorname{Fil}^{n+1} D_K$, we construct a filtered (ϕ, N)-module $E^{(d)}$ as an extension of $K[n+1]$ by D. We define $E_0^{(d)} = D_0 \oplus (K_0[n+1])$, as (ϕ, N)-modules. The filtration on $E_K^{(d)} = E_0^{(d)} \otimes_{K_0} K$ is defined as follows:

$$\operatorname{Fil}^j E_K^{(d)} := \left\{(x,t) \in D_K \oplus K \;\mid\; t \in \operatorname{Fil}^{j-n-1} K,\; x + td \in \operatorname{Fil}^j D\right\}.$$

Consider the isomorphism class of the extension

$$\Xi\colon \quad 0 \to D \xrightarrow{\iota} E^{(d)} \xrightarrow{\pi} K[n+1] \to 0,$$

where the map ι is the canonical inclusion, and the map π is the canonical projection. Note that this sequence is exact and well defined, since

$$\pi(\operatorname{Fil}^j E_K^{(d)}) = \operatorname{Fil}^{j-n-1} K = \operatorname{Fil}^j K[n+1].$$

Moreover, if $d \in \operatorname{Fil}^{n+1} D_K$, then the map

$$1 \mapsto (0,1)$$

splits the extension Ξ in the category of filtered (ϕ, N)-modules. Hence the map

$$D_K/\operatorname{Fil}^{n+1} D_K \to \operatorname{Ext}^1(K[n+1], D)$$

which assigns the extension Ξ to $d \in D/\operatorname{Fil}^{n+1} D$ is well defined.

To end the proof, we need to check that the two assignments are mutually inverse. Starting with $d + \operatorname{Fil}^{n+1} D_K$, the vector space splitting $1 \mapsto (0,1)$ is compatible with the Frobenius and monodromy actions. Also the vector space splitting $1 \mapsto (-d, 1)$ is compatible with the filtrations. We obtain the class of d in $D_K/\operatorname{Fil}^{n+1} D_K$, as wanted.

Conversely, start with an arbitrary extension

$$0 \to D \xrightarrow{\iota} E \xrightarrow{\pi} K[n+1] \to 0.$$

Choose s_1 and s_2 two splittings of π as before, and define $d \in D_K$ such that $\iota_K(d) = s_1(1) - s_2(1)$. Consider now the map $E^{(d)} \to E$ sending

$$(x,t) \mapsto \iota(x) + s_1(t) = \iota(x + td) + s_2(t).$$

The first expression shows that this is a map of (ϕ, N)-modules. The second expression shows that it respects the filtrations. Its inverse is the map

$$y \mapsto \left(\iota^{-1}(y - s_1(\pi(y))), \pi(y)\right) = \left(\iota^{-1}(y - s_2(\pi(y))) - \pi(y)d, \pi(y)\right).$$

Again, the first expression shows that it is respects the Frobenius and monodromy actions, while the second shows that it respects the filtrations. This concludes the proof. $\square$

3.5 Convergent filtered F-isocrystals

Let κ, K_0 and K be as before. In [Ogu] it is show how to extend the constructions in Section 3.2 to formal schemes. Let Z be a formal $\mathcal{O}_K$-scheme. The construction in Section 3.2 is the particular case of $Z = \operatorname{Spf} \mathcal{O}_K$. In this section we follow the exposition of [IS03]. For simplicity, assume that $K = K_0$.

Recall that a p-adic formal $\mathcal{O}_K$-scheme is a formal scheme obtained by gluing affine opens of the form $\operatorname{Spf} R$, where R is a quotient of $\mathcal{O}_K\langle X_1, \dots, X_n\rangle$ for some n.

Definition 3.24. An *enlargement* of Z is a pair (T, z_T) consisting of a flat p-adic formal $\mathcal{O}_K$-scheme T and a morphism of formal $\mathcal{O}_K$-schemes $z_T \colon T_0 \to Z$ (where T_0 is the closed subscheme of T defined by the ideal $p\mathcal{O}_T$, with the reduced scheme structure).

A *morphism of enlargements* of Z, say $(T', z_{T'}) \to (T, z_T)$ is an $\mathcal{O}_K$-morphism $g \colon T' \to T$ such that $z_T \circ g_0 = z_{T'}$.

Definition 3.25. A *convergent isocrystal* $\mathcal{E}$ on Z is the following data:

1. For every enlargement $T = (T, z_T)$ of Z, a coherent $\mathcal{O}_T \otimes_{\mathcal{O}_K} K$-module $\mathcal{E}_T$.
2. For every morphism of enlargements $g \colon (T', z_{T'}) \to (T, z_T)$, an isomorphism of $\mathcal{O}_{T'} \otimes_{\mathcal{O}_K} K$-modules
$$\theta_g \colon g^*(\mathcal{E}_T) \to \mathcal{E}_{T'},$$
such that the collection $\{\theta_g\}$ satisfies the cocycle condition.

Let $\sigma \colon W(\kappa) \to W(\kappa)$ be the Frobenius automorphism. This can be lifted to the absolute Frobenius $F \colon Z \to Z^\sigma$. Given an enlargement (T, z_T) of Z, the pair $(T, F \circ z_T)$ is an enlargement of Z^σ and hence $(T^{\sigma^{-1}}, (F \circ z_T)^{\sigma^{-1}})$ is an enlargement of Z. Define then $F^*\mathcal{E}$ as the isocrystal which on Z which assigns to (T, z_T) the $\mathcal{O}_T \otimes_{\mathcal{O}_K} K$-module:
$$\alpha(\sigma)_* \mathcal{E}_{(T^{\sigma^{-1}}, (F \circ z_T)^{\sigma^{-1}})}.$$
Here, $\alpha(\sigma)$ is the morphism $T^\sigma \to T$ such that the following square commutes:
$$\begin{tikzcd} T^\sigma \arrow[r, "\alpha(\sigma)"] \arrow[d] & T \arrow[d] \\ \operatorname{Spec} \mathcal{O}_K \arrow[r, "\sigma"] & \operatorname{Spec} \mathcal{O}_K. \end{tikzcd}$$

Definition 3.26. A *convergent F-isocrystal* on Z is a convergent isocrystal $\mathcal{E}$ on Z together with an isomorphism of crystals $\Phi\colon F^*\mathcal{E} \to \mathcal{E}$.

Assume from now on that Z is analytically smooth over $\mathcal{O}_K$. The Gauss-Manin connection is a natural connection on $E^{\mathrm{an}} = E_Z^{\mathrm{an}}$. It can be defined as a certain connecting homomorphism in the Hodge to de Rham spectral sequence for Z. A precise definition can be found in [KO68], and the required facts about its properties can be found in [Ogu].

Definition 3.27. A *filtered convergent F-isocrystal* on Z consists of a convergent F-isocrystal $\mathcal{E}$ together with an exhaustive and separated decreasing filtration $\mathrm{Fil}^\bullet E^{\mathrm{an}}$ of coherent $\mathcal{O}_{Z^{\mathrm{an}}}$-submodules, such that

$$\nabla(\mathrm{Fil}^i E^{\mathrm{an}}) \subseteq (\mathrm{Fil}^{i-1} E^{\mathrm{an}}) \otimes_{\mathcal{O}_Z^{\mathrm{an}}} \Omega^1_{Z^{\mathrm{an}}}.$$

This condition is called *Griffiths' transversality* and is required in order to be able to define a filtration on the de Rham cohomology with coefficients in E^{an}.

The category of filtered convergent F-isocrystals on Z is an additive tensor category.

Example 3.28.
1. The identity object $\mathcal{O}_Z$ in this category is the assignment $T \mapsto \mathcal{O}_T \otimes K$. The Frobenius is the canonical one. The connection is the trivial one, given by the usual derivation d. The filtration is given by

$$\mathrm{Fil}^i = \begin{cases} \mathcal{O}_{Z^{\mathrm{an}}} & \text{if } i \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. Let $f\colon X \to Z = \mathrm{Spf}(\mathcal{O}_K)$ be smooth proper morphism of p-adic formal schemes. One can define an F-isocrystal $R^q f_* \mathcal{O}_{X/K}$ using crystalline cohomology sheaves tensored with K. This is a convergent filtered F-isocrystal in a natural way: its analytification $(R^q f_* \mathcal{O}_{X/K})^{\mathrm{an}}$ is a coherent $\mathcal{O}_{Z^{\mathrm{an}}}$-module isomorphic to the relative de Rham cohomology $\mathcal{H}^q_{\mathrm{dR}}(X^{\mathrm{an}}/Z^{\mathrm{an}})$, and the connection is the Gauss-Manin connection ∇. The filtration is given by the Hodge

filtration, induced from the Hodge to de Rham spectral sequence, as explained in [KO68].

3.6 Filtration, Frobenius and monodromy

Let $\mathfrak{X} \to \mathrm{Spec}(\mathcal{O}_K)$ be a proper semistable curve with connected fibers. Suppose that its generic fiber X is smooth and projective, that the irreducible components $C_1, \ldots, C_r$ of the special fiber C are smooth and geometrically connected, and that there are at least two of them. Assume also that the singular points of C are κ-rational ordinary double points.

Denote by X^{an} the rigid analytification of X. We want to describe an admissible covering of X^{an}. Consider the special fiber C of X, and let $\mathfrak{G} = (\mathfrak{V}(\mathfrak{G}), \vec{\mathfrak{E}}(\mathfrak{G}))$ be the (oriented) intersection graph of C: there is one vertex for each irreducible component C_i, and the oriented edges are triples $e = (x, C_i, C_j)$, where x is a singular point of C, and C_i and C_j are the two components on which x lies. We set $o(e) = C_i$ and $t(e) = C_j$, and write $\bar{e}$ for the opposite edge (x, C_j, C_i).

For each vertex $v = C_i$ of $\mathfrak{G}$, let $U_v := \mathrm{red}^{-1}(C_i)$ be the tube associated to it. Here $\mathrm{red}\colon X^{\mathrm{an}} \to C(\bar{k})$ is the reduction map. For each edge $e = (x, C_i, C_j)$, let A_e be the wide open annulus $\mathrm{red}^{-1}(x) = U_{o(e)} \cap U_{t(e)}$, together with the orientation given by e. This gives an admissible covering of X^{an}:

$$X^{\mathrm{an}} = \bigcup_{v \in \mathfrak{V}(\mathfrak{G})} U_v.$$

Define an involution $\overline{(\cdot)}$ on $\vec{\mathfrak{E}}(\mathfrak{G})$ which maps an edge $e = (x, C_i, C_j)$, to

$$\bar{e} := (x, C_j, C_i).$$

Write $\mathfrak{E}(\mathfrak{G})$ for the set of unoriented edges of $\mathfrak{G}$, which can be thought as the set of equivalence classes of $\vec{\mathfrak{E}}(\mathfrak{G})$ by this involution.

Let E be a coherent locally free sheaf of $\mathcal{O}_X$-modules, with a connection

$$\nabla\colon E \to E \otimes_{\mathcal{O}_X} \Omega^1_X,$$

and filtration $\mathrm{Fil}^\bullet E$ by $\mathcal{O}_X$-submodules satisfying Griffiths transversality. That is, such that

$$\nabla(\mathrm{Fil}^i E) \subseteq (\mathrm{Fil}^{i-1} E) \otimes_{\mathcal{O}_X} \Omega^1_X.$$

Assume from now on that E comes from a convergent filtered F-isocrystal $\mathcal{E}$ on the special fiber X of $\mathfrak{X}$. Consider the de Rham cohomology of X with coefficients in E. This is defined as the hyper-cohomology of the complex of sheaves:

The de Rham cohomology of X with coefficients in E, which will be defined in the following subsections, can be given the structure of a filtered (ϕ, N)-module. Moreover, if S is a finite set of points of X and $U{:=}X\backslash S$, one can also give this structure to $H^1_{\mathrm{dR}}(U, E)$. This construction is detailed in [CI03, Section 2], and we recall it in the next two subsections. One needs to assume that the filtered F-isocrystal $\mathcal{E}$ is *regular*, which is a condition on the characteristic polynomials of Frobenius acting on various crystalline cohomology groups. For the precise definition, see [CI03, Definition 2.3].

3.6.1 $H^1_{\mathbf{dR}}(X, E)$ as a filtered (ϕ, N)-module

The (algebraic) de Rham cohomology of X with coefficients in E, denoted by $H^*_{\mathrm{dR}}(X, E)$, is defined to be the hypercohomology of the complex of sheaves of $\mathcal{O}_X$-modules:

$$0 \to E \xrightarrow{\nabla} E \otimes \Omega^1_X \to 0.$$

By rigid-analytic GAGA, this coincides with the rigid-analytic cohomology. We describe explicitly the space $H^1_{\mathrm{dR}}(X, E)$, using the admissible covering described above: an element $x \in H^1_{\mathrm{dR}}(X, E)$ can be represented by a 1-hypercocycle

$$\omega = \left(\{\omega_v\}_{v \in \mathfrak{V}(\mathfrak{G})}; \{f_e\}_{e \in \vec{\mathfrak{E}}(\mathfrak{G})}\right),$$

where the $\omega_v \in (E^{\mathrm{an}} \otimes \Omega^1_{X^{\mathrm{an}}})(U_v)$, and $f_e \in E^{\mathrm{an}}(A_e)$ satisfy

$$\omega_{o(e)}|_{A_e} - \omega_{t(e)}|_{A_e} = \nabla(f_e), \text{ and } f_{\bar{e}} = -f_e.$$

Two such 1-hypercocycles represent the same element $x \in H^1_{\mathrm{dR}}(X, E)$ if their difference is of the form

$$\left(\{\nabla(f_v)\}_{v\in\mathfrak{V}(\mathfrak{G})}; \{f_{o(e)}|_{A_e} - f_{t(e)}|_{A_e}\}_{e\in\vec{\mathfrak{E}}(\mathfrak{G})}\right),$$

for some family $\{f_v\}_{v\in\mathfrak{V}(\mathfrak{G})}$ with $f_v \in E^{\mathrm{an}}(U_v)$.

Assume from now on that the admissible opens U_v and A_e appearing in the covering are acyclic for coherent sheaf cohomology. Consider the maps induced by inclusion:

$$f\colon \coprod_{v\in\mathfrak{V}(\mathfrak{G})} U_v \to X, \qquad\qquad g\colon \coprod_{e\in\vec{\mathfrak{E}}(\mathfrak{G})} A_e \to X.$$

They give an exact sequence of sheaves on X^{an}:

$$0 \to E^{\mathrm{an}} \to f_* f^* E^{\mathrm{an}} \to g_* g^* E^{\mathrm{an}} \to 0,$$

which induces the Mayer-Vietoris long exact sequence:

$$\begin{aligned} 0 &\to H^0_{\mathrm{dR}}(X, E) \to \oplus_{v\in\mathfrak{V}(\mathfrak{G})} H^0_{\mathrm{dR}}(U_v, E^{\mathrm{an}}) \to \oplus_{e\in\mathfrak{E}(\mathfrak{G})} H^0_{\mathrm{dR}}(A_e, E^{\mathrm{an}}) \to \\ &\to H^1_{\mathrm{dR}}(X, E) \to \oplus_{v\in\mathfrak{V}(\mathfrak{G})} H^1_{\mathrm{dR}}(U_v, E^{\mathrm{an}}) \to \oplus_{e\in\mathfrak{E}(\mathfrak{G})} H^1_{\mathrm{dR}}(A_e, E^{\mathrm{an}}) \to \cdots \end{aligned}$$

We extract a short exact sequence

$$0 \to (H^0_{\mathfrak{E}})^- / H^0_{\mathfrak{V}} \xrightarrow{\iota} H^1_{\mathrm{dR}}(X, E) \xrightarrow{\gamma} \ker\left(H^1_{\mathfrak{V}} \to H^1_{\mathfrak{E}}\right) \to 0, \tag{3.2}$$

where

$$H^i_{\mathfrak{E}} = \bigoplus_{e\in\vec{\mathfrak{E}}(\mathfrak{G})} H^i_{\mathrm{dR}}(A_e, E^{\mathrm{an}}), \qquad\qquad H^i_{\mathfrak{V}} = \bigoplus_{v\in\mathfrak{V}(\mathfrak{G})} H^i_{\mathrm{dR}}(U_v, E^{\mathrm{an}}),$$

and the superscript $^-$ indicates the subspace of $H^0_{\mathfrak{E}}$ consisting of elements $\{f_e\}_e$ such that $f_{\bar{e}} = -f_e$ for all $e \in \vec{\mathfrak{E}}(\mathfrak{G})$. The maps ι and γ are given by the following recipe:

1. Let $\{f_e\}_{e\in\vec{\mathfrak{E}}(\mathfrak{G})}$ with $f_e \in H^0_{\mathrm{dR}}(A_e, E^{\mathrm{an}})$ satisfying $f_{\bar{e}} = -f_e$. Then ι sends the class of $\{f_e\}$ to the 1-hypercocycle $(\{0\}_v; \{f_e\})$. Note that this is indeed a hypercocycle, since $\nabla f_e = 0$.

2. Let $(\{\omega_v\}_v; \{f_e\}_e)$ be a 1-hypercocycle representing a class $x \in H^1_{\mathrm{dR}}(X, E)$. Then γ sends x to the class of $\{\omega_v\}_v$ in $\bigoplus_v H^1_{\mathrm{dR}}(U_v, E^{\mathrm{an}})$.

In the following paragraphs we describe the structure as a filtered (ϕ, N)-module of $H^1_{\mathrm{dR}}(X, E)$. The Hodge filtration is defined as

$$\mathrm{Fil}^i H^1_{\mathrm{dR}}(X, E) := \mathrm{img}\left(\mathbb{H}^1(X, \mathrm{Fil}^i E \xrightarrow{\nabla} \mathrm{Fil}^{i-1} E \otimes \Omega^1_X) \to \mathbb{H}^1(X, E \otimes \Omega^\bullet)\right).$$

where the map $\mathbb{H}^1(X, \mathrm{Fil}^i E \xrightarrow{\nabla} \mathrm{Fil}^{i-1} E \otimes \Omega^1_X) \to \mathbb{H}^1(X, E \otimes \Omega^\bullet)$ is induced by functoriality from the inclusion of complexes

$$\begin{tikzcd} \mathrm{Fil}^i E \arrow[r, "\nabla"] \arrow[d, hook] & \mathrm{Fil}^{i-1} E \otimes_{\mathcal{O}_X} \Omega^1_X \arrow[r] \arrow[d, hook] & \cdots \\ E \arrow[r, "\nabla"] & E \otimes_{\mathcal{O}_X} \Omega^1_X \arrow[r] & \cdots \end{tikzcd}$$

Note that this filtration coincides with the one induced from the Hodge to de Rham spectral sequence computing $H^*_{\mathrm{dR}}(X, E)$.

The Frobenius operator is defined by first splitting the exact sequence in Equation (3.2) and defining it in the outer terms. To define the splitting, we will use the Coleman integrals, so fix once and for all a branch of the p-adic logarithm. The map ι admits a retraction P defined as follows: let $x \in H^1_{\mathrm{dR}}(X, E)$ be represented by the 1-hypercocycle $(\{\omega_v\}_v; \{f_e\}_e)$. For any $v \in \mathfrak{V}(\mathfrak{G})$, define F_v to be a Coleman primitive of ω_v, as introduced in Chapter 1. Then the map P assigns to x (the class of) the family $\{g_e\}_{e \in \vec{\mathfrak{E}}(\mathfrak{G})}$, where

$$g_e := f_e - (F_{o(e)}|_{A_e} - F_{t(e)}|_{A_e}).$$

Note that this map is well defined because the integrals F_v are defined up to a rigid horizontal section of $E^{\mathrm{an}}|_{U_v}$.

There is an action of Frobenius on the left and right terms of the exact sequence (3.2). That is, there are lattices inside the space $H^0_{\mathrm{dR}}(A_e, E^{\mathrm{an}})$ and inside $H^1_{\mathrm{dR}}(U_v, E^{\mathrm{an}})$, and respective actions of Frobenius. Concretely, if $e = (x, C_i, C_j)$, then $H^0_{\mathrm{cris}}(x, E^{\mathrm{an}})$ is a K_0-lattice with a natural Frobenius. Also, $H^1_{\mathrm{dR}}(U_v, E^{\mathrm{an}})$ has a natural lattice and action of Frobenius induced from the action on $\mathcal{E}$.

Using the splitting P we obtain a lattice inside $H^1_{\mathrm{dR}}(X, E)$, together with a Frobenius that will be called Φ.

Lastly, we define the monodromy operator N. Associated to each open annulus A_e such that $(E^{\mathrm{an}}|_{A_e})$ has only constant horizontal sections, there is a natural annular residue map $\mathrm{res}_e = \mathrm{res}_{A_e}$ as in Definition 1.21:

$$\mathrm{res}_e \colon H^1_{\mathrm{dR}}(A_e, E^{\mathrm{an}}) \to H^0_{\mathrm{dR}}(A_e, E^{\mathrm{an}}) \cong (E^{\mathrm{an}}|_{A_e})^{\nabla=0}.$$

For a fixed edge $e_0 \in \vec{\mathfrak{E}}(\mathfrak{G})$ there is a natural map $h_{e_0} \colon H^1_{\mathrm{dR}}(X, E) \to H^1_{\mathrm{dR}}(A_{e_0}, E^{\mathrm{an}})$ which sends $(\{\omega_v\}_v; \{f_e\}_e)$ to the class of $\omega_{o(e_0)}|_{A_e}$. Note that this coincides with the class of $\omega_{t(e_0)}|_{A_e}$. The monodromy operator N on $H^1_{\mathrm{dR}}(X, E)$ is defined as:

$$N := \iota \circ (\oplus_e (\mathrm{res}_e \circ h_e)) : H^1_{\mathrm{dR}}(X, E) \to H^1_{\mathrm{dR}}(X, E).$$

The following expected lemma is reassuring.

Lemma 3.29. *The operator N defined above satisfies $N\Phi = p\Phi N$.*

Proof. Let $(\{\omega_v\}_v; \{f_e\}_e)$ be a 1-hypercocycle representing a class $x \in H^1_{\mathrm{dR}}(X, E)$. We first compute $\Phi N x$: note that $N(x)$ is the class of the 1-hyper-cocycle

$$(\{0\}_v; \{\mathrm{res}_e\, \omega_{o(e)}\}_e).$$

Then γ sends this element to 0, so Φ acts as $\iota \circ \phi \circ P$. That is, $\Phi N(x)$ is the class of the 1-hypercocycle $(\{0\}_v; \{\phi(\mathrm{res}_e\, \omega_{o(e)})\})$.

Next we compute $N\Phi x$. Note that $N \circ \iota = 0$, so that this is:

$$N\Phi x = (N \circ t \circ \phi \circ \gamma)(x),$$

where t is the right inverse to γ corresponding to the retraction P. Then $\gamma(x)$ is the class of $\{\omega_v\}_v$, and ϕ acts on it component-wise, to get the class of $\{\phi\omega_v\}_v$. The map t sends this to a 1-hypercocycle of the form $(\{\phi\omega_v\}_v; \{f'_e\}_e)$, for some family $\{f'_e\}_e$ which is irrelevant to us. Lastly, N acts on it by taking the residues of $\phi\omega_v$, to get the class of $(\{0\}_v; \{\mathrm{res}_e\, \phi\omega_{o(e)}\}_e)$.

The lemma follows now from the following claim:

Claim.

$$(\mathrm{res}_e)\circ\phi = p(\phi\circ\mathrm{res}_e).$$

Proof. Let $\omega = \sum a_n t^n dt$ be a local expression for a differential form, where t is a local parameter on the annulus corresponding to e. Then:

$$\phi\omega = \sum \phi(a_n)t^{pn}pt^{p-1}dt = p\sum \phi(a_n)t^{p(n+1)-1}dt.$$

The coefficient of $t^{-1}dt$ in this expression is precisely $p\phi(a_{-1}) = p\phi(\mathrm{res}_e\,\omega)$. $\square$

$\square$

3.6.2 $H^1_{\mathbf{dR}}(U,E)$ as a filtered (ϕ,N)-module

Let S be a finite set of K-rational points on X which are smooth (when considered as points on $\mathfrak{X}$), and which specialize to pairwise different smooth points on C.

Let $U = X\setminus S$. One can define, in a similar way as in the previous section, a structure of a filtered (ϕ,N)-module on $H^1_{\mathrm{dR}}(U,E)$. The monodromy operator is defined as in the previous subsection. To define the Frobenius, one needs to work with logarithmic isocrystals. There is again an exact sequence

$$0\to (H^0_{\mathfrak{E}})^-/H^0_{\mathfrak{V}} \xrightarrow{\iota} H^1_{\mathrm{dR}}(U,E) \xrightarrow{\gamma} \ker\left(H^1_{\mathfrak{V}}\to H^1_{\mathfrak{E}}\right)\to 0,$$

where this time

$$H^i_{\mathfrak{E}} = \bigoplus_{e\in\vec{\mathfrak{E}}(\mathfrak{G})} H^i_{\mathrm{dR}}(A_e,E^{\mathrm{an}}), \qquad H^i_{\mathfrak{V}} = \bigoplus_{v\in\mathfrak{V}(\mathfrak{G})} H^i_{\mathrm{dR}}(U_v\setminus S,E^{\mathrm{an}}).$$

The left-most term is the same as before, because the zeroth cohomology does not change by removing a finite set of points. So to define the Frobenius on $H^1_{\mathrm{dR}}(U,E)$ one has to define it on the right-most term. This is done in [CI03, Section 5], where the de Rham cohomology $H^1_{\mathrm{dR}}(U_v\setminus S,E^{\mathrm{an}})$ is described in terms of the log-crystalline cohomology with coefficients in j^*E of the component $C_i = v$ of

C, where j is the canonical morphism of formal log-schemes $j\colon (\widehat{\mathfrak{X}}, \text{log structure}) \to (\widehat{\mathfrak{X}}, \text{trivial})$.

The Gysin sequence

$$0 \longrightarrow H^1_{\mathrm{dR}}(X,E) \longrightarrow H^1_{\mathrm{dR}}(U,E) \xrightarrow{\oplus \operatorname{res}_x} \bigoplus_{x\in S} \mathcal{E}_x[1]$$

becomes in this way an exact sequence of filtered (ϕ, N)-modules.

Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a smooth proper morphism, and let Y be the generic fiber of $\mathfrak{Y}$. The relative de Rham cohomology

$$\mathcal{H}^q_{\mathrm{dR}}(Y/X) := R^q f_* \mathcal{O}_{\widehat{\mathfrak{Y}}/K}$$

can be given the structure of a convergent filtered F-isocrystal, as in Example 2. Consider the $G_{\mathbb{Q}}$-representation

$$H^1_{\mathrm{et}}(\overline{X}, R^q f_* \mathbb{Q}_p),$$

and the filtered (ϕ, N)-module

$$H^1_{\mathrm{dR}}\left(X, \mathcal{H}^q_{\mathrm{dR}}(Y/X)\right)$$

defined above. The following result relates these two objects. Its proof can be found in [CI03, Theorem 7.5].

Theorem 3.30 (Faltings, Coleman-Iovita)**.** *Using the previous notations:*

1. *The representation $H^1_{et}(\overline{X}, R^q f_* \mathbb{Q}_p)$ is semistable, and there is a canonical isomorphism of filtered (ϕ, N)-modules*

$$\mathrm{D}_{st}\left(H^1_{et}(\overline{X}, R^q f_* \mathbb{Q}_p)\right) \cong H^1_{dR}\left(X, \mathcal{H}^q_{dR}(Y/X)\right).$$

2. *More generally, let S be a finite set of smooth sections of $f\colon \mathfrak{X} \to \operatorname{Spec}(\mathcal{O}_K)$, which specialize to pairwise different (smooth) points on C, and let $U = X \setminus S$, $\overline{U} = U \otimes_K \overline{K}$, and $Y_{\overline{x}}$ be the geometric fiber of $f\colon Y \to X$ over $x \in S$. Then there is an exact sequence of semistable Galois representations*

$$0 \to H^1_{et}(\overline{X}, R^q f_* \mathbb{Q}_p) \to H^1_{et}(\overline{U}, R^q f_* \mathbb{Q}_p) \to \bigoplus_{s\in S} H^q_{et}(Y_{\overline{x}}, \mathbb{Q}_p(-1)),$$

which becomes isomorphic to the sequence

$$0 \to H^1_{dR}(X,E) \to H^1_{dR}(U,E) \to \bigoplus_{x\in S} \mathcal{E}_x[1]$$

after applying the functor D_{st} *(and setting* $\mathcal{E} = \mathcal{H}^q_{\mathrm{dR}}(Y/X)$*).*

3.7 The filtered (ϕ, N)-module $H^1_{\mathbf{dR}}(X_\Gamma, E(V))$

We want to specialize the constructions made in the previous sections to the situation in our work. We will assume that the curve X is a certain quotient of the p-adic upper-half plane, and we will restrict also the class of filtered convergent F-isocrystals that we consider. Let V be an object of $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{GL}_2 \times \mathrm{GL}_2)$. In [IS03, Section 4] the authors associate to V a convergent filtered F-isocrystal on the canonical formal $\mathbb{Z}_p^{\mathrm{ur}}$-model of the upper-half plane $\widehat{\mathcal{H}}$, which is denoted $\mathcal{E}(V)$. Also, for every $\mathbb{Q}_{p^2}$-rational point $\Psi \in \mathrm{Hom}(\mathbb{Q}_{p^2}, M_2(\mathbb{Q}_p))$ of $\widehat{\mathcal{H}}$ they compute the stalk $\mathcal{E}(V)_\Psi$ as a filtered (ϕ, N)-module $V_\Psi \in \mathrm{MF}_{\mathbb{Q}_p^{\mathrm{ur}}}(\phi, N)$. The assignment $V \mapsto \mathcal{E}(V)$ is an exact tensor functor.

The previous construction can be descended to give isocrystals on Mumford curves: if Γ is a discrete cocompact subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$, let X_Γ be the associated Mumford curve over $\mathbb{Q}_p^{\mathrm{ur}}$, so that $X_\Gamma^{\mathrm{an}} = \Gamma\backslash\mathcal{H}_p$. Denote the new filtered isocrystal on X_Γ by the same symbol $\mathcal{E}(V)$ as well.

Let $E(V)$ be the coherent locally free $\mathcal{O}_{X_\Gamma}$-module with connection and filtration corresponding to $\mathcal{E}(V)$, so that $\mathcal{E}(V) = E(V)^{\mathrm{an}}$.

In [IS03] the authors give a concrete description of the filtered (ϕ, N)-module $H^1_{\mathrm{dR}}(X_\Gamma, E(V))$ and, if $U \subseteq X_\Gamma$ is an open subscheme as before, also of the filtered (ϕ, N)-module $H^1_{\mathrm{dR}}(U, E(V))$. This is possible because both the curve X_Γ and the coefficients $E(V)$ are known explicitly. We will assume that Γ is torsion free. We can reduce to this situation as follows: choose $\Gamma' \subset \Gamma$ a free normal subgroup of finite index. The group Γ/Γ' acts on the filtered (ϕ, N)-modules $H^1_{\mathrm{dR}}(X_\Gamma, E(V))$

and $H^1_{\mathrm{dR}}(U, E(V))$ as automorphisms preserving the operators and the filtration. Hence it induces a structure of filtered (ϕ, N)-module on

$$H^1_{\mathrm{dR}}(X_\Gamma, E(V)) := H^1_{\mathrm{dR}}(X_{\Gamma'}, E(V))^{\Gamma/\Gamma'},$$

and similarly for $H^1_{\mathrm{dR}}(U, E(V))$.

3.7.1 The structure of $H^1_{\mathbf{dR}}(X_\Gamma, E(V))$

The fact that $\mathcal{H}_p$ is a Stein space in the rigid-analytic sense allows for the computation of $H^1_{\mathrm{dR}}(X_\Gamma, E(V))$ as group hyper-cohomology, via the Leray spectral sequence. More precisely, the K-vector space $H^1_{\mathrm{dR}}(X_\Gamma, E(V))$ can be computed as the first group hyper-cohomology:

$$H^1_{\mathrm{dR}}(X, E(V)) = \mathbb{H}^1(\Gamma, \Omega^\bullet \otimes V),$$

where $\Omega^\bullet$ is the de Rham complex

$$\Omega^\bullet : \qquad 0 \to \mathcal{O}_{\mathcal{H}_p}(\mathcal{H}_p) \to \Omega^1_{\mathcal{H}_p}(\mathcal{H}_p) \to 0.$$

Concretely, the elements in $H^1_{\mathrm{dR}}(X, E(V))$ are represented by pairs (ω, f_γ), where ω belongs to $\Omega^1(\mathcal{H}_p) \otimes V$ and f_γ is a $\mathcal{O}_{\mathcal{H}_p}(\mathcal{H}_p) \otimes V$-valued 1-cocycle for Γ. They are required to satisfy the relation

$$\gamma\omega - \omega = df_\gamma, \quad \text{for all } \gamma \in \Gamma.$$

Recall the definition of the Γ-representations $C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})$ and $C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})$ and of their harmonic sub-representations $C^0_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})$ and $C^1_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})$ attached to the Bruhat-Tits tree $\mathcal{T}$ as introduced in Definition 3.1. We define a map $\varepsilon\colon C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma \to H^1(\Gamma, V_{\mathbb{Q}_p^{\mathrm{ur}}})$, as follows: given a 1-cocycle $f \in C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma$, the element $\varepsilon(f)$ is defined as the cohomology class of the 1-cocyle

$$\gamma \mapsto \gamma F(\gamma^{-1}(\star)),$$

where $\star \in \mathfrak{V}(\mathcal{T})$ is a choice of a vertex of $\mathcal{T}$, and $F \in C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})$ satisfies $\partial F = f$ and $F(\star) = 0$. One easily checks that this definition does not depend on the choice of the vertex $\star$.

Proposition 3.31. *The map ε induces an isomorphism:*

$$\varepsilon\colon C^1(V_{\mathbb{Q}_p^{ur}})^\Gamma / C^0(V_{\mathbb{Q}_p^{ur}})^\Gamma \to H^1(\Gamma, V_{\mathbb{Q}_p^{ur}}).$$

Proof. Consider the short exact sequence

$$0 \to V_{\mathbb{Q}_p^{\mathrm{ur}}} \to C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}}) \xrightarrow{\partial} C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}}) \to 0.$$

The map ε is the map induced from the connecting homomorphism δ in the Γ-cohomology long exact sequence:

$$0 \to C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma \xrightarrow{\partial} C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma \xrightarrow{\delta} H^1(\Gamma, V_{\mathbb{Q}_p^{\mathrm{ur}}}) \to H^1(\Gamma, C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})) \to \cdots$$

Since Γ is torsion-free, $H^1(\Gamma, C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})) = 0$, and the result follows. $\square$

Note that there is a canonical isomorphism

$$C^1_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma \cong C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma / C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma,$$

and we will identify these two spaces from now on. Let $\gamma \mapsto f_\gamma$ be a 1-cocycle on Γ. The group $C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})$ is Γ-acyclic, so that there is a 0-harmonic cocycle $F \in C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})$ satisfying, for all $\gamma \in \Gamma$,

$$j(f_\gamma) = \gamma F - F.$$

Consider then $\partial(F) \in C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})$. It is fixed by Γ, since:

$$(\gamma\partial(F)) - \partial(F) = \partial(\gamma F - F) = \partial(j(f_\gamma)) = 0.$$

Then the class of the 1-hypercocycle given by $(\{0\}_v; \{F(o(e)) - F(t(e))\}_e)$ is an element of $H^1_{\mathrm{dR}}(X_\Gamma, E(V))$. We thus obtain an injection:

$$\iota\colon H^1(\Gamma, V_{\mathbb{Q}_p^{\mathrm{ur}}}) \to H^1_{\mathrm{dR}}(X_\Gamma, E(V)).$$

Next we construct a map $I\colon H^1_{\mathrm{dR}}(X_\Gamma, E(V)) \to C^1_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma$ which is due to Schneider. It is called “Schneider integration” in [IS03], [dS89] and [dS06], and it is induced from the map $\Omega^{\mathrm{II}}_{X_\Gamma^{\mathrm{an}}} \otimes_{\mathcal{O}_{X_\Gamma}} V \to C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})$:

$$\omega \mapsto (e \mapsto \mathrm{res}_e(\omega)).$$

Because of the residue theorem, the image lies actually in $C^1_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma$.

Lemma 3.32 (de Shalit)**.** *Suppose that* Γ *is arithmetic. Then the sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\Gamma, V_{\mathbb{Q}_p^{ur}}) & \xrightarrow{\iota} & H^1_{dR}(X, E(V)) & \xrightarrow{I} & C^1_{har}(V_{\mathbb{Q}_p^{ur}})^\Gamma & \longrightarrow & 0 \\ & & [f_\gamma] & \longmapsto & [(0, f_\gamma)] & & & & \\ & & & & [(\omega, f_\gamma)] & \longmapsto & (e \mapsto \mathrm{res}_e(\omega)) & & \end{array} \tag{3.3}$$

is exact.

There is a retraction P of ι, given by Coleman integration. This assigns to (ω, f_γ) the 1-cocycle

$$\gamma \mapsto f_\gamma + \gamma F_\omega - F_\omega,$$

where F_ω is a Coleman primitive for ω as in Theorem 1.42. Note that the $V_{\mathbb{Q}_p^{\mathrm{ur}}}$-valued function $\gamma F_\omega - F_\omega$ is constant, so that we can think of it as a well-defined element of $V_{\mathbb{Q}_p^{\mathrm{ur}}}$.

The splitting P thus defines actions of Frobenius on the left and right terms of the exact sequence (3.3), as follows: there is a natural action ϕ_1 of Frobenius on $H^1(\Gamma, V_{\mathbb{Q}_p^{\mathrm{ur}}})$. Define an action ϕ_2 on $C^1_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma$ as $\phi_2{:=}p(\varepsilon^{-1} \circ \phi_1 \circ \varepsilon)$, so that the following equality holds:

$$\varepsilon\phi_2 = p\phi_1\varepsilon.$$

We have now all the maps needed in the definition of the Frobenius and monodromy operators. Define first N to be the composition $\iota \circ (-\varepsilon) \circ I$. Since $I \circ \iota = 0$ it follows that $N^2 = 0$. Actually, Lemma 3.32 implies that $\ker N = \operatorname{img} N$. Let T be the right-inverse to I corresponding to P:

$$\begin{tikzcd} 0 \arrow[r] & H^1(\Gamma, V_{\mathbb{Q}_p^{\mathrm{ur}}}) \arrow[r, "\iota"] \arrow[loop above, "\phi_1"] & H^1_{\mathrm{dR}}(X_\Gamma, E(V)) \arrow[r, "I"] \arrow[l, bend right, dashed, "P"'] & C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma / C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma \arrow[r] \arrow[l, bend right, dashed, "T"'] \arrow[loop above, "\phi_2"] & 0. \end{tikzcd}$$

Define the Frobenius operator Φ on $H^1_{\mathrm{dR}}(X_\Gamma, E(V))$ as:

$$\Phi(\omega){:=}\iota\phi_1(P\omega) + T(\phi_2(I\omega)).$$

We check that:

$$
\begin{aligned}
N\Phi(\omega) &= N\iota\phi_1 P\omega + NT\phi_2 I\omega \\
&= NT\phi_2 I\omega && \text{(since } N\iota = 0\text{)} \\
&= -\iota\varepsilon IT\phi_2 I\omega && \text{(definition of } N\text{)} \\
&= -\iota\varepsilon\phi_2 I\omega && \text{(since } T \text{ is a splitting of } I\text{)} \\
&= -p\iota\phi_1\varepsilon I\omega && (\varepsilon\phi_2 = p\phi_1\varepsilon) \\
&= p\iota\phi_1 P\iota(-\varepsilon)I\omega && \text{(since } P \text{ is a retraction of } \iota\text{)} \\
&= p\Phi(N\omega) && \text{(since } IN = 0\text{).}
\end{aligned}
$$

Remark 3.33. The definition of Φ has been made so that it is compatible with the maps P and ι, and such that it satisfies $N\Phi = p\Phi N$. There is a unique such definition, for if Φ_1 and Φ_2 satisfied these two conditions, then their difference f would satisfy:

1. $f\iota = 0 = Pf$,
2. $Nf = pfN = pf\iota(-\varepsilon)I = 0$.

But the second condition implies that $If = 0$, because $\iota \circ (-\varepsilon)$ is injective. Now just write

$$\mathrm{Id}_{H^1_{\mathrm{dR}}(X_\Gamma, E(V))} = \iota P + TI,$$

so $f = \iota Pf + TIf = 0 + 0 = 0$.

3.7.2 The structure of $H^1_{\mathbf{dR}}(U, E(V))$

Let S be a finite set of points of X_Γ, and let $U = X_\Gamma \setminus S$ be the open subscheme obtained by removing the points in S. The space $H^1_{\mathrm{dR}}(U, E(V))$ is identified with the space of V-valued differential forms on X_Γ^{an} which are of the second kind when restricted to U. The monodromy is defined in the same way as before. The Frobenius is defined so that the Gysin sequence

$$0 \longrightarrow H^1_{\mathrm{dR}}(X_\Gamma, E(V)) \longrightarrow H^1_{\mathrm{dR}}(U, E(V)) \xrightarrow{\oplus_{x\in S}\mathrm{res}_x} \bigoplus_{x\in S} V_{\Psi_x}[1] \tag{3.4}$$

is a sequence of (ϕ, N)-modules, and such that P is compatible with the Frobenii.

3.7.3 Poincaré duality on $H^1_{\mathbf{dR}}(X_\Gamma, E(V))$

Let V be a finite-dimensional representation of Γ over K endowed with a Γ-invariant perfect pairing $\langle \cdot, \cdot, \rangle_V$. We first describe a pairing $\langle \cdot, \cdot \rangle_\Gamma$:

$$\langle \cdot, \cdot \rangle_\Gamma \colon C^1_{\text{har}}(V)^\Gamma \otimes H^1(\Gamma, V) \to K,$$

given as follows: choose a free subgroup $\Gamma' \subset \Gamma$ of finite index, and let $\mathfrak{F}$ be a good fundamental domain for Γ' as in Definition 1.15. Let $b_1, \ldots, b_g, c_1, \ldots, c_g$ be the free edges for $\mathfrak{F}$. For and $f \in C^1_{\text{har}}(V)^\Gamma$ and $[z] \in H^1(\Gamma, V)$, the pairing is given by the formula:

$$\langle [z], f \rangle_\Gamma = \frac{1}{[\Gamma \colon \Gamma']} \sum_{i=1}^{g} \langle z(\gamma_i), f(c_i) \rangle_V.$$

The previous pairing induces a pairing on $H^1_{\text{dR}}(X_\Gamma, E(V))$, and we are interested in a formula for it, which we now proceed to describe. Let $x, y \in H^1_{\text{dR}}(X_\Gamma, E(V))$. E. de Shalit computed first a formula for this pairing in [dS89] and [dS19], and Iovita-Spieß proved it in a more conceptual way which allowed for a generalization, in [IS03]. They obtained the equality:

$$\langle x, y \rangle_{X_\Gamma} = \langle P(x), I(y) \rangle_\Gamma - \langle I(x), P(y) \rangle_\Gamma. \tag{3.5}$$

In order to compute the pairing we will later need the following result:

Proposition 3.34. *Let $f \in C^1_{har}(V)^\Gamma$ and $[z] \in H^1(\Gamma, V)$. Let e be an edge of $\mathcal{T}$ and let $\gamma \in \Gamma$. Then:*

$$\langle z(\gamma), \operatorname{res}_e \omega \rangle_V = -\langle \operatorname{res}_e z(\gamma), F_\omega \rangle_V.$$

3.7.4 Pairings between $H^1_{\mathbf{dR}}(U, E(V))$ and $H^1_{\mathbf{dR,c}}(U, E(V))$

In this subsection we make explicit some of the constructions carried out in [IS03, Appendix].

Let $U = X \setminus \{x\}$, where x is a closed point of X defined over the base field K. Write $j\colon U \to X = X_\Gamma$ for the canonical inclusion. Let z be a lift of x to $\mathcal{H}_p(K)$, taken inside a good fundamental domain $\mathcal{F}$. We assume that the stabilizer of z under the action of Γ is trivial. Let $\mathrm{Ind}^\Gamma(V)$ be the Γ-representation given by $\mathrm{Maps}(\Gamma, V)$, with Γ-action:

$$(\gamma \cdot f)(\tau) {:=} \gamma f(\gamma^{-1}\tau).$$

Let $\mathrm{ad}\colon V \to \mathrm{Ind}^\Gamma(V)$ be defined as the constant map: $\mathrm{ad}(v)(\tau){:=}v$. Consider the complex $\mathcal{K}^\bullet(V)$, concentrated on degrees 0 and 1, defined as:

$$\mathcal{K}^\bullet(V): \qquad V \xrightarrow{\mathrm{ad}} \mathrm{Ind}^\Gamma(V).$$

Consider also the complex $C^\bullet(V)$ defined as follows:

$$C^\bullet(V): \qquad \mathcal{O}_{\mathcal{H}_p}(\mathcal{H}_p) \otimes V \xrightarrow{(d,\mathrm{ev}_z)} \Omega^1(\mathcal{H}_p) \otimes V \oplus \mathrm{Ind}^\Gamma(V).$$

Definition 3.35. The *cohomology with compact support* on U with coefficients in $E(V)$ is the hypercohomology group:

$$H^1_{\mathrm{dR,c}}(U, E(V)) \cong \mathbb{H}^1(\Gamma, C^\bullet(V)).$$

Fact. *The inclusion j induces natural maps:*

$$j_*\colon H^1_{dR,c}(U, E(V)) \to H^1_{dR}(X, E(V))$$

and

$$j^*\colon H^1_{dR}(X, E(V)) \to H^1_{dR}(U, E(V)).$$

There is a pairing $\langle \cdot, \cdot \rangle_U$, on:

$$H^1_{dR,c}(U, E(V)) \times H^1_{dR}(U, E(V)) \to K,$$

induced from the cup-product. It satisfies:

$$\langle j_* y_1, y_2 \rangle_X = \langle y_1, j^* y_2 \rangle_U.$$

The exact triangle:

$$\mathcal{K}^\bullet(V) \to C^\bullet(V) \to C^1_{\mathrm{har}}(V)[-1] \to \mathcal{K}^\bullet(V)[1]$$

induces a short exact sequence:

$$0 \to \mathbb{H}^1(\Gamma, \mathcal{K}^\bullet(V)) \xrightarrow{\iota_{U,c}} H^1_{\mathrm{dR,c}}(U, E(V)) \xrightarrow{I_{U,c}} C^1_{\mathrm{har}}(V)^\Gamma \to 0.$$

The splitting $P_{U,c}$ is defined as follows. Fix a branch of the p-adic logarithm. Let $\mathcal{F}(V)$ be the subspace of those V-valued locally-analytic functions on $\mathcal{H}_p$ which are primitives of elements of $\Omega^1(\mathcal{H}_p) \otimes V$. There is an exact sequence:

$$0 \to V \to \mathcal{F}(V) \xrightarrow{d} \Omega^1(\mathcal{H}_p) \otimes V \to 0,$$

and one immediately checks that this implies that the complex

$$\mathcal{F}(V) \to \Omega^1(\mathcal{H}_p) \otimes V \oplus \mathrm{Ind}^\Gamma(V)$$

is quasi-isomorphic to $\mathcal{K}^\bullet$.

$$P_{U,c}\colon H^1_{\mathrm{dR,c}}(U, E(V)) = \mathbb{H}^1(\Gamma, C^\bullet) \to \mathbb{H}^1(\Gamma, \mathcal{K}^\bullet).$$

We define now a Γ-module $C_U(V)$. There is a surjective map

$$\delta\colon C^1(V) \to C^0(V),$$

defined by:

$$\delta(f)(v) := \sum_{o(e)=v} f(e).$$

Let $v_0 := \mathrm{red}(z)$, and define $\chi\colon \mathrm{Ind}^\Gamma(V) \to C^0(V)$ by:

$$\chi(f)(v) := \begin{cases} f(\gamma) & \text{if } v = \gamma v_0, \text{ for some } \gamma \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

The Γ-module $C_U(\Gamma)$ is defined to be the kernel of the map:

$$C^1(V) \bigoplus \mathrm{Ind}^\Gamma(V) \to C^0(V),$$

mapping $(f, g) \mapsto \delta(f) - \chi(g)$. The map $I_U\colon H^1_{\mathrm{dR}}(U, E(V)) \to C_U(V)^\Gamma$ is naturally induced from the map $\tilde{I}_U\colon \Omega^1(\mathcal{H}_p)(\log(|z|)) \otimes V \to C_U(V)$, defined by:

$$\tilde{I}_U(\omega)(e, \gamma) := (\mathrm{res}_e(\omega), \mathrm{res}_{\gamma(z)}(\omega)).$$

Also, the map ι_U is induced from the natural inclusion

$$V \to \mathcal{O}_{\mathcal{H}_p}(\mathcal{H}_p)(\log(|z|)) \otimes V.$$

Finally, we define a splitting P_U is defined by the same formula as the one defining P. We end this section by recalling the explicit description of the pairing

$$\langle \cdot, \cdot \rangle_{\Gamma,U}\colon H^1(\Gamma, \mathcal{K}^\bullet(V)) \otimes C_U(V)^\Gamma \to K.$$

Proposition 3.36 (Iovita-Spieß)**.** *Let $x \in H^1(\Gamma, \mathcal{K}^\bullet(V))$ be represented by (ζ, f), such that*

$$ad \circ \zeta = \partial(f),$$

with $\zeta \in Z^1(\Gamma, V)$ a one-cocycle and $f \in \mathrm{Ind}^\Gamma(V)$ satisfying

$$(\partial f)(\gamma) = \gamma f - f.$$

Let $(g, g') \in C^1(V) \oplus \mathrm{Ind}^\Gamma(V)$ be an element in $C_U(V)^\Gamma$, so that $\delta(g) = \chi(g')$. Choose a free subgroup $\Gamma' \subset \Gamma$ of finite index, and let $\mathfrak{F}$ be a good fundamental domain for Γ' as in Definition 1.15. Let $b_1, \ldots, b_g, c_1, \ldots, c_g$ be the free edges for $\mathfrak{F}$. Then:

$$\langle [(\zeta, f)], (g, g') \rangle_{\Gamma,U} = \frac{1}{[\Gamma\colon \Gamma']} \sum_{i=1}^{g} \langle \zeta(\gamma_i), g(c_i) \rangle + \langle f(1), g'(1) \rangle.$$

Proof. See [IS03, Appendix]. $\square$

We have constructed a commutative diagram with exact split rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}^1(\Gamma, \mathcal{K}^\bullet(V)) \arrow[r, "\iota_{U,c}"] \arrow[d, "q_*"] & H^1_{\mathrm{dR,c}}(U, E(V)) \arrow[r, "I_{U,c}"] \arrow[d, "j_*"] \arrow[l, dashed, bend right, "P_{U,c}"'] & C^1_{\mathrm{har}}(V)^\Gamma \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^1(\Gamma, V) \arrow[r, "\iota"] \arrow[d, equal] & H^1_{\mathrm{dR}}(X, E(V)) \arrow[r, "I"] \arrow[d, "j^*"] \arrow[l, dashed, bend right, "P"'] & C^1_{\mathrm{har}}(V)^\Gamma \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & H^1(\Gamma, V) \arrow[r, "\iota_U"] & H^1_{\mathrm{dR}}(U, E(V)) \arrow[r, "I_U"] \arrow[l, dashed, bend right, "P_U"'] & C_U(V)^\Gamma \arrow[r] & 0.
\end{tikzcd} \tag{3.6}$$

Here the bent arrows mean splittings of the corresponding maps, and the vertical dotted arrow means the natural induced map on the quotient.

3.8 A special case of interest

In [IS03] the authors apply the previous constructions to a filtered isocrystal on $\mathcal{H}_p$ denoted by $\mathcal{E}(M_2)$. It is shown in [CI03, Lemma 5.10] that $\mathcal{E}(M_2)$ is regular, and therefore one can define a structure of a filtered (ϕ, N)-module on its cohomology groups. Here we make the construction explicit. Consider first the $\mathbb{Q}_p$-vector space of 2×2 matrices M_2. Define two commuting left actions of GL_2 on M_2 by:

$$\rho_1(A)(B){:=}AB \qquad\qquad \rho_2(A)(B){:=}B\overline{A},$$

for $A \in \mathrm{GL}_2$ and $B \in M_2$. The matrix $\overline{A}$ is such that $A\overline{A} = \det A$. This gives a representation:

$$(M_2, \rho_1, \rho_2) \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{GL}_2 \times \mathrm{GL}_2).$$

The isocrystal $\mathcal{E}(M_2)$ is constructed as an isocrystal on the canonical formal model $\widehat{\mathcal{H}}$ over $\mathbb{Z}_p^{\mathrm{ur}}$ of $\mathcal{H}_p$. We are also interested in its fibers over $\mathbb{Q}_{p^2}$-rational points $\Psi \in \mathrm{Hom}(\mathbb{Q}_{p^2}, M_2(\mathbb{Q}_p))$. Let $\mathcal{O}_{\widehat{\mathcal{H}}}$ be the isocrystal attached to $\widehat{\mathcal{H}}$. As an isocrystal $\mathcal{E}(M_2)$ is:

$$M_2(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\widehat{\mathcal{H}}}.$$

We need to the define the Frobenius and filtration. First, let ϕ be the action on $M_2(\mathbb{Q}_p)$:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -p \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -b & -pa \\ -d & -pc \end{smallmatrix}\right).$$

On $\mathcal{O}_{\widehat{\mathcal{H}}}$ there is a Frobenius action $\Phi_{\mathcal{O}_{\widehat{\mathcal{H}}}}$ as well. We define the Frobenius on the tensor product through these two actions.

The next lemma helps us in describing the filtration for $\mathcal{E}(M_2)$.

Lemma 3.37. *The map $(M_2, \rho_1) \to V_1 \oplus V_1$ defined by:*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (av_x + cv_1, bv_x + dv_1)$$

is a GL_2*-equivariant isomorphism. Here* $\{v_x, v_1\}$ *is the dual basis to* $\{x, 1\}$*.*

The filtration is given in degree 1 by:

$$F^1 M_2 = \left\{ \left(\begin{smallmatrix} zf(z) & zg(z) \\ f(z) & g(z) \end{smallmatrix} \right) \mid f, g \in \mathcal{O}_{\mathcal{H}_p} \right\}.$$

3.8.1 The stalks

Let $\Psi \in \mathrm{Hom}(\mathbb{Q}_{p^2}, M_2(\mathbb{Q}_p))$. As a $\mathbb{Q}_p^{\mathrm{ur}}$-vector space, the stalk $\mathcal{E}(M_2)_\Psi$ is just $M_2(\mathbb{Q}_p^{\mathrm{ur}})$. The Frobenius acts by ϕ on M_2 and by σ on $\mathbb{Q}_p^{\mathrm{ur}}$. That is:

$$\Phi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} -\sigma(b) & -p\sigma(a) \\ -\sigma(d) & -p\sigma(c) \end{smallmatrix}\right).$$

The filtration is defined in terms of ρ_1: for each j, let V_j be defined as

$$V_j := \{A \in M_2(\mathbb{Q}_p^{\mathrm{ur}}) \mid \Psi(x)A = x^j \sigma(x)^{1-j} A, \forall x \in \mathbb{Q}_{p^2}\}.$$

Define then:

$$\mathrm{Fil}^i_\Psi M_2(\mathbb{Q}_p^{\mathrm{ur}}) := \bigoplus_{j \geq i} V_j.$$

This is an exhaustive and separated filtration on $M_2(\mathbb{Q}_p^{\mathrm{ur}})$. The monodromy is trivial.

3.9 The p-adic Abel-Jacobi map

Let X be a smooth projective variety over a field K. Suppose given a closed immersion $i \colon Z \hookrightarrow X$ and an open immersion $j \colon U \hookrightarrow X$, such that X is the disjoint union of $i(Z)$ and $j(U)$. Let $\mathcal{F}$ be a sheaf on the étale site of X. Then $i_* i^! \mathcal{F}$ is the largest subsheaf of $\mathcal{F}$ which is zero outside Z.

The group

$$\Gamma(X, i_* i^! \mathcal{F}) = \Gamma(Z, i^! \mathcal{F}) = \ker(\mathcal{F}(X) \to \mathcal{F}(U))$$

is called the group of *sections of $\mathcal{F}$ with support on Z*. The functor which maps a sheaf $\mathcal{F}$ to $\Gamma(Z, i^!\mathcal{F})$ is left-exact, so it makes sense to consider its right-derived functors.

Definition 3.38. The functors

$$H^k_{|Z|}(X, \mathcal{F}) := \mathcal{F} \mapsto R^k\Gamma(Z, i^!\mathcal{F})$$

are called the *étale cohomology groups of $\mathcal{F}$ with support on Z*.

Proposition 3.39. *Let $\mathcal{F}$ be a sheaf on the étale site of X. There is a long exact sequence*

$$\begin{gathered} 0 \to (i^!\mathcal{F})(Z) \to F(X) \to F(U) \to \cdots \\ \cdots \to H^k_{et}(X, \mathcal{F}) \to H^k_{et}(U, \mathcal{F}) \to H^{k+1}_{|Z|}(X, \mathcal{F}) \to \cdots \end{gathered}$$

Proof. This follows immediately by applying the cohomology functor to the exact sequence of sheaves on the étale site of X:

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0.$$

□

Let $\overline{K}$ be the separable closure of K, and let $\overline{X} = X \otimes_K \overline{K}$ be the base change of X to $\overline{K}$. Assume also that $\overline{Z}$ is smooth over $\overline{K}$. Let c be the codimension of Z in X. That is, each of the irreducible components of $\overline{Z}$ is of codimension c inside the corresponding component of $\overline{X}$.

Let $\mathcal{F}$ be a locally constant torsion sheaf on $\overline{X}$, such that its torsion is coprime to $\mathrm{char}(K)$. In our applications, $\mathrm{char}\, K = 0$, so this condition will be void. As a special case of cohomological purity, (see [Mil80] VI.5.1), we have:

Theorem 3.40. *For every $k \in \mathbb{Z}$ there is a canonical isomorphism*

$$H^k_{|\overline{Z}|}(\overline{X}, \mathcal{F}) \cong H^{k-2c}_{et}(\overline{Z}, i^*\mathcal{F}(-c)).$$

Corollary 3.41. *For $0 \leq k \leq 2c-2$, $H^k_{et}(\overline{X},\mathcal{F}) \cong H^k_{et}(\overline{U},\mathcal{F})$. Moreover, if $d = \dim(X)$, there is a long exact sequence:*

$$\begin{aligned} 0 \to H^{2c-1}_{et}(\overline{X},\mathcal{F}) \to H^{2c-1}_{et}(\overline{U},\mathcal{F}) \to H^{2c}_{|\overline{Z}|}\left(\overline{X},\mathcal{F}\right) \xrightarrow{i_*} H^{2c}_{et}(\overline{X},\mathcal{F}) \to \cdots \\ \cdots \to H^{2d}_{|\overline{Z}|}\left(\overline{X},\mathcal{F}\right) \to H^{2d}_{et}(\overline{X},\mathcal{F}) \to H^{2d}_{et}\left(\overline{U},\mathcal{F}\right) \to 0. \end{aligned} \tag{3.7}$$

Remark 3.42. In the previous corollary we could replace the group $H^{2c}_{|\overline{Z}|}\left(\overline{X},\mathcal{F}\right)$ with $H^0_{\mathrm{et}}\left(\overline{Z},i^*\mathcal{F}(-c)\right)$, and the group $H^{2d}_{|\overline{Z}|}\left(\overline{X},\mathcal{F}\right)$ with $H^{2(d-c)}_{\mathrm{et}}\left(\overline{Z},i^*\mathcal{F}(-c)\right)$.

3.9.1 The l-adic Abel-Jacobi Map

Assume now that K is a field of characteristic 0, and let ℓ be a prime. Let X be a smooth projective variety over K, and let $\mathrm{CH}^c(X)$ be the Chow group of X, consisting of codimension-c cycles with *rational* coefficients, modulo rational equivalence. The Chow group will be revisited again in Chapter 5 when discussing a category of relative motives with arbitrary coefficients, but here we work with a simpler setting. Consider the locally-constant sheaves $\mathcal{F}_n = \mathbb{Z}/\ell^n\mathbb{Z}(c)$ in the previous section, and take projective limits with respect to n, to get $\mathbb{Z}_\ell$-valued cohomology. Inverting ℓ we get $\mathbb{Q}_\ell$-valued cohomology, which will be denoted with H_{et} as well.

The Gysin map i_* in Equation 3.7 induces by restriction to rational cycles the *cycle class map* (see [Mil80, Chapter VI.9]):

$$\mathrm{cl}\colon\ \mathrm{CH}^c(X) \to H^{2c}_{\mathrm{et}}\left(\overline{X},\mathbb{Q}_\ell(c)\right)^{G_K},$$

where $\overline{X}{:=}X\otimes_K\overline{K}$. Let $\mathrm{CH}^c_0(X){:=}\ker\mathrm{cl}$. Given a class $[Z]\in\mathrm{CH}^c_0(X)$, represented by a cycle Z, consider the short exact sequence of G_K-modules:

$$0 \to H^{2c-1}_{\mathrm{et}}\left(\overline{X},\mathbb{Q}_\ell(c)\right) \to H^{2c-1}_{\mathrm{et}}\left(\overline{X}\setminus|\overline{Z}|,\mathbb{Q}_\ell(c)\right) \to H^{2c}_{|\overline{Z}|}\left(\overline{X},\mathbb{Q}_\ell(c)\right)_0 \to 0, \tag{3.8}$$

where

$$H^{2c}_{|\overline{Z}|}\left(\overline{X},\mathbb{Q}_\ell(c)\right)_0 {:=} \ker\left(H^{2c}_{|\overline{Z}|}\left(\overline{X},\mathbb{Q}_\ell(c)\right) \xrightarrow{i_*} H^{2c}_{\mathrm{et}}\left(\overline{X},\mathbb{Q}_\ell(c)\right)\right)$$

is the kernel of the *Gysin map* i_*.

Consider the map $\alpha\colon \mathbb{Q}_\ell \mapsto H^{2c}_{|Z|}\left(\overline{X}, \mathbb{Q}_\ell(c)\right)_0$ which sends

$$1 \mapsto \mathrm{cl}^{\overline{X}}_{\overline{Z}}(\overline{Z}) \in H^{2c}_{|\overline{Z}|}\left(\overline{X}, \mathbb{Q}_\ell(c)\right).$$

Set $\overline{U}{:=}\overline{X} \setminus |\overline{Z}|$. Pulling back the exact sequence (3.8) by α we obtain an extension

$$\begin{tikzcd}
0 \arrow[r] & H^{2c-1}_{\mathrm{et}}\left(\overline{X}, \mathbb{Q}_\ell(c)\right) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \mathbb{Q}_\ell \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & H^{2c-1}_{\mathrm{et}}\left(\overline{X}, \mathbb{Q}_\ell(c)\right) \arrow[r] & H^{2c-1}_{\mathrm{et}}\left(\overline{U}, \mathbb{Q}_\ell(c)\right) \arrow[r] & H^{2c}_{|\overline{Z}|}\left(\overline{X}, \mathbb{Q}_\ell(c)\right)_0 \arrow[r] & 0.
\end{tikzcd} \tag{3.9}$$

Definition 3.43. The *l-adic étale Abel-Jacobi map* is the map

$$\mathrm{AJ}^{\mathrm{et}}_\ell\colon \mathrm{CH}^c_0(X) \to \mathrm{Ext}^1_{\mathrm{Rep}(G_K)}\left(\mathbb{Q}_\ell, H^{2c-1}_{\mathrm{et}}\left(\overline{X}, \mathbb{Q}_\ell(c)\right)\right),$$

which assigns to a class $[Z]$ the class of the extension (3.9) in the category of continuous representations of the Galois group G_K.

3.9.2 The case $\ell = p$

Assume now that the variety X is defined over a p-adic field K. Fix then $\ell = p$. Bloch and Kato in [BK90] and Nekovar in [Nek93] have defined, for any Galois representation V of G_K, a subspace

$$H^1_{\mathrm{st}}(K, V){:=}\ker\left(H^1_{\mathrm{et}}(K, V) \to H^1_{\mathrm{et}}(K, V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})\right).$$

When V is semistable, this is identified with the group of extension classes of V by $\mathbb{Q}_p$ in the category of semistable representations of G_K.

The following result can be found in [Nek00] and will be used in a crucial way in this work.

Lemma 3.44. *The image of* AJ^{et}_p *is contained in*

$$H^1_{st}\left(K, H^{2c-1}_{et}\left(\overline{X}, \mathbb{Q}_p(c)\right)\right) \cong \mathrm{Ext}^1_{\mathrm{Rep}_{st}(G_K)}\left(\mathbb{Q}_p, H^{2c-1}_{et}\left(\overline{X}, \mathbb{Q}_p(c)\right)\right).$$

As seen in Section 3.3, the Fontaine functors D_{st} and V_{st} give a canonical comparison isomorphism:

$$\mathrm{Ext}^1_{\mathrm{Rep}_{\mathrm{st}}(G_K)}\left(\mathbb{Q}_p, H^{2c-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p(c))\right) \cong \mathrm{Ext}^1_{\mathrm{MF}_K^{\mathrm{ad},\phi,N}}\left(K[c], \mathrm{D}_{\mathrm{st}}(H^{2c-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p))\right),$$

which will make the computation of the Abel-Jacobi map possible.

Chapter 4

The anti-cyclotomic p-adic L-function

This chapter is based mostly on [BDIS02]. In Section 4.1 we relate modular forms on a Shimura curve to certain harmonic cocycles. In Section 4.2 these are in turn related to distributions. Section 4.3 ties the distributions back to modular forms. In Section 4.4 the anti-cyclotomic p-adic L-function attached to a modular form f and to a quadratic imaginary field K is defined, in the case that p is inert in K. This is the object that will be geometrically interpreted. Section 4.5 ends the chapter by showing how this L-function interpolates special values of classical L-functions.

4.1 Modular forms and harmonic cocycles

Recall the definitions of harmonic cocycles as given in Section 3.1, as well as the definition of the representation $V_n = \mathcal{P}_n^\vee$. Let B be a definite quaternion algebra unramified at p, and let $R \subset B$ be an Eichler $\mathbb{Z}[1/p]$-order of level N^+. Fix an isomorphism $\iota\colon B \otimes \mathbb{Q}_p \xrightarrow{\sim} M_2(\mathbb{Q}_p)$, which induces a group homomorphism on the respective groups of units. Let $\Gamma = \iota(R_1^\times) \subseteq \mathrm{SL}_2(\mathbb{Q}_p)$, where $R_1^\times$ is the subgroup of $R^\times$ of units of reduced norm 1.

Let f be a rigid-analytic modular form of weight $n+2$ on Γ, as defined in Section 2.5. To f one can associate a harmonic cocycle $c_f \in C^1_{\mathrm{har}}(V_n)$, defined by:

$$c_f(e)(r) := \mathrm{res}_e(f(z)r(z)dz), \quad r \in \mathcal{P}_n.$$

Here res_e is the p-adic annular residue along the oriented wide open annulus $U(e)$ corresponding to the edge $e \in \vec{\mathfrak{E}}(\mathcal{T})$. The p-adic residue formula implies that c_f is a cocycle.

There is a pairing on Γ-invariant harmonic cocycles of weight $n+2$ on $\mathcal{T}$, defined by:

$$\langle c_1, c_2 \rangle := \sum_{e \in \Gamma \backslash \mathfrak{E}(\mathcal{T})} w_e \langle c_1(e), c_2(e) \rangle,$$

where the pairing on the right was defined in the previous section, and w_e is the size of the stabilizer of e in Γ. It can be checked that this pairing is non-degenerate, and that the Hecke operators T_l, for $l \nmid N$ are self-adjoint with respect to it.

Definition 4.1. An eigenform $f \in M_{n+2}(\Gamma)$ is said to be *normalized* if its associated cocycle c_f satisfies $\langle c_f, c_f \rangle = 1$.

4.2 Harmonic cocycles and distributions

Let X be a compact subset of $\mathbb{P}^1(\mathbb{C}_p)$. Note that the space $\mathbb{P}^1(\mathbb{C}_p)$ has a natural non-archimedean metric: given $\alpha = [a_1 : a_2]$ and $\beta = [b_1 : b_2]$ with $(a_1, a_2) = (b_1, b_2) = 1$, we set $d(\alpha, \beta) := p^{-\mathrm{ord}_p(a_1 b_2 - a_2 b_1)}$. For each $N \geq 1$, define the affinoid subdomain

$$X[N] := \{x \in \mathbb{P}^1(\mathbb{C}_p) \mid \exists y \in X, d(x, y) \leq p^{-N}\} \subseteq \mathbb{P}^1(\mathbb{C}_p).$$

The $\mathbb{Q}_p$-algebra of rigid analytic functions on $X[N]$, which here we write $A(X[N])$, is a Banach algebra over $\mathbb{Q}_p$ for the spectral norm, and we let

$$A(X) := \varinjlim_N A(X[N]),$$

where the transition maps are just restrictions. This is called the $\mathbb{Q}_p$-algebra of *locally analytic functions on* X. Write also $\mathcal{A} := A(\mathbb{P}^1(\mathbb{Q}_p))$, and denote by $\mathcal{A}_n$ the set of $\mathbb{C}_p$-valued functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally analytic except for a pole of order at most n at ∞.

Given a Γ-invariant harmonic cocycle $c \in C^1_{\mathrm{har}}(V_n)^\Gamma$, we wish to associate to it a distribution on the space $\mathcal{A}_n$.

Schneider defined in [Sch84] a continuous linear functional μ_c on the space of locally analytic functions which are supported on compact subsets $U \subset \mathbb{Q}_p$. The distribution μ_c is characterized by the formula:

$$\mu_c(r \cdot \chi_{U(e)}) := \int_{U(e)} r(x) d\mu_c(x) := c(e)(r),$$

for all $e \in \mathfrak{E}_\infty(\mathcal{T})$ satisfying $\infty \notin U(e)$, and all $r \in \mathcal{P}_n$. In [Tei90], Teitelbaum extended uniquely the distribution μ_c to the space $\mathcal{A}_n$.

4.3 Distributions associated to modular forms

Let f be a rigid analytic modular form of weight $n+2$ on Γ. By combining the constructions of the two previous sections one associates to f a distribution μ_f on $\mathcal{A}_n$.

Definition 4.2. The distribution associated to f is given by

$$\mu_f(P \cdot \chi_{U(e)}) := \int_{U(e)} P(x) d\mu_f(x) := c_f(e)(P(X)),$$

where c_f is the Γ-invariant harmonic cocycle associated to f, the polynomial P belongs to $\mathcal{P}_n$, and e is an end in $\mathfrak{E}_\infty(\mathcal{T})$ such that $\infty \notin U(e)$.

This distribution extends uniquely to $\mathcal{A}_n$, and one obtains:

Lemma 4.3. *If $P \in \mathcal{P}_n$, then*

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} P(x) d\mu_f(x) = 0.$$

Proof. Write first

$$\mathbb{P}^1(\mathbb{Q}_p) = \coprod_{i=0}^{p} U(e_i),$$

where $e_0, \ldots, e_p$ are the $p+1$ edges leaving the origin vertex v_0. Then:

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} P(x) d\mu_f(x) = \left(\sum_{i=0}^{p} c_f(e_i) \right) (P(X)) = 0,$$

because c_f is a harmonic cocycle. □

The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts also on $\mathcal{A}_n$ with weight n, by the rule:

$$(\varphi * \beta)(x) := (cx+d)^n \varphi(\beta \cdot x), \quad \varphi \in \mathcal{A}_n, \text{ and } \beta \in \mathrm{PGL}_2(\mathbb{Q}_p).$$

The following proposition allows one to recover a modular form f from its associated distribution.

Proposition 4.4 (Teitelbaum)**.** *Let f be a rigid analytic modular form of weight $n+2$ on Γ, and let μ_f be the associated distribution on $\mathbb{P}^1(\mathbb{Q}_p)$. Then*

$$f(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_f(t).$$

Proof. See [Tei90, Theorem 3]. □

4.4 The definition

The goal in this section is to construct a p-adic distribution that allows the definition of the anti-cyclotomic p-adic L-function, as done in [BDIS02]. The construction is slightly different depending on the splitting of p in K, and we will only need the case where p is inert in K, which we will assume from now on. Assume also that the primes dividing N^- are inert in K. Fix an isomorphism $\iota \colon B_p \to M_2(\mathbb{Q}_p)$.

4.4.1 Oriented Embeddings

Let $\mathcal{O}_K$ be the ring of integers of K, and let $\mathcal{O} = \mathcal{O}_K[1/p]$ be its ring of p-integers. Let $R \subseteq B$ be an Eichler $\mathbb{Z}[1/p]$-order of level N^+.

Definition 4.5. An *orientation* of R is a surjective ring homomorphism

$$\mathfrak{o} \colon R \to (\mathbb{Z}/N^+\mathbb{Z}) \times \prod_{l|N^-} \mathbb{F}_{l^2},$$

and the pair $(R, \mathfrak{o})$ is called an *oriented Eichler order*.

Similarly, an orientation of $\mathcal{O}$ is a surjective ring homomorphism

$$\mathcal{O} \to (\mathbb{Z}/N^+\mathbb{Z}) \times \prod_{l|N^-} \mathbb{F}_{l^2}.$$

Remark 4.6. Note that to give an orientation of $\mathcal{O}$ is equivalent to the choice of a prime ideal of K above each prime l dividing N^+, and an identification of the residue field of K at l with $\mathbb{F}_{l^2}$ for each l dividing N^-.

Fix an orientation for R and for $\mathcal{O}$.

Definition 4.7. An *oriented optimal embedding of $\mathcal{O}$ into R* is an embedding $\Psi \colon K \to B$ of an oriented Eichler order $(R, \mathfrak{o})$ into the quaternion algebra B such that

1. $\Psi(K) \cap R = \Psi(\mathcal{O})$, so that Ψ induces an embedding of $\mathcal{O}$ into R, and
2. Ψ is compatible with the chosen orientations for R and $\mathcal{O}$.

The group $R^\times$ acts by conjugation on the set of oriented optimal embeddings, as well as its subgroup $\Gamma{:=}R_1^\times$, of elements of reduced norm equal to 1. Denote by $\mathrm{emb}(\mathcal{O}, R)$ the set of oriented optimal embeddings of $\mathcal{O}$ into R, modulo this action.

Let $\Delta{:=}\,\mathrm{Pic}(\mathcal{O})$ be the Picard group of projective modules of rank one over $\mathcal{O}$. There is a natural action of Δ on $\mathrm{emb}(\mathcal{O}, R)$ which is described in detail [BDIS02, Section 2.3]. We omit here the exposition of this action, which would require us to introduce notation coming from class field theory that is not needed in the sequel.

4.4.2 The partial p-adic L-function

What will be called the partial p-adic L-function depends on a pair $(\Psi, \star)$, of an optimal embedding $\Psi\colon K \to B$ and a base-point $\star \in \mathbb{P}^1(\mathbb{Q}_p)$. Fix such a pair. The embedding Ψ induces an embedding $\Psi\colon K_p \to B_p$ (where we write $K_p := K \otimes \mathbb{Q}_p$). Recall that we have fixed an isomorphism $\iota\colon B_p \to M_2(\mathbb{Q}_p)$. The composition $\iota \circ \Psi$ induces an embedding of $K_p^\times/\mathbb{Q}_p^\times$ into $\mathrm{PGL}_2(\mathbb{Q}_p)$, which is also noted $\iota \circ \Psi$. This gives an action $*$ of $K_p^\times/\mathbb{Q}_p^\times$ on the boundary $\mathbb{P}^1(\mathbb{Q}_p)$ of $\mathcal{H}_p$:

$$\alpha * x := (\iota \circ \Psi)(\alpha)(x),$$

for $\alpha \in K_p^\times/\mathbb{Q}_p^\times$ and $x \in \mathbb{P}^1(\mathbb{Q}_p)$. Since p is assumed to be inert in K, this action is simply-transitive.

The base point $\star \in \mathbb{P}^1(\mathbb{Q}_p)$ gives an identification

$$\eta_{\Psi,\star}\colon K_p^\times/\mathbb{Q}_p^\times \xrightarrow{\cong} \mathbb{P}^1(\mathbb{Q}_p),$$

by sending 1 to $\star$. The torus $\iota \circ \Psi(K_p^\times)$ has two fixed points in $\mathcal{H}_p$, which are denoted z_0 and $\overline{z}_0$. They belong to K_p and are interchanged by $\mathrm{Gal}(K_p/\mathbb{Q}_p)$. In fact, having fixed an embedding of $H(\mu_M) \hookrightarrow \overline{\mathbb{Q}}$ (recall that H is the Hilbert class field of K), it is shown in [BD98, Section 5] how to distinguish z_0 from $\overline{z}_0$. At the cost of an ambiguity in the sign of the subsequent formulas, we can omit this subtlety.

There is a natural homeomorphism $G \cong K_p^\times/\mathbb{Q}_p^\times$, where $G := K_{p,1}^\times$ denotes the subgroup of $K_p^\times$ of elements of norm 1. This identification induces an isomorphism

$$\eta_{\Psi,\star}\colon G \xrightarrow{\cong} \mathbb{P}^1(\mathbb{Q}_p).$$

When $\star = \infty$, this is given explicitly by:

$$\eta_{\Psi,\infty}(\alpha) = \frac{z_0\alpha - \overline{z}_0}{\alpha - 1}, \qquad \eta_{\Psi,\infty}^{-1}(x) = \frac{x - \overline{z}_0}{x - z_0}.$$

The function $\eta_{\Psi,\star}$ induces in turn a continuous isomorphism:

$$(\eta_{\Psi,\star})_* = (\eta_{\Psi,\star}^{-1})^*\colon A(G) \to \mathcal{A},$$

from the ring of locally-analytic functions on G to $\mathcal{A}$.

Consider the polynomial $P^{\frac{n}{2}}_{\Psi(\sqrt{-D_K})}$, where $P_{\Psi(\sqrt{-D_K})}$ has been defined in Equation (3.1) of Chapter 3. Given $\varphi \in A(G)$, we define

$$\mu_{f,\Psi,\star}(\varphi) := \mu_f \left(P^{\frac{n}{2}}_{\Psi(\sqrt{-D_K})} \cdot (\eta^{-1}_{\Psi,\star})^*(\varphi) \right).$$

This is a locally analytic distribution on G. Now, fix a branch $\log_p$ of the p-adic logarithm such that $\log_p(p) = 0$. This induces a homomorphism $\log\colon K_p^\times \to K_p$ which vanishes at the roots of unity, thus giving a homomorphism $G \to K_p$. For $s \in \mathbb{Z}_p$ and $x \in G$, define then

$$x^s := \exp(s \log x).$$

Let $[\Psi] \in \mathrm{emb}(\mathcal{O}, R)$ and let $\star \in \mathbb{P}^1(\mathbb{Q}_p)$ be a fixed base point.

Definition 4.8 ([BDIS02, Definition 2.20])**.** The *partial p-adic L-function* attached to the datum $(f, (\Psi, \star))$ is the function of the p-adic variable $s \in \mathbb{Z}_p$ defined by:

$$L_p(f, \Psi, \star, s) := \int_G x^{s - \frac{n+2}{2}} d\mu_{f,\Psi,\star}(x).$$

4.4.3 Definition of the p-adic L-function

We have so far constructed a distribution on G, to which we attach a partial p-adic L-function. In order to define the anticyclotomic p-adic L-function we need to consider anticyclotomic extensions of number fields, which we recall now.

Definition 4.9. An abelian extension L/K is called *anticyclotomic* if it is Galois over $\mathbb{Q}$ and if the involution in $\mathrm{Gal}(K/\mathbb{Q})$ acts (by conjugation) as -1 on $\mathrm{Gal}(L/K)$.

Let K_∞ denote the maximal anticyclotomic extension of K unramified outside p. Let H be the Hilbert class field of K. Let K_n be the ring class field of K of conductor p^n (so that $K_0 = H$). We have a tower of extensions:

$$\mathbb{Q} \subset K \subset H \subset K_1 \subset \cdots \subset K_n \subset \cdots .$$

Assume for simplicity that $\mathcal{O}_K^\times = \{\pm 1\}$. By class field theory, the p-adic group G is isomorphic to $\mathrm{Gal}(K_\infty/H)$. Let $G^n{:=}\,\mathrm{Gal}(K_\infty/K_n)$, and let $\Delta{:=}\,\mathrm{Gal}(H/K)$. Write also $\tilde{G}{:=}\,\mathrm{Gal}(K_\infty/K)$. These fit into an exact sequence:

$$1 \to G \to \tilde{G} \to \Delta \to 1,$$

and in [BDIS02, Lemma 2.13] it is shown how the natural action of Δ on $\mathrm{emb}(\mathcal{O}, R)$ lifts to an action of $\tilde{G}$ on the same set. The logarithm $\log_p$ extends uniquely to $\tilde{G}$, and thus one can define x^s for $s \in \mathbb{Z}_p$ and $x \in G$.

Let $\alpha \in \tilde{G}$ be an element of $\tilde{G}$. Given a function $\varphi\colon \tilde{G} \to \mathbb{C}_p$, denote by φ_α the function $G \to \mathbb{C}_p$ sending x to $\varphi(\alpha x)$.

For each $\delta \in \Delta$, fix once and for all a lift α_δ of δ to $\tilde{G}$.

Definition 4.10. A function $\varphi\colon \tilde{G} \to \mathbb{C}_p$ is *locally analytic* if $\varphi_{\alpha_\delta} \in A(G)$ for all $\delta \in \Delta$. The set of locally-analytic functions on $\tilde{G}$ is denoted by $A(\tilde{G})$.

Define a distribution $\mu_{f,K}$ on $A(\tilde{G})$ by the formula:

$$\mu_{f,K}(\varphi){:=}\sum_{\delta\in\Delta} \mu_{f,\Psi_\delta,\star_\delta}(\varphi_{\alpha_\delta}),$$

where $(\Psi_\delta, \star_\delta){:=}\alpha_\delta(\Psi, \star)$.

Finally, define the anticyclotomic p-adic L-function:

Definition 4.11 ([BDIS02, Definition 2.19])**.** The *anticyclotomic p-adic L-function* attached to the modular form f and the field K is:

$$L_p(f, K, s){:=}\int_{\tilde{G}} \alpha^{s-\frac{n+2}{2}} d\mu_{f,K}(\alpha), \quad s \in \mathbb{Z}_p.$$

4.5 Interpolation of classical values

Let f_∞ be a classical cusp form of weight $n+2$ on $\Gamma_0(N)$. Assume now that f_∞ is pN^--new, and let $\mathcal{O}_c$ be the order of K of conductor c and $G_c = \mathrm{Pic}(\mathcal{O}_c)$ be the Picard group of rank one projective $\mathcal{O}_c$-modules.

Let ε_K be the quadratic Dirichlet character attached to K, and for each positive integer m let $r_{\mathfrak{a}}(m)$ be the number of integral ideals of norm m in the ideal class of $\mathfrak{a}$. For a character $\chi\colon G_c \to \mathbb{C}^\times$, set:

$$L(f_\infty/K, \chi, s) := \sum \chi(\mathfrak{a}) L(f_\infty/K, \mathfrak{a}, s),$$

where the sum runs over all classes in G_c, and where:

$$L(f_\infty/K, \mathfrak{a}, s) := \left(\sum_{m=1,(m,N)=1}^{\infty} \frac{\varepsilon_K(m)}{m^{2s-n-1}} \right) \left(\sum_{m=1}^{\infty} \frac{a_m r_{\mathfrak{a}}(m)}{m^s} \right)$$

is the partial L-function associated to the class $[\mathfrak{a}] \in G_c$. Here, ε_K is the quadratic character associated to K, and the coefficients a_m are those in the q-expansion of f_∞:

$$f_\infty = \sum_{m=1}^{\infty} a_m q^m, \quad a_1 = 1.$$

Although $L(f_\infty/K, \mathfrak{a}, s)$ does not have Euler products, the complete L-function $L(f_\infty/K, \chi, s)$ does have an Euler product and a functional equation relating s to $n+2-s$.

Let $f \in M_{n+2}(X_\Gamma)$ be the modular form on the Shimura curve X_Γ associated to f_∞ via Theorem 2.39. The following proposition gives the interpolation property for the p-adic distribution $\mu_{f,K}$ and the p-adic L-function defined above.

Proposition 4.12 ([BDIS02, Section 2.5])**.** *Assume that n is even. Let $\chi\colon \tilde{G} \to \mathbb{C}_p^\times$ be a continuous finite-order character. Then:*

$$\left| \int_{\tilde{G}} \chi(x) d\mu_{f,K}(x) \right|^2 = M(\chi)(-D_K)^{\frac{n+1}{2}} \frac{L(f_\infty/K, \chi, \frac{n+2}{2})}{(f_\infty, f_\infty)},$$

where $M(\chi)$ is a simple explicit constant (it is one if χ is unramified), and $(\cdot,\cdot)$ is the Pettersson inner product.

Part II

Results

Chapter 5

A motive

In this chapter we define a certain Chow motive $\mathcal{D}_n$, which will allow for the geometric interpretation of the p-adic L-function defined in the previous chapter. We will then calculate its realizations. Section 5.1 introduces some notions on Chow motives that will be used in this chapter and the following. Section 5.2 recalls the motive described in [IS03]. In Section 5.4 this definition is modified in the spirit of [BDP09], thus yielding the motive $\mathcal{D}_n$. The goal of the final Section 5.5 is to compute the realizations of $\mathcal{D}_n$.

5.1 Relative motives with coefficients

In this section we introduce the category of relative Chow motives with coefficients in an arbitrary field. We follow the exposition given in [Kün01, Section 2].

Let K be a field of characteristic 0. Let S be a smooth quasiprojective connected scheme over K. For simplicity, assume that S is of dimension 1, as this is the only situation that we will need in the following. Denote by $\mathbf{Sch}(S)$ the category of smooth projective schemes $X \to S$. Let F be any field of characteristic 0.

Definition 5.1. The *ith Chow group* of X, written $\mathrm{CH}^i(X)$, is the group of algebraic cycles on X of codimension i, modulo rational equivalence.

Definition 5.2. The *Chow ring* of X, written $\mathrm{CH}(X)$, is the ring of algebraic cycles on X, modulo rational equivalence. The product is given by intersection of cycles.

There is an obvious decomposition, as abelian groups:

$$\mathrm{CH}(X) = \bigoplus_{i=0}^{d+1} \mathrm{CH}^i(X),$$

where d is the relative dimension of X over S. Write also

$$\mathrm{CH}(X, F) := \mathrm{CH}(X) \otimes_{\mathbb{Z}} F.$$

Definition 5.3. Given X, Y two smooth projective S-schemes, the *ring of S-correspondences with coefficients in F* is defined as:

$$\mathrm{Corr}_S(X, Y) := \mathrm{CH}(X \times_S Y, F).$$

For $\alpha \in \mathrm{CH}(X_1 \times_S X_2, F)$ and $\beta \in \mathrm{CH}(X_2 \times_S X_3, F)$, the composition of α and β is defined as:

$$\beta \circ \alpha := \mathrm{pr}_{13,*}\left(\mathrm{pr}_{12}^*(\alpha) \cdot \mathrm{pr}_{23}^*(\beta)\right),$$

where pr_{ij} is the projection of $X_1 \times_S X_2 \times_S X_3$ to $X_i \times X_j$.

Definition 5.4. A *projector* on X over S is an idempotent in the ring of relative correspondences $\mathrm{Corr}_S(X, X) = \mathrm{CH}(X \times_S Y, F)$. If p belongs to the ith graded piece $\mathrm{CH}^i(X \times_S X, F)$ we say that p is of degree i.

We first introduce the category $\mathbf{Mot}(S, F)$ of Chow motives over S with coefficients in F, with respect to ungraded correspondences. Its objects are pairs (X, p), where $X \to S$ is in $\mathbf{Sch}(S)$, and p is a projector with coefficients in F. We set:

$$\mathrm{Hom}_{\mathbf{Mot}(S,F)}\left((X, p), (Y, q)\right) := q \circ \mathrm{CH}(X \times_S Y, F) \circ p,$$

and composition is induced by composition of correspondences. For $i \in \mathbb{Z}$, we say that $q \circ \alpha \circ p \in \mathrm{Hom}_{\mathbf{Mot}(S)}\left((X, p), (Y, p)\right)$ is homogeneous of degree i if

$$q \circ \alpha \circ p \in \bigoplus_{\nu} \mathrm{CH}^{d_\nu + i}(X_\nu \times_S Y, F),$$

where $X = \coprod_\nu X_\nu$ is the decomposition of X into connected components and $d_\nu = \dim(X_\nu/S)$. This makes $\mathrm{Hom}_{\mathbf{Mot}(S,F)}$ into a graded ring, with multiplication given by composition. Also, given (X,p) and (Y,q) two objects in $\mathbf{Mot}(S,F)$, we can define $(X,p)\oplus_S(Y,q){:=}(X\coprod Y, p\coprod q)$ and $(X,p)\otimes_S(Y,q){:=}(X\times_S Y, p\otimes_S q)$.

Fact. *The category of motives* $\mathbf{Mot}(S,F)$ *is an additive, pseudo-abelian* F*-linear tensor-category.*

Next, we define the category $\mathbf{Mot}^0_+(S,F)$ of effective relative Chow motives with coefficients. Its objects are those objects (X,p) in $\mathbf{Mot}(S,F)$ such that p is homogeneous of degree 0. As morphisms one takes the degree-zero morphisms:

$$\mathrm{Hom}_{\mathbf{Mot}^0_+(S,F)}\left((X,p),(Y,q)\right) := \left(\mathrm{Hom}_{\mathbf{Mot}(S,F)}\left((X,p),(Y,q)\right)\right)^0.$$

Given an S-scheme X, one associates to it an object of $\mathbf{Mot}^0_+(S,F)$:

$$h(X){:=}(X,\Delta_X),$$

where Δ_X is the diagonal of X in $X\times_S X$. Given a map $f\colon Y\to X$ of S-schemes, we can consider (the class of) the transpose of its graph $[\,{}^t\Gamma_f]$ as an element of $\mathrm{CH}(X\times_S Y)$. Concretely, we consider the map

$$\gamma_f\colon X\to X\times_S Y,\quad \gamma_f = \mathrm{Id}\times f,$$

and we set $[\Gamma_f]{:=}(\gamma_f)_*[X]$. This can be seen as a morphism

$$[\Gamma_f]\in \mathrm{CH}^d(X\times_S Y,F) = \mathrm{Hom}_{\mathbf{Mot}^0_+(S)}\left(h(X),h(Y)\right),$$

and hence $[\,{}^t\Gamma_f]\in \mathrm{Hom}_{\mathbf{Mot}^0_+(S,F)}((h(Y),h(X))$. Assigning to f the morphism $[\,{}^t\Gamma_f]$ we obtain a contravariant embedding of categories:

$$h\colon\ \mathbf{Sch}(S)\to \mathbf{Mot}^0_+(S,F).$$

Lastly, define the category $\mathbf{Mot}^0(S,F)$ of *relative Chow motives over* S *with coefficients over* F. Its objects are triples (X,p,i), where X is a smooth projective

S-scheme, p is a projector on X, and i is an integer. Given (X, p, i) and (Y, q, j) two such objects, we define

$$\mathrm{Hom}_{\mathbf{Mot}^0(S,F)}\left((X,p,i),(Y,q,j)\right) := \mathrm{Hom}^{j-i}_{\mathbf{Mot}(S,F)}\left((X,p),(Y,q)\right).$$

Composition is again induced from composition of correspondences. In this way, the category $\mathbf{Mot}^0_+(S, F)$ can be seen as a full subcategory of $\mathbf{Mot}^0(S, F)$.

Fact. *The category* $\mathbf{Mot}^0(S, F)$ *is an additive, pseudo-abelian* F*-category with a canonical tensor product given by:*

$$(X, p, m) \otimes (Y, q, n) := = (X \times_S Y, p \otimes q, m + n).$$

There is also duality: given $M = (X, p, m)$, with X pure of relative dimension n over S, define $M^\vee := (X, p^t, n - m)$. Then we have:

$$\mathrm{Hom}(P \otimes M, N) = \mathrm{Hom}(P, M^\vee \otimes N).$$

Twisting in $\mathbf{Mot}^0(S, F)$ is defined by

$$(X, p, m)(n) := (X, p, m + n).$$

One has a form of Poincaré duality: if d is the relative dimension of X over S, then:

$$h(X)^\vee = h(X)(d).$$

One also has direct sums, which can be described explicitly. Consider first the Lefschetz motive $L_S := (\mathbb{P}^1_S, \pi_2, 0)$, where π_2 is the Künneth projector onto R^2, coming from any section to $\mathbb{P}^1_S \to S$. Since for $m \le 0$ the motive (X, p, m) is isomorphic to $(X, p, 0) \otimes_S L_S^{-m}$, and since the direct sum for motives of degree 0 is easy:

$$(X, p, 0) \oplus (Y, q, 0) := (X \coprod Y, p \coprod q, 0),$$

we can define the direct sum for general objects in $\mathbf{Mot}^0(S, F)$ as follows: let $r \ge \max(m, n)$. Then $(X, p, m) \oplus_S (Y, q, n)$ is, by definition:

$$\left(\left(X \times_S (\mathbb{P}^1_S)^{r-m}\right) \coprod \left(Y \times_S (\mathbb{P}^1_S)^{r-n}\right), \left(p \otimes \pi_2^{\otimes(r-m)}\right) \coprod \left(q \otimes \pi_2^{\otimes(r-n)}\right), r\right).$$

The importance of Chow motives lies in their universality for the *realization functors*. For us, this means that given a motive (X, p, i), the correspondence p induces a projector on any Weil cohomology $H^*(X)$, and therefore we obtain functors H^* from the category $\mathbf{Mot}^0(S, F)$ to the same category where $H^*(X)$ would live, by sending (X, p, i) to $pH^*(X)$. These functors are called *realization functors*. We will concentrate on the l-adic étale and de Rham realizations, in the next sections.

5.2 The motive $\mathcal{M}_n^{(M)}$ of Iovita and Spieß

5.2.1 Decomposition of the universal abelian surface

Fix $M \geq 3$, and let $X_M/\mathbb{Q}$ be the Shimura curve parametrizing abelian surfaces with quaternionic multiplication by $\mathcal{R}^{\max} \subseteq \mathcal{B}$ and level-M structure, as described in Section 2.2. Let $\pi\colon \mathcal{A} \to X_M$ be the universal abelian surface with quaternionic multiplication. Consider the relative motive $h(\mathcal{A})$ as an object of $\mathbf{Mot}(X_M, F)$, where h is the contravariant functor

$$h\colon \mathbf{Sch}(X_M) \to \mathbf{Mot}^0_+(X_M, F)$$

from the category of smooth and proper schemes over X_M to the category of Chow motives with coefficients in F, as explained above. In general, the realization functors of a motive give the corresponding cohomology groups, as graded vector spaces with extra structures, and one cannot isolate the ith cohomology groups at the motivic level, without assuming what are known as "standard conjectures". If the underlying scheme has extra endomorphisms then one can hope to annihilate some of these groups and thus obtain only the desired degree. The following result establishes this for abelian schemes:

Theorem 5.5 (Deninger-Murre, Künnemann). *The motive $h(\mathcal{A})$ admits a canon-*

ical decomposition

$$h(\mathcal{A}) = \bigoplus_{i=0}^{4} h^i(\mathcal{A}),$$

with $h^i(\mathcal{A}) \cong \bigwedge^i h^1(\mathcal{A})$ *and* $h^i(\mathcal{A})^\vee \cong h^{4-i}(\mathcal{A})(2)$*.*

Proof. This is originally proved in [DM91, Theorem 3.1, Proposition 3.3] using the so called "Fourier theory for abelian schemes". An explicit closed formula is given in [Kün01]. □

5.2.2 Definition of $\mathcal{M}_n^{(M)}$ for even n

Fix an integer $M \geq 3$. In [IS03, Appendix] the authors define a motive $\mathcal{M}_n^{(M)}$ for *even* $n \geq 2$. In this section we recall this construction. Let e_2 be the unique nonzero idempotent in $\mathrm{End}(\bigwedge^2 h^1(\mathcal{A})) = \mathrm{End}(h^2(\mathcal{A}))$ such that

$$x \cdot e_2 = \mathrm{nrd}(x) e_2, \text{ for all } x \in \mathcal{B}.$$

Define ε_2 to be the projector in the ring $\mathrm{Corr}_{X_M}(\mathcal{A}, \mathcal{A})$ such that

$$(\mathcal{A}, \varepsilon_2) = \widetilde{\mathcal{M}}_2^{(M)} := \ker(e_2).$$

Set m as $n/2$ and define $\widetilde{\mathcal{M}}_n^{(M)} := \mathrm{Sym}^m \widetilde{\mathcal{M}}_2^{(M)}$. There is a symmetric pairing, given by the cup-product,

$$h^2(\mathcal{A}) \otimes h^2(\mathcal{A}) \to \bigwedge^4 h^1(\mathcal{A}) \cong \mathbb{Q}(-2).$$

Let $\langle \cdot, \cdot \rangle$ be its restriction to $\widetilde{\mathcal{M}}_2^{(M)} \otimes \widetilde{\mathcal{M}}_2^{(M)}$. It induces a *Laplace operator*

$$\Delta_m \colon \widetilde{\mathcal{M}}_n^{(M)} \to \widetilde{\mathcal{M}}_{n-2}^{(M)}(-2),$$

given symbolically by

$$\Delta_m(x_1 x_2 \cdots x_m) = \sum_{1 \leq i < j \leq m} \langle x_i, x_j \rangle x_1 \cdots \widehat{x}_i \cdots \widehat{x}_j \cdots x_m.$$

Lemma 5.6 ([IS03, Section 10.1])**.** $\ker(\Delta_m)$ *exists (as a motive).*

Sketch of proof. We will rewrite $\ker(\Delta_m)$ as the kernel of a certain projector, and use the fact that, even if the category of motives is not abelian, at least it has kernels of projectors.

Let λ_{m-2} be the morphism

$$\lambda_{m-2}\colon \widetilde{\mathcal{M}}_{n-2}^{(M)} \to \widetilde{\mathcal{M}}_{n}^{(M)}(2)$$

given symbolically by $\lambda_{m-2}(x_1x_2\cdots x_{m-2}) = x_1x_2\cdots x_{m-2}\mu$, where

$$\mu\colon \mathbb{Q} \to \widetilde{\mathcal{M}}_{2}^{(M)}(2)$$

is the dual of $\langle\cdot,\cdot\rangle$ twisted by 2.

Clearly $\Delta_m \circ \lambda_{m-2}$ is an isomorphism. Let then

$$\mathrm{pr} := \lambda_{m-2} \circ (\Delta_m \circ \lambda_{m-2})^{-1} \circ \Delta_m,$$

which is a projector, so $\ker \Delta_m = \ker(\mathrm{pr})$ can be written as the kernel of a projector. $\square$

Define the correspondence ε_n in $\mathrm{Corr}_{X_M}(\mathcal{A}^m, \mathcal{A}^m)$ to be such that

$$(\mathcal{A}^m, \varepsilon_n) = (\mathcal{M}_n)^{(M)} := \ker(\Delta_m).$$

5.3 Extending $\mathcal{M}_n^{(M)}$ to all weights

We propose a uniform construction of $\mathcal{M}_n^{(M)}$ for all integers $n \geq 1$, which is expected to coincide with the definition given above when n is even. Fix a quadratic field F splitting $\mathcal{B}$, and fix an embedding $q\colon F \hookrightarrow \mathcal{B}$. For instance, F could be taken to be the imaginary quadratic field K. The construction we give is not as natural as the definition given for even n, since it depends on the choice of the embedding $q\colon F \hookrightarrow \mathcal{B}$.

The $\mathcal{R}^{\max}$-action on $\mathcal{A}$ induces an embedding $\mathcal{B} \hookrightarrow \mathrm{End}(h^1(\mathcal{A}))$ which gives an action of $F^\times$ on $h^1(\mathcal{A})$. Let e_1' be the unique nonzero idempotent in $\mathrm{End}(h^1(\mathcal{A}))$ such that for all $x \in F^\times$ one has $x \cdot e_1' = xe_1'$. Define the projector $\varepsilon_1' \in \mathrm{Corr}_{X_M}(\mathcal{A}, \mathcal{A})$ via the formula

$$(\mathcal{A}, \varepsilon_1') = e_1'.$$

More concretely, write $F = \mathbb{Q}(\alpha)$ where $\alpha \in F$ is chosen to have trace 0. The projector ε_1' is defined by the formula:

$$\frac{1}{2\alpha}\left(q(\alpha) + \alpha\right).$$

For each integer $n \geq 1$, define $\varepsilon_n' \in \mathrm{Corr}_{X_M}(\mathcal{A}^n, \mathcal{A}^n)$ by

$$\mathcal{M}_n'^{(M)} := (\mathcal{A}^n, \varepsilon_n') := \mathrm{Sym}^n(\mathcal{A}, \varepsilon_1').$$

The projector ε_n' can be expressed in a more concrete way: let $\mathfrak{S}_n$ be the symmetric group on n letters, which acts on $\mathcal{A}^n$ by permuting the copies. Also the action of ε_1' can be extended to $\mathcal{A}^n$ by making it act on each of the copies. Then:

$$\varepsilon_n' = \left(\frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} \sigma\right) \circ \varepsilon_1'.$$

Note that ε_n' defined by the above formula is a projector, because ε_1' commutes with $\sum_{\sigma \in \mathfrak{S}_n} \sigma$.

Example 5.7. Let $\alpha \in \mathcal{B} \setminus \mathbb{Q}$ be an element of trace 0. Assume for simplicity that $\mathrm{nrd}(\alpha) = 1$, and let $F := \mathbb{Q}(\alpha)$. Then F embeds in $\mathcal{B}$ via

$$a + b\alpha \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right).$$

Consider then $H^1_{\mathrm{dR}}(\mathcal{A}_\tau) \cong M_2$. The projector ε_1' will act on $H^1_{\mathrm{dR}}(\mathcal{A}_\tau)$ so that

$$\varepsilon_1'\left(H^1_{\mathrm{dR}}(\mathcal{A}_\tau) \otimes F\right) = \left\{\left(\begin{smallmatrix} x & \alpha x \\ y & \alpha y \end{smallmatrix}\right)\right\},$$

since this is the subspace of matrices $A \in M_2(F)$ satisfying

$$A\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) = (a + b\alpha)A,$$

for all $a, b \in \mathbb{Q}$.

Let $\mathcal{M}$ denote the motive $h^1(\mathcal{A})$. There is a commutative diagram:

$$\begin{tikzcd}
\mathcal{M}\otimes\mathcal{M} \arrow[r, equal] \arrow[d, "\varepsilon_1'\otimes\varepsilon_1'"] & \mathcal{M}\otimes\mathcal{M} \arrow[d, "p_A"] \\
\mathcal{M}_1^{(M)}\otimes\mathcal{M}_1^{(M)} \arrow[d, "p_S"] & \bigwedge^2\mathcal{M} \arrow[d, "\varepsilon_2"] \\
\mathrm{Sym}^2\,\mathcal{M}_1^{(M)} \arrow[r, "\psi"] & \left(\bigwedge^2\mathcal{M}\right)_- .
\end{tikzcd}$$

Here, the projectors p_A and p_S are the natural projectors onto the alternating square and the symmetric square, respectively. The map ψ is given symbolically by:

$$x\cdot y\mapsto\frac{1}{2}(x\wedge\beta y+y\wedge\beta x).$$

In [Bes95, Theorem 5.8 (ii)] it is shown that this map is an isomorphism, for the complex de Rham realization. The second statement, again for the de Rham realization, is proven in [Bes95, Theorem 5.8 (iii)]. This leads us to the following conjecture:

Conjecture 5.1. *There is an isomorphism of motives:*

$$\mathrm{Sym}^2(\mathcal{A},\varepsilon_1')\cong(\mathcal{M}_2)^{(M)}.$$

Moreover, for each even integer n, there is an isomorphism of motives:

$$\mathrm{Sym}^n(\mathcal{A},\varepsilon_1')\cong(\mathcal{M}_n)^{(M)}$$

5.4 The motive $\mathcal{D}_n$

5.4.1 Symmetric powers of a CM curve

Fix A an abelian surface with quaternionic multiplication. Assume also that A has CM. By Remark 2.33, A is isomorphic to $E\times E$. Fix such an isomorphism.

Let $\mathfrak{S}_n$ be the symmetric group on n letters, and consider the wreath product $\Xi_n:=\mu_2\wr\mathfrak{S}_n$, which can be described as the semidirect product

$$\Xi_n:=(\mu_2)^n\rtimes\mathfrak{S}_n,$$

with $\sigma \in \mathfrak{S}_n$ acting on $(\mu_2)^n$ by $(x_1, \ldots, x_n)^\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. This is isomorphic to the group of signed permutation matrices of degree n.

The group Ξ_n acts on E^n as follows: each of the copies of μ_2 acts by multiplication by -1 on the corresponding copy of E, and $\mathfrak{S}_n$ permutes the n copies.

Let $j\colon \Xi_n \to \{\pm 1\}$ be the homomorphism which sends $-1 \in \mu_2$ to -1, and which is the sign character on $\mathfrak{S}_n$, and let

$$\varepsilon_E := \frac{1}{2^n(n)!} \sum_{\sigma \in \Xi_n} j(\sigma)\sigma \in \mathbb{Q}[\mathrm{Aut}(E^n)],$$

which is an idempotent in the rational group ring of $\mathrm{Aut}(E^n)$.

By functoriality, ε_E induces a projector in $\mathrm{Corr}_{X_M}(E^n, E^n)$, inducing an endomorphism on the different cohomology groups.

Lemma 5.8.

1. *The image of ε_E action on $H^*_{et}(E^n, \mathbb{Q}_l)$ is*

$$\varepsilon_E H^*_{et}(E^n, \mathbb{Q}_l) = \mathrm{Sym}^n\, H^1_{et}(E, \mathbb{Q}_l).$$

2. *The image of ε_E acting on $H^*_{dR}(E^n)$ is*

$$\varepsilon_E H^*_{dR}(E^n) = \mathrm{Sym}^n\, H^1_{dR}(E).$$

Proof. Denote by H either H_{dR} or H_{et}. First note that -1 acts as the identity on $H^0(E, \mathbb{Q}_l)$ and $H^2(E, \mathbb{Q}_l)$ and as -1 on $H^1(E, \mathbb{Q}_l)$. Therefore all terms in the Künneth decomposition

$$H^*(E^n) = \bigoplus_{(i_1, \ldots, i_n)} H^{i_1}(E) \otimes \cdots \otimes H^{i_n}(E)$$

vanish under the action of ε_E except for $H^1(E)^{\otimes n}$. The action of $\mathfrak{S}_n$ on this factor is the permutation twisted by the sign character, and thus it induces the projection onto $\mathrm{Sym}^n\, H^1_?(E)$.

See also the discussion in [BDP09, Lemma 1.8]. □

5.4.2 A new motive

We want to generalize the construction of [IS03] in the spirit of [BDP09]. Let n be a positive even integer, and set $m{:=}n/2$.

Definition 5.9. The motive $\mathcal{D}_n^{(M)}$ over X_M is defined as:

$$\mathcal{D}_n^{(M)}{:=}(\mathcal{A}^m \times E^n, \varepsilon_n^{(M)}){:=}\mathcal{M}_n^{(M)} \otimes (E^n, \varepsilon_E),$$

where $E^n \to X_M$ is seen as a constant family $E^n \times X_M$, with fibers E^n.

We descend this construction to the Shimura curve X. For that, consider the group $G = (\mathcal{R}^{\max}/M\mathcal{R}^{\max}) \cong \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$, which acts canonically (through X-automorphisms) on X_M, on $\mathcal{A}^m$ and on E^n. Hence we can consider the projector

$$p_G{:=}\frac{1}{|G|}\sum_{g\in G} g \in \mathrm{Corr}_X(\mathcal{A}^m \times E^n, \mathcal{A}^m \times E^n).$$

The projector p_G commutes with both ε_n and ε_E. In fact, p_G acts trivially on E^n. So the composition of these projectors is also a projector, which will be denoted ε.

Definition 5.10. The *generalized Kuga-Sato motive* $\mathcal{D}_n$ is defined to be

$$\mathcal{D}_n{:=}(\mathcal{A}^m \times E^n, \varepsilon){:=}p_G\left(\mathcal{D}_n^{(M)}\right) = p_G\left(\mathcal{M}_n^{(M)}\right) \otimes (E^n, \varepsilon_E).$$

5.5 Realizations

We are interested in the p-adic étale and de Rham realizations of $\mathcal{D}_n$.

5.5.1 The p-adic étale realization

Consider the p-adic étale sheaf $R^2\pi_*\mathbb{Q}_p$, which has fibers at each geometric point $\tau \to X_M$ given by $H^2_{\mathrm{et}}(\mathcal{A}_\tau, \mathbb{Q}_p)$. We want to work with a subsheaf of $R^2\pi_*\mathbb{Q}_p$. For this, note that the action of $\mathcal{R}^{\max}$ on $\mathcal{A}$ induces an action of $\mathcal{B}^\times$ on $R^2\pi_*\mathbb{Q}_p$.

Consider the p-adic étale sheaf

$$\mathbb{L}_2 := \bigcap_{b\in\mathcal{B}^\times} \ker\left(b - \mathrm{nrd}(b)\colon R^2\pi_*\mathbb{Q}_p \to R^2\pi_*\mathbb{Q}_p\right) \subseteq R^2\pi_*\mathbb{Q}_p,$$

which is the subsheaf on which $\mathcal{B}^\times$ acts as the reduced norm nrd of $\mathcal{B}$. It is a 3-dimensional locally-free sheaf on X_M. Set m to be $n/2$, and consider the map $\Delta_m\colon \mathrm{Sym}^m\,\mathbb{L}_2 \to \left(\mathrm{Sym}^{m-2}\,\mathbb{L}_2\right)(-2)$ given by the Laplace operator. That is,

$$\Delta_m(x_1\cdots x_m) = \sum_{1\le i<j\le m}(x_i,x_j)x_1\cdots\widehat{x}_i\cdots\widehat{x}_j\cdots x_m,$$

where $(\cdot,\cdot)$ is the non-degenerated pairing induced from the cup product and the trace: $(x,y)=\mathrm{tr}(x\cup y)$. Define also

$$\mathbb{L}_n := \ker\Delta_m,$$

and

$$\mathbb{L}_{n,n} := \mathbb{L}_n \otimes \mathrm{Sym}^n\, H^1_{\mathrm{et}}(E,\mathbb{Q}_p).$$

The following lemma gives the p-adic étale realization of the motive $\mathcal{D}_n$.

Lemma 5.11. *Consider $\mathcal{D}_n$ as an absolute motive over $\mathbb{Q}$. Let $H_p(-)$ be the p-adic realization functor. Then:*

$$H_p(\mathcal{D}_n) \cong H^1_{et}\left(\overline{X_M},\mathbb{L}_{n,n}\right)^G = H^1_{et}\left(\overline{X_M},\mathbb{L}_n\right)^G \otimes \mathrm{Sym}^n\, H^1_{et}(E,\mathbb{Q}_p).$$

Proof. First, note that the p-adic realization of the motive $\mathcal{D}_n^{(M)}$, as thought of as in the derived category, is the complex of $\mathbb{Q}_p$-sheaves

$$\mathbb{L}_n[-n]\otimes \mathrm{Sym}^n\, H^1_{\mathrm{et}}(E,\mathbb{Q}_p).$$

concentrated in degree $-n$. Then, we just need to compute:

$$\begin{aligned} H_p(\mathcal{D}_n) &= (p_G)_*\left(H^*\left(\overline{X_M},\mathbb{L}_n[-n]\otimes\mathrm{Sym}^n\, H^1_{\mathrm{et}}(E,\mathbb{Q}_p)\right)\right)\\ &= H^{*-2n}_{\mathrm{et}}\left(\overline{X_M},\mathbb{L}_n\right)^G\otimes\mathrm{Sym}^n\, H^1_{\mathrm{et}}(E,\mathbb{Q}_p). \end{aligned}$$

which follows from the cohomology of $\mathbb{L}_n$ being concentrated in degree 1 and from the Künneth formula. □

5.5.2 The odd case

Since all prime divisors v dividing the discriminant of $\mathcal{B}$ are non-split (actually, inert) in K, the field K splits $\mathcal{B}$. Let $q\colon K \hookrightarrow \mathcal{B}$ be an inclusion, and let q' be its conjugate (that is, the composition $q \circ c$ where c is the nontrivial automorphism of K). Let $\beta \in \mathcal{B}_0$ be such that $\beta \circ q = q' \circ \beta$. Let K_p be the completion of K at p. The inclusion q induces an action of $K^\times$ on $R^1\pi_* K_p$. At the same time, $R^1\pi_* K_p$ has a natural structure of K-module. Let $\mathbb{L}'_1$ be the subsheaf of $R^1\pi_* K_p$ on which $K^\times$ acts by multiplication.

For each integer $n \geq 1$, define

$$\mathbb{L}'_n := \operatorname{Sym}^n \mathbb{L}'_1.$$

Similarly to the construction for motives, we have:

Conjecture 5.2. *The map* $\psi\colon \mathbb{L}'_2 = \operatorname{Sym}^2 \mathbb{L}'_1 \xrightarrow{\sim} \mathbb{L}_2$ *defined by*

$$xy \mapsto \frac{1}{2}(x \wedge \beta y + y \wedge \beta x)$$

is an isomorphism. Moreover, the induced map

$$\operatorname{Sym}^m \psi\colon \operatorname{Sym}^m \left(\operatorname{Sym}^2 \mathbb{L}'_1\right) \to \operatorname{Sym}^m \mathbb{L}_2$$

identifies $\mathbb{L}'_n = \operatorname{Sym}^n \mathbb{L}'_1$ *with* $\ker \Delta_m = \mathbb{L}_n$.

5.5.3 Semistability and the de Rham realization

The Hodge filtration on $H^1_{\mathrm{dR}}(E) := H^1_{\mathrm{dR}}(E, \mathbb{Q}_p)$, which is equivalent to the exact sequence

$$0 \to H^0(E, \Omega^1_{E/\mathbb{Q}_p}) \to H^1_{\mathrm{dR}}(E) \to H^1\left(E, \mathcal{O}_E\right) \to 0$$

induces a filtration on $\operatorname{Sym}^n H^1_{\mathrm{dR}}(E)$. Write H_j for its jth step:

$$H_j := \operatorname{Fil}^j \left(\operatorname{Sym}^n H^1_{\mathrm{dR}}(E)\right).$$

The definition of the filtration of the symmetric powers gives:

$$H_j = \begin{cases} \operatorname{Sym}^n H^1_{\mathrm{dR}}(E) & \text{if } j \leq 0 \\ \operatorname{img}\left(\operatorname{Sym}^j H^0(E, \Omega^1_E) \otimes \operatorname{Sym}^{n-j} H^1_{\mathrm{dR}}(E)\right) & \text{if } 1 \leq j \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Recall the uniformization result of Theorem 2.35. It provides with a map $\pi\colon \mathcal{H}_p \to (X_M)^{\mathrm{an}}_{\mathbb{Q}_p^{\mathrm{ur}}}$. Recall also the filtered isocrystal $\mathcal{E}(M_2)$ defined in Section 3.8.

Theorem 5.12 (Faltings, Iovita-Spieß)**.** *There is a canonical isomorphism of filtered isocrystals on $\mathcal{H}_p$:*

$$\pi^* \mathcal{H}^1_{dR}(\mathcal{A}/X_M) \cong \mathcal{E}(M_2).$$

This isomorphism takes the $\mathcal{B}^\times_{\mathbb{Q}_p^{ur}}$-action on the left-hand side to the action by ρ_2 in the right-hand side.

Proof. See [IS03, Lemma 5.10]. □

Consider the representation (V_n, ρ_1) of GL_2 constructed in Section 3.1, and let ρ_2 be the one-dimensional representation of GL_2 given by $\det^m$. Then the pair (V_n, ρ_1, ρ_2) induces a convergent filtered F-isocrystal $\mathcal{V}_n = \mathcal{E}(V_n\{m\})$ as described in the first paragraph of Section 3.7 and in [IS03, Section 4].

Lemma 5.13 (Iovita-Spieß)**.** *The filtered F-isocrystal $\mathcal{V}_n$ is regular.*

Sketch of proof. Note that X_Γ is a Mumford curve, and that $\det^m$ is pure of weight $2m$: an element $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ in the center of GL_2 acts by multiplication by:

$$\left(\det \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\right)^m = a^{2m}.$$

Then it is shown in [IS03], after Lemma 4.3, how this implies that $\mathcal{V}_n$ is regular. □

A simple computation using the compatibility of the isomorphism of Theorem 5.12 with tensor products gives the following consequence:

Corollary 5.14.

$$\bigcap_{x\in\mathcal{B}^\times} \ker\left((x-\mathrm{nrd}(x))\colon \mathcal{E}\left(\wedge^2 M_2\right)\to\mathcal{E}\left(\wedge^2 M_2\right)\right)\cong\mathcal{V}_2.$$

We believe that one has a similar result for odd n, but we will not formulate a precise statement for it.

There is a map from the space of modular forms on X_Γ of weight $n+2$ to $\mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(X_\Gamma,\mathcal{V}_n)$, given by $f(z)\mapsto\omega_f{:=}f(z)\,\mathrm{ev}_z\otimes dz$, where ev_z is the functional $R(X)\mapsto R(z)$. Identifying these spaces, one obtains the filtration of $H^1_{\mathrm{dR}}(X_\Gamma,\mathcal{V}_n)$:

Proposition 5.15 ([IS03, Proposition 6.1])**.** *The filtration of $H^1_{dR}(X_\Gamma,\mathcal{V}_n)$ is given by:*

$$\mathrm{Fil}^j H^1_{dR}(X_\Gamma,\mathcal{V}_n)=\begin{cases} H^1_{dR}(X_\Gamma,\mathcal{V}_n) & \text{if } j\le 0,\\ M_k(\Gamma) & \text{if } 1\le j\le n+1,\\ 0 & \text{otherwise.}\end{cases}$$

Define the convergent filtered F-isocrystal $\mathcal{V}_{n,n}$ as:

$$\mathcal{V}_{n,n}{:=}\mathcal{V}_n\otimes\mathrm{Sym}^n H^1_{\mathrm{dR}}(E).$$

Understanding the structure of $\mathrm{D}_{\mathrm{st}\mathbb{Q}_p^{\mathrm{ur}}}(H_p(\mathcal{D}_n))$ will allow us to compute the Abel-Jacobi map in an explicit way. Write $H^{2n+1}_{\mathrm{dR}}(\mathcal{D}_n)$ for the filtered (ϕ,N)-module $\mathrm{D}_{\mathrm{st},\mathbb{Q}_p^{\mathrm{ur}}}(H_p(\mathcal{D}_n))$. The following key result is a consequence of the facts shown so far.

Theorem 5.16. *The $G_{\mathbb{Q}_p}$-representation $H^{2n+1}_p(\mathcal{D}_n)$ is semistable, and there is a (canonical up to scaling) isomorphism of filtered (ϕ,N)-modules*

$$\mathrm{D}_{st}(H^{2n+1}_p(\mathcal{D}_n))=H^{2n+1}_{dR}(\mathcal{D}_n)\cong H^1_{dR}(X_\Gamma,\mathcal{V}_{n,n})=H^1_{dR}(X_\Gamma,\mathcal{V}_n)\otimes\,\mathrm{Sym}^n H^1_{dR}(E).$$

Moreover, writing Fil^j for $\mathrm{Fil}^j H^{2n+1}_{dR}(\mathcal{D}_n)$ we have:

$$\mathrm{Fil}^j=\begin{cases} H^1_{dR}\left(X_\Gamma,\mathcal{V}_{n,n}\right) & \text{if } j\le 0,\\ H^1_{dR}\left(X_\Gamma,\mathcal{V}_n\right)\otimes H_j+M_k(\Gamma)\otimes\mathrm{Sym}^n H^1_{dR}(E) & \text{if } 1\le j\le n+1\\ M_k(\Gamma)\otimes H_{j-n-1} & \text{if } n+2\le j\le 2n+1,\\ 0 & \text{otherwise.}\end{cases}$$

In particular,

$$\operatorname{Fil}^{n+1} H^{2n+1}_{dR}(\mathcal{D}_n) \cong M_k(\Gamma) \otimes \operatorname{Sym}^n H^1_{dR}(E).$$

Proof. To prove semistability, we can extend the base to $\mathbb{Q}_p^{\mathrm{ur}}$. In this case the curve X is isomorphic to a disjoint union of Mumford curves, and hence it is semistable.

By Corollary 5.14, there is an isomorphism:

$$\bigcap_{x \in \mathcal{B}^\times} \ker\left(x - \operatorname{nrd}(x) \colon \mathcal{E}\left(\wedge^2 M_2\right) \to \mathcal{E}\left(\wedge^2 M_2\right)\right) \cong \mathcal{V}_2.$$

Applying Theorem 3.30 and functoriality, we see that the filtered (ϕ, N)-module

$$\mathrm{D}_{\mathrm{st},\mathbb{Q}_p^{\mathrm{ur}}}\left(H^1_{\mathrm{et}}(\overline{X_M}, \mathbb{L}_n) \otimes \operatorname{Sym}^n H^1_{\mathrm{et}}(E, \mathbb{Q}_p)\right)$$

is isomorphic to

$$H^1_{\mathrm{dR}}((X_M)_{\mathbb{Q}_p^{\mathrm{ur}}}, \mathcal{V}_n) \otimes \operatorname{Sym}^n H^1_{\mathrm{dR}}(E/\mathbb{Q}_p).$$

This isomorphism can then be descended to $\mathcal{D}_n$ by taking G-invariants.

Putting together Proposition 5.15 with Equation (5.1) we obtain the formula for the filtration. □

5.6 The p-adic Abel-Jacobi for $\mathcal{D}_n$

Consider the generalized Kuga-Sato motive $\mathcal{D}_n = (\mathcal{A}^m \times E^n, \varepsilon)$ as in Definition 5.10. The construction of the p-adic Abel-Jacobi map of Section 3.9 can be easily extended to the motive $\mathcal{D}_n$, by applying the projector at the appropriate places. This can be done for each realization, and we are interested in the de Rham realization of $\mathcal{D}_n$, which we have computed to be

$$H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}).$$

This fits in a short exact sequence as in Theorem 3.30:

$$0 \to H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \to H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n}) \to \bigoplus_{z \in S} (\mathcal{V}_{n,n})_z[1], \tag{5.2}$$

where S is a finite set of points in X_Γ lying in distinct residue classes, and U is the complement in X_Γ of S. In Chapter 6 we will define certain cycles on $\mathcal{A}^m \times E^n$ which are supported on a fiber above a point $P \in X$. These cycles are of codimension $n+1$, and therefore sending 1 to their cycle class yields a map:

$$K[n+1] \to (\mathcal{V}_{n,n})_z[1].$$

Pulling back the extension (5.2) we obtain another extension:

$$0 \to H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \to E \to K[n+1] \to 0.$$

Using Lemma 3.22, and the fact that the space

$$\mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})$$

is self-orthogonal, we obtain:

$$H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) / \mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \cong \left(M_{n+2}(\Gamma) \otimes \mathrm{Sym}^n H^1_{\mathrm{dR}}(E)\right)^\vee$$

The composition map will be denoted AJ_K:

$$\mathrm{AJ}_K \colon \mathrm{CH}^{n+1}(\mathcal{D}_n) \to \left(M_{n+2}(\Gamma) \otimes \mathrm{Sym}^n H^1_{\mathrm{dR}}(E)\right)^\vee.$$

In the next chapter we will compute this map in certain cases.

Chapter 6

Geometric interpretation of the values of $L_p'(f, K, s)$

This chapter contains the main result of this thesis. In Section 6.1 we obtain a formula for the values of the derivative of the p-adic L-function, in terms of Coleman integrals on the p-adic upper-half plane. In Section 6.2 we define a family of cycles on the motive $\mathcal{D}_n$ introduced in the previous chapter, whose image under the p-adic Abel-Jacobi map will be calculated. Finally, in Section 6.3 we calculate this image and explain the main result.

6.1 Values of $L_p'(f, K, s)$ in terms of Coleman integration on $\mathcal{H}_p$

In Chapter 4 we saw how to attach to an eigenform f of weight $n+2$ a distribution $\mu_{f,K}$ on the group $\widetilde{G} = \mathrm{Gal}(K_\infty/K)$. We defined L_p as:

$$L_p(f, K, s) := \int_{\widetilde{G}} \alpha^{s-\frac{n+2}{2}} d\mu_{f,K}(\alpha), \quad s \in \mathbb{Z}_p,$$

Recall also the p-adic group $G = \mathrm{Gal}(K_\infty/H_p) \cong K_{p,1}^\times$, which fits in the exact sequence:

$$1 \to G \to \widetilde{G} \to \mathrm{Gal}(H_p/K) \to 1,$$

where the right-most group if finite. If p is inert in K, then $L_p(f, K, j+1) = 0$ for all $0 \leq j \leq n$. One is then interested in the first derivative. Write first

$$L'_p(f, K, j+1) = \int_{\widetilde{G}} \log(\alpha)\alpha^{j-\frac{n}{2}} d\mu_{f,K}(\alpha) = \sum_{i=1}^{h} L'_p(f, \Psi_i, j+1),$$

where

$$L'_p(f, \Psi_i, j+1) := \int_G \log(\alpha)\alpha^{j-\frac{n}{2}} d\mu_{f,\Psi_i}(\alpha).$$

The following formula is a generalization of [BDIS02, Theorem 3.5] which, although immediate, is not currently present in the literature:

Theorem 6.1. *For all j with $0 \leq j \leq n$, the following equality holds:*

$$L'_p(f, \Psi, j+1) = \int_{\overline{z}_0}^{z_0} f(z)(z - z_0)^j (z - \overline{z}_0)^{n-j} dz,$$

where the right hand side is to be understood as a Coleman integral on $\mathcal{H}_p$.

Proof. Start by manipulating the expression for $L'_p(f, \Psi_i, j+1)$:

$$\begin{aligned} L'_p(f, \Psi_i, j+1) &= \int_G \log(\alpha)\alpha^{j-\frac{n}{2}} d\mu_{f,\Psi_i}(\alpha) \\ &= \int_{\mathbb{P}_1(\mathbb{Q}_p)} \log\left(\frac{x - z_0}{x - \overline{z}_0}\right)\left(\frac{x - z_0}{x - \overline{z}_0}\right)^{j-\frac{n}{2}} P_{\Psi_i}^{\frac{n}{2}}(x) d\mu_f(x) \\ &= \int_{\mathbb{P}_1(\mathbb{Q}_p)} \left(\int_{\overline{z}_0}^{z_0} \frac{dz}{z - x}\right)\left(\frac{x - z_0}{x - \overline{z}_0}\right)^{j-\frac{n}{2}} P_{\Psi_i}^{\frac{n}{2}}(x) d\mu_f(x) \end{aligned}$$

where the second equality follows from the change of variables $x = \eta_{\Psi_i}(\alpha)$ and the third from the definition of the logarithm. Note that from Lemma 4.3 it follows that:

$$\int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{\left(\frac{x-z_0}{x-\overline{z}_0}\right)^{j-\frac{n}{2}} P_{\Psi_i}^{\frac{n}{2}}(x)}{z - x} d\mu_f(x) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{\left(\frac{z-z_0}{z-\overline{z}_0}\right)^{j-\frac{n}{2}} P_{\Psi_i}^{\frac{n}{2}}(z)}{z - x} d\mu_f(x), \qquad (6.1)$$

since the expression:

$$\frac{\left(\frac{z-z_0}{z-\overline{z}_0}\right)^{j-\frac{n}{2}} P_{\Psi_i}^{\frac{n}{2}}(z) - \left(\frac{x-z_0}{x-\overline{z}_0}\right)^{j-\frac{n}{2}} P_{\Psi_i}^{\frac{n}{2}}(x)}{z - x}$$

is a polynomial in x of degree at most n. Using Equation (6.1), a change of order of integration and Proposition 4.4 we obtain:

$$\begin{aligned}
L'_p(f,\Psi_i,j+1) &= \int_{\mathbb{P}_1(\mathbb{Q}_p)}\int_{\overline{z}_0}^{z_0} \frac{dz}{z-x}\left(\frac{x-z_0}{x-\overline{z}_0}\right)^{j-\frac{n}{2}} P_{\Psi_i}^{\frac{n}{2}}(x)d\mu_f(x)\\
&= \int_{\overline{z}_0}^{z_0}\left(\int_{\mathbb{P}_1(\mathbb{Q}_p)}\frac{d\mu_f(x)}{z-x}\right)\left(\frac{z-z_0}{z-\overline{z}_0}\right)^{j-\frac{n}{2}} P_{\Psi_i}^{\frac{n}{2}}(z)dz\\
&= \int_{\overline{z}_0}^{z_0} f(z)\left(\frac{z-z_0}{z-\overline{z}_0}\right)^{j-\frac{n}{2}} P_{\Psi_i}^{\frac{n}{2}}(z)dz.
\end{aligned}$$

A justification for the validity of the change of the order of integration can be found in the proof given in [Tei90, Theorem 4]. □

6.2 Cycles on $\mathcal{D}_n$

6.2.1 Definition of the cycles

We will define a cycle class (or rather a family of them) in $\mathrm{CH}^{n+1}(\mathcal{D}_n)$, indexed by isogenies $\varphi\colon E \to E'$.

Let E be an elliptic curve with complex multiplication by $\mathcal{O}$. Recall that $\mathcal{O} = \mathrm{End}_{\mathcal{R}^{\max}}(E)$ is an order in an imaginary quadratic number field K. Consider an isogeny φ from E to another elliptic curve with complex multiplication E', of degree coprime to N^+M. If there is a level-N^+ structure and full level-M structure on E, we obtain the same structures on E', and also on $A' := E' \times E'$, by putting this level structures only on the first copy. Hence we obtain a point $P_{A'}$ in X_M, together with an embedding:

$$i_{A'}\colon A' \to \mathcal{A},$$

defined over K. Let Υ_φ be the cycle

$$\Upsilon_\varphi := ({}^t\Gamma_\varphi)^n \subseteq (E' \times E)^n \cong (A')^m \times E^n \hookrightarrow \mathcal{A}^m \times E^n,$$

where the last inclusion is induced from the canonical embedding $i_{A'}$, and Γ_φ is the graph of φ. Finally, apply the projector ε defined in Chapter 5.

Definition 6.2. The cycle

$$\Delta_\varphi := \varepsilon \Upsilon_\varphi \in \mathrm{CH}^{n+1}(\mathcal{D}_n)$$

is called the *generalized Heegner cycle* attached to the isogeny $\varphi\colon E \to E'$.

Remark 6.3. Since $H_p(\mathcal{D}_n)$ is concentrated in degree $2n+1$, the cycle Δ_φ is null-homologous. Therefore it makes sense to study the image of Δ_φ under any Abel-Jacobi map, in particular the p-adic version discussed in Section 3.9.

6.3 The main theorem

In this section we derive a formula for the image of the Abel-Jacobi map of generalized Heegner cycles, and we compute it in a special setting, thus giving a geometric interpretation of values of the derivative of the anticyclotomic p-adic L-function.

6.3.1 Set-up for the computation

We have defined a family Δ_φ of generalized Heegner cycles of dimension n, attached to isogenies $\varphi\colon E \to E'$. Let $\tilde{P}_{A'}$ be the point of X_M attached to A' through the isogeny φ. The cycle Δ_φ lies in the $(2n+1)$-dimensional scheme $\mathcal{A}^m \times E^n$, and so it has codimension $n+1$. As described in Section 5.6, we obtain a map:

$$\mathrm{AJ}_K\colon \mathrm{CH}^{n+1}(\mathcal{D}_n) \to \left(M_{n+2}(\Gamma) \otimes \mathrm{Sym}^n H^1_{\mathrm{dR}}(E)\right)^\vee,$$

Let ω_f be the differential form associated to a modular form $f \in M_{n+2}(\Gamma)$ as explained in Section 2.5. Fix $\alpha \in \mathrm{Sym}^n H^1_{\mathrm{dR}}(E)$. We want to compute the value:

$$\mathrm{AJ}_K(\Delta_\varphi)(\omega_f \wedge \alpha) \in \mathbb{C}_p.$$

Write $\mathrm{cl}_{P_{A'}}(\Delta_\varphi)$ for the cycle class of Δ_φ on the fiber above $P_{A'} \in X_M$:

$$\mathrm{cl}_{P_{A'}}(\Delta_\varphi) := \mathrm{cl}^{\mathcal{A}^m \times E^n}_{|\Delta_\varphi|}(\overline{\Delta_\varphi}) \in H^{2n+2}_{|\Delta_\varphi|}\left(\mathcal{A}^m \times E^n, \mathbb{Q}_p(n+1)\right).$$

Consider the sequence of filtered (ϕ, N)-modules:

$$0 \to H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \to H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n}) \xrightarrow{\mathrm{res}_{P_{A'}}} (\mathcal{V}_{n,n})_{P_{A'}}[1] \to 0. \tag{6.2}$$

Lemma 6.4. *The sequence of Equation* (6.2) *is exact.*

Proof. From its construction, the sequence is left-exact. The cokernel of the map $\mathrm{res}_{P_{A'}}$ injects in:

$$H^2_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \cong H^0_{\mathrm{dR}}(X_\Gamma, \mathcal{V}^\vee_{n,n}),$$

where the isomorphism is given by Serre duality. But $\mathcal{V}_{n,n}$ (and therefore $\mathcal{V}^\vee_{n,n}$) does not have Γ-invariants, since it is isomorphic to n copies of the standard representation of Γ. Therefore the cokernel of $\mathrm{res}_{P_{A'}}$ vanishes, as desired. □

Remark 6.5. Here is where we needed to exclude the case of weight 2, which would correspond to $n = 0$: in that case $\mathrm{res}_{P_{A'}}$ is always zero, since the restriction map induces an isomorphism

$$H^1_{\mathrm{dR}}(X_\Gamma) \to H^1_{\mathrm{dR}}(U).$$

We argued that the sequence in Equation (6.2) is exact. Its pull-back under the map $1 \mapsto \mathrm{cl}_{P_{A'}}(\Delta_\varphi)$ yields then a short exact sequence:

$$0 \to H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \to E \to K[n+1] \to 0$$

Lemma 3.22 ensures that if one forgets the filtration, the resulting sequence of (ϕ, N)-modules is split, say by a map $s_1 \colon K[n+1] \to E$. If we write $([\eta_1], x)$ for $s_1(1)$, note then:

1. For $([\eta_1], x)$ to be a splitting of the given extension, necessarily $x = 1$, and η_1 has to satisfy:

   (a) $N_U([\eta_1]) = 0$, and

   (b) $\Phi([\eta_1]) = p^{n+1}[\eta_1]$.

2. For $([\eta_1], 1)$ to be in

$$E = H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n}) \times_{(\mathcal{V}_{n,n})_{P_{A'}}[1]} K[n+1],$$

necessarily $\mathrm{res}_{P_{A'}}(\eta_1) = \mathrm{cl}_{P_{A'}}(\Delta_\varphi)$.

So let η_1 be a $\mathcal{V}_{n,n}$-valued 1-hypercocycle on U satisfying the conditions:

$$\mathrm{res}_{P_{A'}}(\eta_1) = \mathrm{cl}_{P_{A'}}(\Delta_\varphi), \qquad N_U([\eta_1]) = 0, \qquad \Phi(\eta_1) = p^{n+1}\eta_1 + \nabla G,$$

where G is a $\mathcal{V}_{n,n}$-valued rigid section over U. Consider next

$$[\eta_2] \in \mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n})$$

such that $\mathrm{res}_{P_{A'}}(\eta_2) = \mathrm{cl}_{P_{A'}}(\Delta_\varphi)$. This element exists as well, because it is the image of 1 under the splitting s_2 of Lemma 3.22. Let

$$[\widetilde{\eta}_\varphi] := [\eta_1 - \eta_2] \in H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n}).$$

Then $[\widetilde{\eta}_\varphi]$ can be extended to all of X_Γ. That is, there is $[\eta_\varphi] \in H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})$ such that

$$j_*([\eta_\varphi]) = [\widetilde{\eta}_\varphi] \equiv [\eta_1] \pmod{\mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n})}.$$

Write $[\eta_\varphi] = \iota(c) + t$, with $t \in \mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})$. Then one can replace $[\eta_2]$ by $[\eta_2] + t$ without changing the properties required for $[\eta_2]$, and hence we can assume that $[\eta_\varphi] = \iota(c)$ for some $c \in H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})$. Recall the maps I and P_U as appearing in diagram of Equation (3.6). We can prove:

Proposition 6.6. *With the previous notation, the following equality holds:*

$$\mathrm{AJ}_K(\Delta_\varphi)([\omega_f] \wedge \alpha) = \langle I([\omega_f] \wedge \alpha), P_U([\eta_2]) \rangle_\Gamma . \tag{6.3}$$

Proof. Using the definition of the Abel-Jacobi map and following the recipe given in Lemma 3.22, together with the pairings on $H^1_{\mathrm{dR}}(X, \mathcal{V}_{n,n})$, we obtain the following equality:

$$\mathrm{AJ}_K(\Delta_\varphi)([\omega_f] \wedge \alpha) = \langle [\omega_f] \wedge \alpha, [\eta_\varphi] \rangle_{X_\Gamma} .$$

The assumption of $\eta_\varphi = \iota(c)$ implies that $I(\eta_\varphi)$ is zero. So the right-hand side can be rewritten, using Equation (3.5) and the diagram of Equation 3.6, as:

$$- \langle I_{U,c}([\omega_f] \wedge \alpha), P_U([\eta_\varphi]) \rangle_\Gamma .$$

Now the result follows from observing that on U one can write $[\eta_\varphi] = [\eta_1] - [\eta_2]$, and that $P_U([\eta_1]) = 0$. $\square$

The following result computes a formula the right-hand side of Equation (6.3):

Theorem 6.7. *Let F_f be a Coleman primitive of ω_f, and let $z_0' \in \mathcal{H}_p$ be a point in the p-adic upper-half plane such that $P_A' = \Gamma_M z_0'$. Then:*

$$\langle I([\omega_f] \wedge \alpha), P_U([\eta_2]) \rangle_\Gamma = \left\langle F_f(z_0') \wedge \alpha, \mathrm{cl}_{z_0'}(\Delta_\varphi) \right\rangle_{V_{n,n}} .$$

Proof. Observe first that the spaces

$$\mathrm{Fil}^{n+1} H^1_{\mathrm{dR,c}}(U, \mathcal{V}_{n,n}) \text{ and } \mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n})$$

are orthogonal to each other. Therefore:

$$0 = \langle [\omega_f] \wedge \alpha, [\eta_2] \rangle_U = \langle P_{U,c}([\omega_f] \wedge \alpha), I_U([\eta_2]) \rangle_{\Gamma,U} - \langle I([\omega_f] \wedge \alpha), P_U([\eta_2]) \rangle_\Gamma ,$$

and hence we obtain:

$$\langle I([\omega_f] \wedge \alpha), P_U([\eta_2]) \rangle_\Gamma = \langle P_{U,c}([\omega_f] \wedge \alpha), I_U([\eta_2]) \rangle_{\Gamma,U} .$$

In order to show that the right-hand side of the previous equation coincides with

$$\left\langle F_f(z_0') \wedge \alpha, \mathrm{cl}_{z_0'}(\Delta_\varphi) \right\rangle_{V_{n,n}} ,$$

we use the explicit formula for the pairing as found in Proposition 3.36. Since η_2 has only nonzero residue at $P_{A'}$, the right hand side of the formula appearing in Proposition 3.36 reduces to pairing the primitive of $\omega_f \wedge \alpha$ with that residue, at the point corresponding to $P_{A'}$, yielding the desired formula. $\square$

6.3.2 Computation and application

From here on, let z_0 be a point in the p-adic upper-half plane $\mathcal{H}_p$ such that the orbit $P_A{:=}\Gamma z_0$ corresponds to $A = E \times E$ in X. Consider the map

$$g = (\mathrm{Id}_E^n, \varphi^n) \colon E^n \to E^n \times (E')^n,$$

Then Δ_φ is the projection via ε of the image $g(E^n)$. The functoriality of the cycle class map gives:

$$\big\langle F_f(z_0') \wedge \alpha, \mathrm{cl}_{z_0'}(\Delta_\varphi)\big\rangle_{V_{n,n}} = \langle \varphi^* F_f(z_0), \alpha\rangle_{V_n},$$

where now the pairing is the natural one in the stalk $V_n = (\mathcal{V}_n)_{z_0}$

Now, to compute this last quantity we first do it on a horizontal basis for

$$\mathrm{Sym}^n H^1_{\mathrm{dR}}(E/K) = (\mathcal{V}_n)_{z_0}.$$

Let $\{u, v\}$ be a horizontal basis for V_1, normalized so that $\langle u, u\rangle = \langle v, v\rangle = 0$ and $\langle u, v\rangle = -\langle v, u\rangle = 1$. This induces a basis $\{v_i{:=}u^i v^{n-i}\}_{0\le i\le n}$ of V_n.

Choose a global regular section ω^n in the lowest piece of the filtration which transforms with respect to Γ_M as of weight n, and scale it so that ω^n corresponds to $\sum_{i=0}^n (-1)^i \binom{n}{i} z^i v_i$, which is a regular section in $\mathrm{Fil}^n \mathcal{V}_n$. Note that $\omega^n = (u - zv)^n$, and so:

$$\langle \omega^n, v_i\rangle = \langle u - zv, u\rangle^i \langle u - zv, v\rangle^{n-i} = z^i.$$

We want to obtain a formula for the Coleman primitive F_f of ω_f. We proceed by differentiating the section $\langle F_f, v_i\rangle$ and using that $\{v_i\}$ is a horizontal basis:

$$d\langle F_f, v_i\rangle = f(z)\langle \omega^n, v_i\rangle dz = z^i f(z) dz.$$

One deduces the formula:

$$\langle F_f(\varphi(z_0)), \sum_i a_i v_i\rangle = \sum_{i=0}^n a_i \int_\star^{\varphi(z_0)} f(z) z^i dz.$$

From now on we concentrate on the stalk of $\mathcal{V}_n$ at z_0. The chosen regular differential ω yields a basis element ω_{z_0} for $H^1_{\mathrm{dR}}(E, K)$. Choose η_{z_0} in the span of $\Phi\omega_{z_0}$ such that

$$\langle \omega_{z_0}, \eta_{z_0} \rangle = 1.$$

This yields a basis for $\mathrm{Sym}^n H^1_{\mathrm{dR}}(E/K)$, namely $\{\omega_{z_0}^j \eta_{z_0}^{n-j}\}_{0 \leq j \leq n}$. We express this basis in terms of the horizontal basis $\{v_i\}$. By construction, we have:

$$\omega_{z_0}^j \eta_{z_0}^{n-j} = (z_0 - \overline{z}_0)^{j-n} \sum_i P_{i,j,n}(z_0, \overline{z}_0) v_i,$$

where

$$P_{i,j,n}(X, Y) = \sum_k \binom{j}{k} \binom{n-j}{k+i-j} (-1)^{n-i} X^k Y^{n-i-k}.$$

The following lemma can be proven by a simple calculation.

Lemma 6.8.

$$\sum_i P_{i,j,n}(z_0, \overline{z}_0) z^i = (z - z_0)^j (z - \overline{z}_0)^{n-j}.$$

The previous lemma gives a formula for a primitive for ω_f:

$$\left\langle \varphi^* F_f(z_0), \omega_{z_0}^j \eta_{z_0}^{n-j} \right\rangle = (z_0 - \overline{z}_0)^{j-n} \int_\star^{\varphi(z_0)} f(z)(z - z_0)^j (z - \overline{z}_0)^{n-j} dz.$$

Note that this equality is only defined up to an "integration constant" in $\mathbb{C}_p$, because in $\mathcal{H}_p$ the sheaf $\mathcal{V}_{n,n}$ is trivial. We can now prove the following application:

Theorem 6.9. *Let $\varphi \colon E \to E'$ be an isogeny of elliptic curves with level-N structure, and let $\overline{\varphi}$ be the morphism $E \to \overline{E}'$ obtained by from φ by applying to E' the nontrivial automorphism of K. Let $\Delta_\varphi^- {:=} \Delta_\varphi - \Delta_{\overline{\varphi}}$, and write $z_0' \in \mathcal{H}_p$ for the point in the p-adic upper-half plane which corresponds to the abelian surface $E' \times E'$. Then there exist a constant $\Omega \in K$ such that, for all $0 \leq j \leq n$:*

$$\mathrm{AJ}_K(\Delta_\varphi^-)(\omega_f \wedge \omega^j \eta^{n-j}) = \Omega^{j-n} L_p'(f, \Psi_{P_{E'}}, j+1).$$

Proof. Set Ω to be $z_0 - \overline{z}_0$. Using the previous results, we obtain first:

$$\mathrm{AJ}_K(\Delta_\varphi^-)(\omega_f \wedge \omega^j \eta^{n-j}) = \left\langle F_f(z_0'), \omega_{z_0}^j \eta_{z_0}^{n-j} \right\rangle - \left\langle F_f(\overline{z}_0'), \omega_{z_0}^j \eta_{z_0}^{n-j} \right\rangle,$$

Therefore, the second term in the previous displayed expression becomes

$$\left\langle F_f(\overline{z}_0'), \omega_{z_0}^{n-j}\eta_{z_0}^{j}\right\rangle = \Omega^{j-n}\int_{\star}^{\overline{z}_0'} f(z)(z-z_0)^j(z-\overline{z}_0)^{n-j}dz.$$

Combining this with the formula for $\left\langle F_f(z_0'), \omega_{z_0}^{j}\eta_{z_0}^{n-j}\right\rangle$ yields:

$$\mathrm{AJ}_K(\Delta_\varphi^-)(\omega_f\wedge\omega^j\eta^{n-j}) = \Omega^{j-n}\int_{\overline{z}_0'}^{z_0'} f(z)(z-z_0')^j(z-\overline{z}_0')^{n-j}dz.$$

The result follows now from Theorem 6.1. □

Note that the integral appearing in the previous theorem coincides with the value at $s=j+1$ of the derivative of the partial p-adic L-function described before. We obtain the following corollary:

Corollary 6.10. *Let H/K be the Hilbert class field of K, and consider a set of representatives $\{\Psi_1,\dots,\Psi_h\}$ for $\mathrm{emb}(\mathcal{O},\mathcal{R})$. For each Ψ_i, let P_i be the corresponding Heegner point on X_H, and let Δ_{Ψ_i} be the cycle corresponding to P_i. Define $\Delta^-{:=}\sum_i\Delta_{\Psi_i}^-$. There exists a constant $\Omega\in K$ such that for all $0\le j\le n$:*

$$\mathrm{AJ}_K(\Delta^-)(\omega_f\wedge\omega^j\eta^{n-j}) = \Omega^{j-n}L_p'(f,K,j+1).$$

Proof. This follows immediately from the expression given in Theorem 6.9 for the partial p-adic L-functions:

$$\mathrm{AJ}_K(\Delta_{\Psi_i}^-)(\omega_f\wedge\omega^j\eta^{n-j}) = \Omega^{j-n}L_p'(f,\Psi_i,j+1).$$

□

Remark 6.11. There is no canonical choice for the regular differential $\omega\in\Omega^1_{E/K}$. If a given ω is changed to $\omega_\lambda{:=}\lambda\omega$, with $\lambda\in K$, we obtain:

$$\mathrm{AJ}_K(\Delta^-)\left(\omega_f\wedge(\omega_\lambda^j\eta_\lambda^{n-j})\right) = \Omega^{j-n}\lambda^{2j-n}L_p'(f,K,j+1).$$

Note in particular that the formula at the central point $j=n/2$ does not depend on the choice of the basis of the differential form ω.

The formula in Corollary 6.10 is the type of result that we were looking for: it relates the values of the derivative of $L_p(f,K,s)$ evaluated at the integer points $s=1,\dots,n+1$ to the image under the p-adic Abel-Jacobi map of a the generalized Heegner cycle Δ^- which is supported above a CM cycle of X.

Chapter 7

Future directions

This thesis opens many directions to be explored in the future. Here we present some of them, ordered by increasing generality.

First of all, the motive $\mathcal{D}_n$ should be studied more carefully, as well as the possible families of nontrivial CM cycles on it. The case of odd weight should yield similar results to those presented in this thesis.

Next, one would like to compute the p-adic Abel-Jacobi image of the cycles Δ_φ (instead of just the cycles Δ_φ^-), and this should be compared to their p-adic heights.

The understanding of the previous questions would shed light on the problem of finding nontrivial families of cycles in the Chow group of certain algebraic varieties, which is a widely studied problem.

Moreover, the theory of p-adic L-functions has grown considerably, but lacks a unified point of view. A much more ambitious project would be to find an elegant theory that encapsulated all these different instances of L-functions, and which made clear which interpolation properties each of them satisfies.

In the remainder of this chapter we give some more detail on the immediate directions that one can pursue from the work presented in this thesis.

7.1 Modular forms of odd weight

If one wants to extend the results of this thesis to modular forms of odd weight, one needs to work on the two sides of the equation: on the one hand, the motive $\mathcal{D}_n$ needs to be extended to odd n. This has partially been done in this thesis, but its realizations need to be completely understood before being able to compute the p-Abel-Jacobi map. On the other hand, the anti-cyclotomic p-adic L-function as defined in [BDIS02] does not contemplate possible nebentypes, thus restricting the construction to even-weight modular forms. One should give a more general construction which allowed nebentypes, and these should be incorporated in the definition of the motive $\mathcal{D}_n$ as well.

7.2 More general cycles

It would be interesting to compute the image of the Abel-Jacobi map for arbitrary cycles on $\mathcal{D}_n$ supported on CM-points of the Shimura curve. Finding explicit formulas for cycles supported at a single point is a more difficult problem than what has been treated in this thesis, since some of the techniques used above cannot be used. However similar computations have been carried over in the split case in [BDP09], and one should be able to adapt them to the setting of this work. A careful choice of these cycles will yield formulae for other p-adic L-functions.

The underlying philosophy is that all these special values should come from geometric data, such as algebraic cycles and Abel-Jacobi map.

7.3 Relation with p-adic heights

The focus of this thesis has been put on the study of the relation of the anti-cyclotomic p-adic L-function to the image of certain cycles under the Abel-Jacobi map. More germane to the original Gross-Zagier formulas would be to instead relate the values of the L-function to p-adic analogues of the Néron-Tate heights of the cycles. The investigation of the relation of the p-adic Abel-Jacobi map appearing in this thesis with the p-adic height pairings as in the articles of Gross and Coleman [CG89] and of Nekovář ([Nek93] and [Nek95]) will certainly be fruitful.

In particular, one should be able to compute the p-adic heights of the cycles constructed in this thesis, or of generalizations of them, and relate them to values of the anti-cyclotomic p-adic L-function, or of its derivative. One could also compute archimedean formulas for the heights that we have defined, and try to relate those to the classical L-functions.

7.4 Nontrivial families of cycles

In [Bes95], the author defines a family of cycles on a variety similar to what we studied in this project, and proves that they span an infinite-dimensional subspace of the Griffiths group of the variety. The author uses the complex Abel-Jacobi map to show the non-triviality of these cycles.

We hope to be able to obtain a similar result by means of the p-adic Abel-Jacobi map. Experiments should be carried out in order to collect evidence supporting this nontriviality statement, and this can be easily done using explicit evaluation of the p-adic measure attached to a modular form f.

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Index