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Doctoral Thesis

On Twisted Triple Products and the Arithmetic of Elliptic Curves

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Abstract

The unifying theme of the thesis is the arithmetic of elliptic curves, more specifically the conjecture of Birch and Swinnerton-Dyer and its generalizations. This subject leverages different aspects of number theory, arithmetic geometry and representation theory, including automorphic representations, Shimura varieties, p-adic modular forms and p-adic L-functions. Simply put, the BSD-conjecture claims the equality of the analytic and the algebraic rank of an elliptic curve. A tantalizing aspect of the conjecture is its offering very subtle information – that goes beyond the Sato-Tate conjecture – on how the size of rational points influences, and is influenced by, the distribution of the number of *mod* ℓ points, for all rational primes ℓ. In the past 50 years gifted mathematicians have done gorgeous work establishing the first cases of the BSD-conjecture ([CW77], [GZ86], [Kol88]), and the intertwining of p-adic methods with Euler systems has become more and more widespread ([Kat04], [LLZ14], [DR17a]), providing a deeper understanding of the objects involved. The work in this thesis is part of this growing thread of investigation.

The idea of this project rests on the exploration of a twisted variant of the setting studied in [DR14],[DR17a] and [DR17b], with an emphasis on understanding the relation between twisted triple product L-functions and Hirzebruch-Zagier cycles. After a brief introduction, Chapter 1 [For19] investigates twisted triple product L-functions with applications to arithmetic statistics. By solving a class of Galois embedding problems over totally real fields, it demonstrates that the analytic rank of a modular elliptic curve of odd non-square conductor grows over a positive proportion of quintic extensions.

Chapter 2 [For17] comprises the construction of twisted triple product p-adic L-functions over totally real fields for nearly ordinary Hida families. When $L/\mathbb{Q}$ is a real quadratic field and p splits in L, a p-adic Gross-Zagier formula expresses values of the p-adic L-functions that are outside the range of interpolation, in terms of the syntomic Abel-Jacobi map of generalized Hirzebruch-Zagier cycles. Novel ideas appear in the treatment of compactifications of Kuga-Sato varieties using Wildeshaus' work on the interior motive [Wil12], and in the choice of the Coleman primitive for the evaluation of the syntomic regulator.

The final chapter of the dissertation features a preview of a work in progress joint with Zhaorong Jin (Princeton) bringing to fruition the preceding pieces. When p splits in the real quadratic field L, we expect to provide a geometric construction of twisted triple product p-adic L-functions using big Hirzebruch-Zagier classes. This geometric construction, together with [For19], should lead to new instances of the BSD-conjecture in rank 0 for rational elliptic curves over certain quintic number fields whose normal closure has Galois group S_5.

Abrégé

Le thème unificateur de la présente thèse est l'arithmétique des courbes elliptiques, et de manière plus précise la conjecture de Birch et Swinnerton-Dyer ainsi que ses généralisations. Ce sujet fait interagir plusieurs aspects de la théorie des nombres, la géometrie arithmétique et la théorie de la représentation, y compris les représentations automorphes, les variétés de Shimura, les formes modulaires p-adiques et les fonctions L p-adiques. Dit de manière simple, la conjecture BSD prédit l'égalité des rangs analytiques et algébriques d'une courbe elliptique. Un aspect captivant de la conjecture est le fait qu'elle offre de l'information subtile – allant plus loin que celle offerte par la conjecture de Sato-Tate – permettant d'expliquer comment le nombre de points rationnels influence, et est influencé par, la distribution du nombre de points modulo ℓ, pour tout nombre premier ℓ. Durant les dernières 50 années, de talentueux mathématiciens ont établis les premiers cas connus de la conjecture BSD ([CW77], [GZ86], [Kol88]), et la combinaison de méthodes p-adiques avec la théorie des systèmes d'Euler est devenue de plus en plus répandue ([Kat04], [LLZ14], [DR17a]), ouvrant les portes à une compréhension plus profonde des objets en question. Le travail de la présente thèse s'insère dans le courant de ces idées.

L'idée derrière le projet repose sur l'étude d'une variante tordue de la situation étudiée dans [DR14],[DR17a] and [DR17b], en mettant l'accent sur le rapport entre les fonctions L associées au triple produit tordu et les cycles de Hirzebruch-Zagier.

Après une brève introduction, le chapitre 1 [For19] étudie les fonctions L associées au triple produit tordu et leurs applications en statistique arithmétique. En résolvant une suite de problèmes de plongement sur des corps totalement réels, il est démontré que le rang analytique d'une courbe elliptique modulaire de conducteur impair et non égal à un carré parfait croît sur une proportion positive d'extensions quintiques.

Le chapitre 2 [For17] comprend la construction de fonctions L p-adiques associées au triple produit tordu sur des corps totalement réels pour des familles de Hida presque ordinaires. Lorsque $L/\mathbb{Q}$ est un corps quadratique réel et p est un nombre premier déployé dans L, une formule de Gross-Zagier p-adique exprime les valeurs des fonctions L p-adiques en dehors de la région d'interpolation classique en termes des images par l'application d'Abel-Jacobi syntomique de cycles de Hirzebruch-Zagier généralisés. Des idées nouvelles interviennent lors du traitement de la compactification de variétés de Kuga-Sato faisant recourt au travail de Wildeshaus sur le motif intérieur [Wil12], ainsi que lors du choix de la primitive de Coleman pour l'évaluation du régulateur syntomique.

Le chapitre final de la thèse donne un aperçu sur un travail en cours en collaboration avec Zhaorong Jin (Princeton), qui représente l'aboutissement des travaux des deux chapitres précédents. Lorsque p est déployé dans le corps quadratique réel L, nous prévoyons donner une construction géometrique des fonctions L p-adiques associées au triple produit tordu faisant usage de grandes classes de Hirzebruch-Zagier. Cette construction géometrique, combinée avec [For19], donnera lieu à de nouveaux cas de la conjecture BSD en rang 0 pour des courbes elliptiques rationelles pour certains corps de nombres S_5-quintiques.

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Finally, my love goes to my family, my friends Alberto, Alex, Dante and Mattia, and my girlfriend Giulia, because without them even mathematics wouldn't make any sense.

Preface

Why algebraic number theory? I like to think about myself as a number theorist of the algebraic clan. I have not always thought about myself this way. At the end of my undergraduate studies I was attracted to abstract and geometric-flavored mathematics, and thus I had decided I wanted to do *derived algebraic geometry*[1]. Afterwards, the opportunity to earn Masters degrees in Montréal and Paris made enroll in the ALGANT[2] program. What I did not know at the time was that ALGANT in Montréal meant NT and that Number Theory is actually a subject one can do research on! The first semester in Montréal was a cultural shock: I thought I knew what mathematics was all about, but people there were always talking about elliptic curves[3], modular forms and L-functions, all words I had never heard of. Luckily, after a while I decided to give it a chance and I became fascinated: I discovered that Number Theory is a field of research that satisfies my attraction to abstraction, generality and all-encompassing powerful results as well as the desire for concrete statements and examples that are easy to share with friends. Number Theory welcomes and needs tools and insights from all fields of mathematics, and it is incredibly satisfying to find connections between seemingly unrelated concepts.

Why elliptic curves? The quest for understanding the arithmetic of elliptic curves has been guided my efforts since the beginning of my doctorate. I chose this topic for simple reasons: the open problems are easy to appreciate, and I love the tools and the techniques that have been most successful in understanding them so far. Furthermore, classical elementary problems in Number Theory, like Fermat's Last Theorem and the Congruent Number Problem, can be better understood using elliptic curves. Indeed, a non-trivial rational solution of the Fermat equation $x^n + y^n = z^n$ for $n \geq 3$ produces an elliptic curve that cannot exist, while an integer m is *congruent*, i.e. it is the area of a right triangle with rational sides, if and only if the elliptic curve $y^2 = x^3 - m^2 x$ has algebraic rank greater than or equal to one.

I recently learned how powerful elliptic curve cryptography (ECC) is, and I will defintely use this application to explain what I work on to people. Cryptographic protocols are used to secure our transactions and the protocol's key length is a crucial parameter that determines processing performance and security level. In general, long keys provide high security, but slow down encryption and decryption of data. Hence, for commercial purposes one looks for the best trade-off: protocols that offer high security with short keys. To have a feeling of how good elliptic curve cryptography is, let us compare the famous RSA protocol, based on the factorization of large integers, and an encryption based on elliptic curves using a 228-bits key. It turns out that the energy required to break a 228-RSA code wouldn't even boil a teaspoon of water; however, to break a 228-ECC code with the current approaches one would need the energy to boil all the water on the planet!

[1] Even though I still do not know what it is.
[2] **AL**gebra, **G**eometry **A**nd **N**umber **T**heory.
[3] I still remember thinking how lame it was to be studying "just" curves!

Contents

Introduction

Let K be a number field. A K-rational elliptic curve $E_{/K}$ is a genus one, smooth and projective curve over K with a choice of K-rational point $P_0 \in E(K)$. Every elliptic curve over K can be described by a Weierstrass equation

$$E_{/K} \;:\quad y^2 = x^3 + Ax + B \qquad A, B \in \mathcal{O}_K \tag{1}$$

where $\Delta_{E/K} = -16(4A^3 + 27B^2) \neq 0$ and P_0 is taken to be the point at infinity in the projectivization. One unique feature of elliptic curves is their being endowed with a commutative group structure which can be geometrically defined by stating that three points on the curve sum to zero if and only if they belong to the same line.

Rational points on curves

The understanding that a curve's topology strongly influences its arithmetic dates back at least to 1922 when Mordell formulated his renowned conjecture. The $\mathbb{C}$-points of a smooth and projective curve defined over a number field K constitute a Riemann surface. Its genus, i.e. the number of doughnut holes, determines a trichotomy in the structure of K-rational points. Firstly, Hilbert and Hurwitz [HH90] proved that the K-rational points of a genus zero curve can be explicitly parametrized so there are either no such points or infinitely many. Secondly, for curves of genus greater than or equal to two, Faltings proved Mordell's intuition right [Fal83], that is to say that those curves can only have finitely many K-rational points. It is still an open and actively researched problem to make Faltings' theorem effective. Thirdly, genus one curves can exhibit both finitely many or infinitely many K-rational points. In this case, the effective enumeration of such points relies on the BSD-conjecture.

Even though the set of K-rational points $E(K)$ of an elliptic curve can be infinite, it is always a finitely generated abelian group. Simply put, finitely many solutions of equation (1) suffice to compute all the other ones by means of the geometric summation law described above. Therefore, there is an isomorphism of abstract groups

$$E(K) \cong \mathbb{Z}^r \oplus E(K)_{\text{tor}} \quad \text{for some } r \in \mathbb{N},$$

allowing the definition of the algebraic rank of $E_{/K}$ as $r_{\text{alg}}(E/K) := r$, the maximal number of linearly independent K-rational points on $E_{/K}$. The torsion subgroup $E(K)_{\text{tor}}$ is relatively well-understood. Mazur [Maz77] and Merel [Mer96] showed that there are only finitely many possibilities for the torsion subgroup of a K-rational elliptic curve given K. Moreover, there has been recent interest in providing complete lists of the possible torsion subgroups in a number of cases ([Dan+18], [Cho+18]), both for the natural appeal of the question and for applications to modularity of elliptic curves over general number fields [Tho19].

One big remaining mystery of the arithmetic of elliptic curves is the algebraic rank, which can divide opinions even when $K = \mathbb{Q}$. For instance, how large can the rank of a rational elliptic curve be? Both options, boundedness or unboundedness of ranks, have good arguments in their favor. On the one hand, there has been a recent revival of the boundedness hypothesis in [Par+18], where a reasonable model predicts that all but finitely many rational elliptic curves have rank ≤ 21. On the other hand, unboundedness of ranks has been shown for elliptic curves over function fields ([TS67], [Ulm02]) providing a good reason to remain undecided.

BSD-conjecture

The question of effectively determining the set of K-rational points of an elliptic curve has a widely accepted conjectural answer: the BSD-conjecture [Ste]. The formulation of the conjecture dates back to the 1960s, when Birch and Swinnerton-Dyer experimentally noticed a remarkable relation between the algebraic rank of an elliptic curve and a multiplicative average of its number of points over different finite fields.

More precisely, given a K-rational elliptic curve $E_{/K}$, for every prime $\mathcal{O}_K$-ideal $\mathfrak{q}$ such that $\mathfrak{q} \nmid \Delta_{E/K}$, one can reduce the Weierstrass equation of E modulo $\mathfrak{q}$ to obtain an elliptic curve $\bar{E}_{/\mathbb{F}_\mathfrak{q}}$ over the residue field $\mathbb{F}_\mathfrak{q} = \mathcal{O}_K/\mathfrak{q}$. Every such curve is much simpler than the original one: for instance, there is a polynomial time[4] algorithm[5] [Sch85] that computes the number $N_\mathfrak{q}(E) := |\bar{E}(\mathbb{F}_\mathfrak{q})|$ of points modulo $\mathfrak{q}$. Heuristically, one could expect that a large algebraic rank would force the sets $\bar{E}(\mathbb{F}_\mathfrak{q})$ to be larger on average because there are natural reduction maps $E(K) \to \bar{E}(\mathbb{F}_\mathfrak{q})$ for all $\mathfrak{q} \nmid \Delta_{E/K}$. Birch and Swinnerton-Dyer turned this heuristic into a quantitative mathematical statement and tested it successfully on a computer. They noticed that an appropriately normalized product of $N_\mathfrak{q}(E)$'s grows as the $r_{\mathrm{alg}}(E/K)$-th power of log,

$$\prod_{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{q}) \leq T} \frac{N_\mathfrak{q}(E)}{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{q})} \quad \overset{?}{\sim}_{+\infty} \quad \big(\log T\big)^{r_{\mathrm{alg}}(E/K)}. \tag{2}$$

Actually, (2) implies what is usually called the BSD-conjecture these days [Gol82], but it has the advantage of being more immediate and easier to appreciate. The modern point of view on the BSD-conjecture relies on the analytic properties of an L-function that can be associated to an elliptic curve as follows. The quantities $\mathsf{a}_\mathfrak{q}(E) := \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{q}) + 1 - |\bar{E}(\mathbb{F}_\mathfrak{q})|$ for all $\mathfrak{q} \nmid \Delta_{E/K}$ can be packaged into a generating series

$$L_S(E/K, s) = \prod_{\mathfrak{q} \nmid \Delta_{E/K}} \Big(1 - \mathsf{a}_\mathfrak{q}(E) \cdot \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{q})^{-s} + \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{q})^{1-2s}\Big)^{-1}$$

which converges for $\mathrm{Re}(s) \gg 0$ and defines a holomorphic function on a half-plane. After an appropriate "completion", the function $L_S(E/K, s)$ is expected to admit holomorphic continuation to the whole complex plane and a functional equation $s \mapsto 2 - s$. Therefore, assuming holomorphic continuation, it is possible to define the analytic rank of $E_{/K}$ as $r_{\mathrm{an}}(E/K) := \mathrm{ord}_{s=1} L_S(E/K, s)$ and phrase the BSD-conjecture as predicting the equality of the two ranks:

[4] In the variable $\log \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{q})$.

[5] Schoof discovered his algorithm after the formulation of the BSD-conjecture. Birch and Swinnerton-Dyer performed their computations on rational CM elliptic curves whose number of points mod $\mathfrak{q}$ can be quickly computed using Hecke characters.

$$r_{\text{an}}(E/K) \stackrel{?}{=} r_{\text{alg}}(E/K). \tag{3}$$

State of the art

The most general result towards the BSD-conjecture follows from the methods of Gross-Zagier [GZ86] and Kolyvagin [Kol88], as extended to totally real fields by Shouwu Zhang and his school [Zha01]. The theorem states that if $E_{/F}$ is a modular elliptic curve over a totally real field F such that either $E_{/F}$ has at least one prime of multiplicative reduction or $[F : \mathbb{Q}]$ is odd, then

$$r_{\text{an}}(E/F) \in \{0,1\} \implies r_{\text{an}}(E/F) = r_{\text{alg}}(E/F). \tag{4}$$

It is important not to forget that the modularity of $E_{/F}$ is the only known way to access the analytic properties of the L-function $L(E/F,s)$. This way, it becomes natural to expect that cycles on Shimura varieties will play a role in any strategy to establish the BSD-conjecture.

The three pillars of this approach are: (i) the existence of a non-constant map $X_{/F} \to E_{/F}$ from a Shimura curve to the elliptic curve, (ii) the existence of CM points on $X_{/F}$ with their significance for Selmer groups, and (iii) formulas for the derivative of certain base-change L-functions of $E_{/F}$ in terms of the height of images of CM points, called Heegner points. These three items are at the same time the strengths and the limitations of the most effective strategy developed so far to prove BSD. Firstly, the strong form of geometric modularity in (i) can only be realized for certain elliptic curves over totally real fields, hence the first pillar topples down right away when considering elliptic curves defined over general number fields[6]. Secondly, suppose we fixed an elliptic curve over a totally real field F and we took a finite extension K/F; what could then one say about the BSD-conjecture for $E_{/K}$? In this case, even though there could still be a modular parametrization, one would lack a way to produce points over extensions of a general K. Indeed, Heegner points are defined over dihedral extensions of F, and therefore miss all non-solvable extensions. Finally, what if we contented ourselves with tackling the BSD-conjecture over totally real fields? In this case all the pillars could still be standing, but the last two would have nothing to say about higher rank situations. The striking feature of CM points is their explicit relation to *first* derivative of L-functions; thus, as soon as the rank is greater than or equal to two, they become torsion.

The equivariant BSD-conjecture

The line of inquiry followed in this thesis is motivated by the equivariant refinement of the BSD-conjecture. Let F be a totally real field and K/F a finite Galois extension. For any elliptic curve $E_{/F}$, the Galois group $G(K/F)$ naturally acts on the $\mathbb{C}$-vector space $E(K) \otimes \mathbb{C}$ generated by the group of K-rational points. Since complex representations of finite groups are semisimple, the representation $E(K) \otimes \mathbb{C}$ decomposes into a direct sum of ϱ-isotypic components $E(K)^{\varrho} = \mathrm{Hom}_{G(K/F)}(\varrho, E(K) \otimes \mathbb{C})$, indexed by irreducible representations $\varrho \in \mathrm{Irr}\big(G(K/F)\big)$, each with its appropriate multiplicity. It then becomes natural to define the algebraic rank of E with respect to some ϱ as

$$r_{\text{alg}}(E, \varrho) := \dim_{\mathbb{C}} E(K)^{\varrho}.$$

[6] However, as Longo showed [Lon06], a lot can be proved using congruences for general rank zero elliptic curves over totally real fields.

On the analytic side, for any $\varrho \in \mathrm{Irr}\big(G(K/F)\big)$ one can define a twisted L-function $L(E,\varrho,s)$ as the L-function associated to the Galois representation $\mathrm{V}_p(E)(1) \otimes \varrho$ of the absolute Galois group of F. The analytic rank of E with respect to some ϱ is then defined as

$$r_{\mathrm{an}}(E,\varrho) := \mathrm{ord}_{s=1} L(E,\varrho,s).$$

The Artin formalism of L-functions can be used to show that the BSD-conjecture for an elliptic curve $E_{/F}$ base-changed to K should be equivalent to the equality of ranks

$$r_{\mathrm{alg}}(E,\varrho) \stackrel{?}{=} r_{\mathrm{an}}(E,\varrho) \quad \text{for all } \varrho \in \mathrm{Irr}\big(G(K/F)\big).$$

The advantage of this point of view resides in the fact that it splits the BSD-conjecture into more manageable pieces. Furthermore, when the considered representation ϱ arises from automorphic forms, it becomes easier to identify the right framework that should be explored in order to prove the equality of the ranks.

Indeed, Bertolini, Darmon and Rotger [BDR15] proved new instances of the equivariant BSD-conjecture in rank zero for rational elliptic curves in the case of ϱ an Artin representations corresponding to a weight one elliptic cuspform. When ϱ is the tensor product of two Artin representations attached to weight one elliptic cuspforms, Darmon and Rotger [DR17a] established the first cases of the BSD-conjecture in rank zero for rational elliptic curves over non-solvable extensions of $\mathbb{Q}$. In the same paper, Darmon and Rotger pushed their ideas further to provide compelling evidence that generalized Kato classes – constructed from diagonal cycles on triple products of modular curves – can be used to access scenarios in which the involved elliptic curves have rank two. It is very exciting to read the recent preprint [CH18], where Castella and Hsieh gather even more evidence in support of the relation between generalized Kato classes and elliptic curves of rank two over quadratic imaginary fields.

The theme of p-adic deformation

Recently Skinner-Urban [SU14], Xin Wan [Wan12] and Skinner [Ski14] were able to establish the first instances of the opposite implication of the BSD-conjecture

$$r_{\mathrm{alg}}(E/\mathbb{Q}) \in \{0,1\} \quad \& \quad \#\text{Ш}(E/\mathbb{Q}) < +\infty \implies r_{\mathrm{an}}(E/\mathbb{Q}) = r_{\mathrm{alg}}(E/\mathbb{Q}), \tag{5}$$

for any rational elliptic curve $E_{/\mathbb{Q}}$ in the rank zero case, and for semistable rational elliptic curves with either at least one odd prime of non-split multiplicative reduction or at least two odd primes of split multiplicative reduction, in the rank one case. The strength of the three works resides in the use of Iwasawa theory[7]: for the rank zero case the non-vanishing of a p-adic L-function directly implies that the analytic rank is zero. For the rank one case, Iwasawa theory is used more subtly to establish that a Heegner point is non-torsion, so that the Gross-Zagier formula itself implies that the analytic rank is one. Interestingly, implication (5) heavily relies on p-adic methods that were not necessary for the method of Gross-Zagier and Kolyvagin (4), and indeed, in recent years, the theme of p-adic deformation has become dominant in the field.

[7]The idea of exploiting p-adic variation of arithmetic objects can be traced back to the seminal work of Coates and Wiles [CW77] on rank zero CM elliptic curves.

On p-adic Gross-Zagier formulas

As we have seen, the relation between the analytic and the algebraic rank of elliptic curves does not seem to be easy to establish in the general case. The insight behind recent successful ideas is that p-adic methods can serve as the third leg of the stool: special values of automorphic L-functions can be interpolated, determining p-adic meromorphic functions which sometimes can be shown to arise from algebraic cycles. This double embodiment of p-adic L-functions, both analytic and geometric, can be used to transform information on special L-values into information on the arithmetic of algebraic varieties over number fields. One crucial step in exposing the dual nature of p-adic L-functions resides in the proof of p-adic Gross-Zagier formulas, i.e. the evaluation of p-adic L-functions outside their range of interpolation in terms of global arithmetic invariants. With the benefit of hindsight, we can trace back the origin of these formulas to Leopoldt's p-adic class number formula:

Theorem 0.0.1. *([Was97], Theorem 5.18) Let χ be an even non-trivial Dirichlet character of conductor $\mathfrak{f}$ and ζ a primitive $\mathfrak{f}$-th root of unity, then*

$$L_p(1,\chi) = -\left(1 - \frac{\chi(p)}{p}\right) \frac{\tau(\chi)}{\mathfrak{f}} \sum_{a=1}^{\mathfrak{f}} \chi(a)^{-1} \log_p(1-\zeta^a),$$

where $\tau(\chi) = \sum_{a=1}^{\mathfrak{f}} \chi(a)\zeta^a$ is a Gauss sum.

The range of interpolation of Kubota-Leopoldt's p-adic L-function $L_p(s,\chi)$ is the set of non-positive integers, and the formula expresses the value at $s = 1$ in terms of p-adic logarithms of cyclotomic units. Since then, several instances of such formulas have had a profound impact in the understanding of the arithmetic of elliptic curves. For example, the BDP-formula [BDP13] by Bertolini, Darmon and Prasanna, generalizing previous work of Rubin [Rub92], is a cornerstone of Skinner's proof of (5) in the rank one case.

The p-adic Gross-Zagier formula proved in this thesis is part of a program initiated by the work Bertolini, Darmon and Rotger, whose aim is to explore new instances of the equivariant BSD-conjecture: it uses higher rank groups and the corresponding higher dimensional Shimura varieties to go beyond the arithmetic of elliptic curves over dihedral extensions of totally real fields.

Chapter 1

Growth of analytic rank over quintic extensions

We have seen how the BSD-conjecture can be thought of as highlighting a statistical property of a single elliptic curve viewed over an infinite collection of finite finite fields, but what happens when we consider families? A very natural question in this context is to wonder about the "probability" that a given rational elliptic curve has rank $n \in \mathbb{N}$. The general belief is that the distribution of ranks should be the minimal one allowed by the distribution of signs of the functional equations [Bek+07]; for instance, it is conjectured that 50% of all rational elliptic curves should have rank 0 and 50% should have rank 1. Recently, Bhargava and Shankar [BS13] established that at least 83.75% of all rational elliptic curves have rank 0 or 1.

In this chapter we consider modular elliptic curves $E_{/F}$ over a totally real field F and we to try to shed some light on the distribution of analytic ranks in the family of base-changes of $E_{/F}$ to quintic extensions of F. We were led to consider this setting by studying twisted triple product L-functions attached to $E_{/F}$ and a Hilbert cuspform of parallel weight one for a totally real quadratic extension L/F. Inspired by [DR17a], we hoped the twisted setting could help us access the arithmetic of elliptic curves over non-solvable extensions in settings when the sign of the functional equation is generically odd. We denote by $G_5(E_{/F}; X)$ the number of quintic extensions K of F such that the norm of the relative discriminant is at most X and the analytic rank of E grows over K, i.e., $r_{\mathrm{an}}(E/K) > r_{\mathrm{an}}(E/F)$. We can then prove the following theorem.

Theorem 1.0.1. *If the modular elliptic curve $E_{/F}$ has odd conductor and at least one prime of multiplicative reduction, then $G_5(E_{/F}; X) \asymp_{+\infty} X$, i.e., there are constants $c_1, c_2 > 0$ such that $c_1 X \leq G_5(E_{/F}; X) \leq c_2 X$ for X large enough.*

As Bhargava, Shankar and Wang [BSW15] showed that the number of quintic extensions of F with norm of the relative discriminant at most X is asymptotic to $c_{5,F}X$ for some positive constant $c_{5,F}$, our result exposes the growth of the analytic rank as a very common circumstance over quintic extensions. Note that the BSD-conjecture implies that inequality $r_{\mathrm{an}}(E/K) \geq r_{\mathrm{an}}(E/F)$[1] always holds and that the strict inequality has to be explained by the presence of a non-torsion point in $E(K)$ linearly independent from $E(F)$. Therefore, we like to think about Theorem 1.0.1 as evidence for the fact that there should be a systematic way to produce non-torsion points over certain S_5-quintic extensions of totally real fields, in analogy with the case of Heegner points over CM fields.

[1]We establish the inequality unconditionally for a positive proportion of quintic fields and modular elliptic curves with odd conductor.

Theorem 1.0.1 is compatible with the conjectures in [DFK04], [DFK07] about the growth of the analytic rank of rational elliptic curves over cyclic quintic extensions. In those works growth is predicted to be a rare phenomenon, cyclic quintic extensions form a thin subset of all quintic extensions: the counting function of cyclic quintic fields is asymptotic to $\alpha X^{1/4}$ for some positive constant $\alpha > 0$ [Wri89]. Finally, we would like to remark that all elliptic curves over a totally real field F with $[F : \mathbb{Q}] \leq 2$ are modular and that, in general, all but finitely many $\overline{\mathbb{Q}}$-isomorphism classes of elliptic curves over a totally real field F are known to be modular ([Wil95], [TW95], [Bre+01], [FLHS15]) making our result widely applicable.

Strategy of the proof

Let F be a totally real field, K/F an S_5-quintic extension with a totally complex Galois closure J such that the subfield of J fixed by A_5 is a totally real quadratic extension L/F. For $E_{/F}$ a modular elliptic curve corresponding to a primitive Hilbert cuspform f_E of parallel weight two, the key idea of the paper is to interpret the ratio of L-functions $L(E/K, s)/L(E/F, s)$ as the twisted triple product L-function attached to f_E and a certain Hilbert cuspform g over L of parallel weight one. Then, the sign $\varepsilon_{K/F}$ of the functional equation of $L(E/K, s)/L(E/F, s)$ is determined by the splitting behaviour in K of the primes of multiplicative reduction of $E_{/F}$, and we can prove the existence of a positive proportion of quintic extensions K/F for which $\varepsilon_{K/F} = -1$ by invoking [BSW15].

The twisted triple product L-function attached to a modular elliptic curve $E_{/F}$ and a cuspform g of parallel weight one over a totally real quadratic extension L/F is the L-function $L(E, \otimes\text{-}\mathrm{Ind}_L^F(\varrho_g), s)$. Here, $\otimes\text{-}\mathrm{Ind}_L^F(\varrho_g)$ denotes the tensor induction of the Artin representation attached to g. The main technical result of our work consists in proving the existence of an eigenform g such that

$$\otimes\text{-}\mathrm{Ind}_L^F(\varrho_g) = \mathrm{Ind}_K^F \mathbb{I} - \mathbb{I},$$

where $\mathbb{I}$ denotes the trivial representation. Thanks to the modularity of totally odd Artin representations [PS16], the problem reduces to finding the solution of a Galois embedding problem as follows. The group $G(J/L) \cong A_5$ does not admit any irreducible 2-dimensional representation, but it has two conjugacy classes of embeddings into $\mathrm{PGL}_2(\mathbb{C})$. Therefore, we look for a lift of the 2-dimensional projective representation of $G_L \twoheadrightarrow G(J/L) \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$ which (i) is totally odd, (ii) has controlled ramification, and (iii) whose tensor induction is $\mathrm{Ind}_K^F \mathbb{I} - \mathbb{I}$. Note that every projective 2-dimensional representation has a minimal lift with index a power of 2 (Lemma 1.1, [Que95]), thus we are led to consider the following Galois embedding problem:

> Given a finite set of primes Σ_0, is it possible to find a Galois extension H/F unramified at Σ_0, containing J/F and such that $1 \to \mathscr{C}_{2^r} \to G(H/F) \to G(J/F) \to 1$ is a non-split extension for some $r \geq 1$?

Here $\mathscr{C}_{2^r}$ denotes the cyclic group of order 2^r considered as an S_5-module via the homomorphism $S_5 \twoheadrightarrow \{\pm 1\} \hookrightarrow \mathrm{Aut}(\mathscr{C}_{2^r})$, taking the non-trivial element of $\{\pm 1\}$ to the automorphism $x \mapsto x^{-1}$. In Theorem 1.2.4, we are able to provide conditions for the Galois embedding problem to have a solution.

1.1 On exotic tensor inductions

Let A, B be groups, $n \in \mathbb{N}$ and $\phi : A \to S_n$ a group homomorphism. The wreath product of B with A is $B \wr A = B^{\oplus n} \rtimes_{\phi} A$, where A acts permuting the factors through ϕ.

Let G be a group and Q a subgroup of index n. Denote by $\pi : G \to S_n$ the action of G on right cosets by right multiplication and let $\{g_1, \dots, g_n\}$ be a set of coset representatives. For any $g \in G$ and $i \in \{1, \dots, n\}$, we denote by $q_i(g)$ the unique element of Q such that $g \cdot g_i = g_{i\pi(g)} \cdot q_i(g)$. The map $\varphi : G \to Q \wr S_n$, given by $g \mapsto (q_1(g), \dots, q_n(g), \pi(g))$, is an injective group homomorphism. Moreover, a different choice of coset representatives produces a homomorphism conjugated to φ by an element of G.

Definition 1.1.1. *Let Q be a subgroup of G of index n, $\varrho : Q \to \mathrm{Aut}(V)$ a representation of Q. We define the tensor induction $\otimes\text{-}\mathrm{Ind}_Q^G(\varrho)$ as the composition of the arrows in the following diagram*

$$\begin{tikzcd} G \arrow[d, "\varphi"'] \arrow[rrrr, dotted, "\otimes\text{-}\mathrm{Ind}_Q^G(\varrho)"] & & & & \\ Q \wr S_n \arrow[rr, "(\varrho, \mathrm{id}_{S_n})"'] & & Aut(V) \wr S_n \arrow[rr, "(\alpha, \psi)"'] & & Aut(V^{\otimes n}), \end{tikzcd}$$

where $\alpha : Aut(V)^{\oplus n} \to Aut(V^{\otimes n})$ is given by $\alpha(f_1, \dots, f_n) = f_1 \otimes \cdots \otimes f_n$, and $\psi : S_n \to Aut(V^{\otimes n})$ by $\sigma \mapsto [\psi(\sigma) : v_1 \otimes \cdots \otimes v_n \mapsto v_{1\sigma} \otimes \cdots \otimes v_{n\sigma}]$.

Example 1.1.2. *Suppose Q is a subgroup of of G index 2 and let $\{1, \theta\}$ be representatives for the right cosets, then*

$$\begin{array}{lll} q_1(g) = g, & q_2(g) = \theta g \theta^{-1} & \text{if } g \in Q \\ q_1(g) = g\theta^{-1}, & q_2(g) = \theta g & \text{if } g \in G \setminus Q. \end{array}$$

Thus,

$$\otimes\text{-}\mathrm{Ind}_Q^G(\varrho)(g) = \begin{cases} \varrho(g) \otimes \rho(\theta g \theta^{-1}) & g \in Q \\ [\varrho(g\theta^{-1}) \otimes \varrho(\theta g)] \circ \psi(12) & g \in G \setminus Q. \end{cases}$$

Proposition 1.1.3. *Let Q be a subgroup of index 2 of G and $\{1, \theta\}$ a set of representatives for the right cosets. Consider (V, ϱ) an irreducible complex 2-dimensional representation of Q with projective image isomorphic to either A_4, S_4 or A_5.*

Then the tensor induction $(V^{\otimes [G:Q]}, \otimes\text{-}\mathrm{Ind}_Q^G(\varrho))$ is reducible if and only if $V^*(\lambda) \cong V^\theta$ for some character $\lambda : Q \to \mathbb{C}^\times$, where $(V^\theta, \varrho^\theta)$ is the representation obtained by conjugation by θ. Moreover, when $\otimes\text{-}\mathrm{Ind}_Q^G(\varrho)$ is reducible its decomposition type is $(3, 1)$.

Proof. If $V^*(\lambda) \cong V^\theta$ then the tensor product $V \otimes V^\theta$ factors as

$$V \otimes V^\theta \cong \mathrm{Ad}^0(V)(\lambda) \oplus \mathbb{C}(\lambda),$$

where $\mathrm{Ad}^0(V)$ is irreducible (Lemma 2.1, [DLR16]). By Frobenius reciprocity,

$$\mathrm{Hom}_G(V^{\otimes [G:Q]}, \mathrm{Ind}_Q^G(\lambda)) = \mathrm{Hom}_Q(V \otimes V^\theta, \mathbb{C}(\lambda)) \neq 0,$$

hence $V^{\otimes [G:Q]}$ is reducible, and since $\left(V^{\otimes [G:Q]}\right)_{|Q} = V \otimes V^\theta$ has decomposition type $(3, 1)$, so does $V^{\otimes [G:Q]}$.

Suppose $V^{\otimes[G:Q]}$ is reducible. We first show that if $V^{\otimes[G:Q]}$ contains a 1-dimensional subrepresentation, then $V^*(\lambda) \cong V^\theta$. Indeed, if $\mathbb{C}(\chi)$ is a subrepresentation of $V^{\otimes[G:Q]}$ then

$$0 \neq \mathrm{Hom}_G(V^{\otimes[G:Q]}, \mathbb{C}(\chi)) \hookrightarrow \mathrm{Hom}_Q(V \otimes V^\theta, \mathbb{C}(\chi_{|Q})).$$

Therefore, the tensor product $V \otimes V^\theta(\chi_{|Q}^{-1})$ has a non-zero Q-invariant vector, i.e.

$$0 \neq \mathrm{H}^0(Q, V \otimes V^\theta(\chi_{|Q}^{-1})) = \mathrm{Hom}_Q(V^*(\chi_{|Q}), V^\theta),$$

which implies that $V^*(\chi_{|Q}) \cong V^\theta$ given the irreducibility of V. Then we can apply the previous step to compute the decomposition type of $V^{\otimes[G:Q]}$. We conclude the proof by showing that $V^{\otimes[G:Q]}$ cannot have decomposition type $(2,2)$. Indeed, suppose $V^{\otimes[G:Q]}$ is of type $(2,2)$, then at least one of the irreducible components decomposes into a sum of characters when restricted to Q (Lemma 2.2, [DLR16]), but then (Lemma 2.1, [DLR16]) produces a contradiction. □

1.2 Galois embedding problems

Cohomological computation

Let F be a totally real number field, Σ_0 a finite set of places of F disjoint from the set Σ_∞ of archimedean places and the set Σ_2 of places above 2. For Σ the complement of Σ_0, we let $G_{F,\Sigma}$ denote the Galois group of the maximal Galois extension F_Σ of F unramified outside Σ. We consider L/F a totally real quadratic extension unramified outside Σ, and for all $r \geq 1$ we give $\mathscr{C}_{2^r}$ the structure of $G_{F,\Sigma}$-module via the homomorphism $G_{F,\Sigma} \twoheadrightarrow G(L/F) \hookrightarrow \mathrm{Aut}(\mathscr{C}_{2^r})$ taking the non-trivial element of $G(L/F)$ to the automorphism $x \mapsto x^{-1}$.

We denote by

$$\mathscr{M}_2 = \lim_{\to,\, r} \mathscr{C}_{2^r}$$

the $G_{F,\Sigma}$-module obtained by taking the direct limit with respect to the natural inclusions $\mathscr{C}_{2^r} \to \mathscr{C}_{2^{r+1}}$. Let $\mathscr{C}'_{2^r}$ be the dual Galois module $\mathrm{Hom}_{\mathrm{Gr}}(\mathscr{C}_{2^r}, \mathcal{O}_\Sigma^\times)$, where $\mathcal{O}_\Sigma$ is the ring of Σ-integers in F_Σ. As a G_L-module $\mathscr{C}'_{2^r}$ is isomorphic to the group μ_{2^r} of 2^rth-roots of unity with the natural Galois action, hence the field $L_r = L(\mu_{2^r})$ trivializes $\mathscr{C}'_{2^r}$.

We are interested in analyzing the maps between the various kernels

$$Ш^1(G_{F,\Sigma}, \mathscr{C}'_{2^r}) = \ker\left(\mathrm{H}^1(G_{F,\Sigma}, \mathscr{C}'_{2^r}) \longrightarrow \prod_{v \in \Sigma} \mathrm{H}^1(F_v, \mathscr{C}'_{2^r})\right).$$

Proposition 1.2.1. *For all $r \geq 2$ the map $(j'_r)_* : Ш^1(G_{F,\Sigma}, \mathscr{C}'_{2^r}) \to Ш^1(G_{F,\Sigma}, \mathscr{C}'_{2^{r-2}})$, induced by the dual of the natural inclusion $j_r : \mathscr{C}_{2^{r-2}} \to \mathscr{C}_{2^r}$, is zero.*

Proof. We claim that the restriction

$$\mathrm{H}^1(G_{L_r,\Sigma}, \mathscr{C}'_{2^r}) \to \prod_{w \in \Sigma(L_r)} \mathrm{H}^1(L_{r,w}, \mathscr{C}'_{2^r})$$

is injective, where the product is taken over all places of L_r above a place in Σ. Indeed, if $\phi : G_{L_r} \to \mathscr{C}'_{2^r}$ is in the kernel of the restriction map, then the field fixed by $\ker\phi$ is a Galois extension of L_r in which the primes that split completely have density 1. Cebotarev's density theorem implies that such extension is L_r itself. By examining the commutative diagram

$$\begin{tikzcd}
 & & \mathrm{H}^1(G_{L_r,\Sigma},\mathscr{C}'_{2^r}) \arrow[r, hook] & \prod\limits_{w\in\Sigma(M_r)} \mathrm{H}^1(L_{r,w},\mathscr{C}'_{2^r}) \\
0 \arrow[r] & \text{Ш}^1(G_{F,\Sigma},\mathscr{C}'_{2^r}) \arrow[r] & \mathrm{H}^1(G_{F,\Sigma},\mathscr{C}'_{2^r}) \arrow[r] \arrow[u] & \prod\limits_{v\in\Sigma} \mathrm{H}^1(F_v,\mathscr{C}'_{2^r}) \arrow[u] \\
0 \arrow[r] & \text{Ш}^1(L_r/F,\mathscr{C}'_{2^r}) \arrow[r] \arrow[u, hook] & \mathrm{H}^1(L_r/F,\mathscr{C}'_{2^r}) \arrow[r] \arrow[u, hook] & \prod\limits_{v\in\Sigma} \mathrm{H}^1(L_{r,w}/F_v,\mathscr{C}'_{2^r}), \arrow[u, hook]
\end{tikzcd}$$

we see that $\text{Ш}^1(G_{F,\Sigma},\mathscr{C}'_{2^r}) = \text{Ш}^1(L_r/F,\mathscr{C}'_{2^r})$.

We claim that $\text{Ш}^1(G_{F,\Sigma},\mathscr{C}'_{2^r})$ is killed by multiplication by 4. Clearly, it suffices to prove that $\mathrm{H}^1(L_r/F,\mathscr{C}'_{2^r})$ is killed by multiplication by 4. Considering the inflation-restriction exact sequence

$$0 \longrightarrow \mathrm{H}^1(L/F,(\mathscr{C}'_{2^r})^{G(L_r/L)}) \longrightarrow \mathrm{H}^1(L_r/F,\mathscr{C}'_{2^r}) \longrightarrow \mathrm{H}^1(L_r/L,\mathscr{C}'_{2^r}).$$

and the fact that both $\mathrm{H}^1(L/F,(\mathscr{C}'_{2^r})^{G(L_r/L)})$ and $\mathrm{H}^1(L_r/L,\mathscr{C}'_{2^r})$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (Lemma 9.1.4 and Proposition 9.1.6, [NSW08]), the claim follows.

There is a natural factorization of multiplication by 4 on $\mathscr{C}'_{2^r}$,

$$\begin{tikzcd}
\mathscr{C}'_{2^r} \arrow[rr, "{[4]'}"] \arrow[dr, "j'_r"'] & & \mathscr{C}'_{2^r} \\
 & \mathscr{C}'_{2^{r-2}} \arrow[ur, "(4)'"'] &
\end{tikzcd} ,$$

which induces the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(G_{F,\Sigma},\mathscr{C}'_{2^r}) \arrow[rr, "{[4]'_*}"] \arrow[dr, "(j'_r)_*"'] & & \mathrm{H}^1(G_{F,\Sigma},\mathscr{C}'_{2^r}) \\
 & \mathrm{H}^1(G_{F,\Sigma},\mathscr{C}'_{2^{r-2}}) \arrow[ur, "(4)'_*"'] &
\end{tikzcd} .$$

Hence, to complete the proof we need to show that $\text{Ш}^1(G_{F,\Sigma},\mathscr{C}'_{2^{r-2}})$ does not intersect the kernel of $(4)'_*$ because it would provide the required inclusion $\text{Ш}^1(G_{F,\Sigma},\mathscr{C}'_{2^r}) \subset \ker(j'_r)_*$. The exact sequence of $G_{F,\Sigma}$-modules

$$1 \longrightarrow \mathscr{C}'_{2^{r-2}} \xrightarrow{(4)'} \mathscr{C}'_{2^r} \longrightarrow \mathscr{C}'_{2^2} \longrightarrow 1,$$

induces the exact sequence of cohomology groups

$$1 \longrightarrow C_2 = \mathrm{H}^0(G_{F,\Sigma},\mathscr{C}'_{2^2}) \xrightarrow{\delta} \mathrm{H}^1(G_{F,\Sigma},\mathscr{C}'_{2^{r-2}}) \xrightarrow{(4)'_*} \mathrm{H}^1(G_{F,\Sigma},\mathscr{C}'_{2^r})$$

because any complex conjugation in $G_{F,\Sigma}$ acts by inversion. Hence, $\delta(\mathrm{H}^0(G_{F,\Sigma}, \mathscr{C}'_{2^2})) = \ker(4)'_*$. Finally, for every real place $v \in \Sigma_\infty$ the connecting homomorphism

$$\delta_v : C_2 = \mathrm{H}^0(\mathbb{R}, \mathscr{C}'_{2^2}) \hookrightarrow \mathrm{H}^1(\mathbb{R}, \mathscr{C}'_{2^r-2})$$

is injective. In particular, the non-trivial class of $\delta(\mathrm{H}^0(G_{F,\Sigma}, \mathscr{C}'_{2^2}))$ is not locally trivial at the real places. □

Lemma 1.2.2. *Let v be a place of F, then the local Galois cohomology group $\mathrm{H}^2(F_v, \mathscr{M}_2)$ is trivial.*

Proof. If v splits in L/F then G_{F_v} acts trivially on $\mathscr{M}_2$ and we can refer to Tate's Theorem (Theorem 4, [Ser77]). If v is inert or ramified (so non-archimedean under our assumptions), then G_K has cohomological dimension 2 and $\mathrm{H}^2(F_v, \mathscr{M}_2)$ is 2-divisible. We conclude by noting that multiplication by 2 factors through $\mathrm{H}^2(L_v, \mathscr{M}_2)$ which is trivial because $\mathscr{M}_2$ is a trivial G_{L_v}-module. □

Theorem 1.2.3. *Let F be a totally real number field, Σ_0 a finite set of places of F disjoint from the set Σ_∞ of archimedean places and the set Σ_2 of places above 2. For Σ the complement of Σ_0, we consider L/F a totally real quadratic extension unramified outside Σ. Then $\mathrm{H}^2(G_{F,\Sigma}, \mathscr{M}_2) = 0$.*

Proof. By Lemma 1.2.2, it suffices to show that the restriction map

$$\mathrm{H}^2(G_{F,\Sigma}, \mathscr{M}_2) \to \bigoplus\nolimits_{v \in \Sigma} \mathrm{H}^2(F_v, \mathscr{M}_2)$$

is injective. For every $r \geq 2$, consider the exact sequence

$$0 \longrightarrow \text{Ш}^2(G_{F,\Sigma}, \mathscr{C}_{2^r}) \longrightarrow \mathrm{H}^2(G_{F,\Sigma}, \mathscr{C}_{2^r}) \longrightarrow \bigoplus_{v \in \Sigma} \mathrm{H}^2(F_v, \mathscr{C}_{2^r}).$$

Poitou-Tate duality (Theorem 8.6.7, [NSW08]) gives us a commuting diagram

$$\begin{tikzcd}
\text{Ш}^1(G_{F,\Sigma}, \mathscr{C}'_{2^r}) \arrow[dd, "j'_{r*}"] & \times & \text{Ш}^2(G_{F,\Sigma}, \mathscr{C}_{2^r}) \arrow[dr] & \\
& & & \mathbb{Q}/\mathbb{Z} \\
\text{Ш}^1(G_{F,\Sigma}, \mathscr{C}'_{2^r-2}) & \times & \text{Ш}^2(G_{F,\Sigma}, \mathscr{C}_{2^r-2}) \arrow[uu, "j_{r*}"] \arrow[ur] &
\end{tikzcd},$$

which in combination with Proposition 1.2.1, shows that

$$j_{r*} : \text{Ш}^2(G_{F,\Sigma}, \mathscr{C}_{2^r-2}) \to \text{Ш}^2(G_{F,\Sigma}, \mathscr{C}_{2^r})$$

is zero because the pairings are perfect. Finally, direct limits are exact and commute with direct sums, so

$$0 = \lim_{r,\to} \text{Ш}^2(G_{F,\Sigma}, \mathscr{C}_{2^r}) \longrightarrow \mathrm{H}^2(G_{F,\Sigma}, \mathscr{M}_2) \longrightarrow \bigoplus_{v \in \Sigma} \mathrm{H}^2(F_v, \mathscr{M}_2)$$

is exact. □

Galois embedding problem

Let $n \geq 4$, $r \geq 1$ be integers. We consider the cyclic group $\mathscr{C}_2$ of order 2 endowed with the trivial action of the symmetric group S_n. It is a classical computation that

$$\mathrm{H}^2(S_n, \mathscr{C}_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \qquad \text{and} \qquad \mathrm{H}^2(A_n, \mathscr{C}_2) \cong \mathbb{Z}/2\mathbb{Z}.$$

We consider a class

$$[\omega]: \quad 1 \to \mathscr{C}_2 \to \Omega \to S_n \to 1 \quad \in \quad \mathrm{H}^2(S_n, \mathscr{C}_2)$$

that does not belong to the kernel of the restriction map $\mathrm{H}^2(S_n, \mathscr{C}_2) \to \mathrm{H}^2(A_n, \mathscr{C}_2)$. Let F be a totally real field. A S_n-Galois extension J/F, ramified at a finite set Σ_{ram} of places of F, determines a surjection $e : G_{F,\Sigma} \twoheadrightarrow S_n$ where Σ is the complement of any finite set Σ_0 of places of F disjoint from $\Sigma_{\mathrm{ram}} \cup \Sigma_\infty \cup \Sigma_2$. We denote by $L = J^{A_n}$ the fixed field by A_n.

Theorem 1.2.4. *Suppose the quadratic extension L/F cut out by A_n is totally real. For all $[\omega] \in \mathrm{H}^2(S_n, \mathscr{C}_2)$ restricting to the universal central extension of A_n it is possible to embed J/F into a Galois extension H/F unramified outside Σ, such that the Galois group $G(H/F)$ represents the non-trivial extension $i_{r*}[\omega]$ of S_n by the S_n-module $\mathscr{C}_{2^r}$ for some $r \gg 0$.*

Proof. Let $i_r : \mathscr{C}_2 \hookrightarrow \mathscr{C}_{2^r}$ be the natural inclusion. The obstruction to the solution of the Galois embedding problem is encoded in the cohomology class $e^* i_{r*}[\omega] \in \mathrm{H}^2(G_{F,\Sigma}, \mathscr{C}_{2^r})$. Indeed, the triviality of the cohomology class is equivalent to the existence of a continuous homomorphism $\gamma : G_{F,\Sigma} \to \Omega_r$ such that the following diagram commutes

$$\begin{tikzcd}
e^* i_{r*}[\omega]: & 1 \arrow[r] & \mathscr{C}_{2^r} \arrow[r] \arrow[d, equal] & e^*\Omega_r \arrow[r] \arrow[d] & G_{F,\Sigma} \arrow[r] \arrow[d, "e"] \arrow[dl, dotted, "\gamma"'] & 1 \\
i_{r*}[\omega]: & 1 \arrow[r] & \mathscr{C}_{2^r} \arrow[r] & \Omega_r \arrow[r] & S_n \arrow[r] & 1.
\end{tikzcd}$$

The homomorphism γ need not be surjective, but it still defines a non-trivial extension of S_n by a submodule of $\mathscr{C}_{2^r}$ as Ω_r is a non-trivial extension. The non-triviality of the class $i_{r*}[\omega]$ follows by the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{H}^2(S_n, \mathscr{C}_2) \arrow[r, "i_{r*}"] \arrow[d] & \mathrm{H}^2(S_n, \mathscr{C}_{2^r}) \arrow[d] \\
\mathrm{H}^2(A_n, \mathscr{C}_2) \arrow[r, "i_{r*}"] & \mathrm{H}^2(A_n, \mathscr{C}_{2^r})
\end{tikzcd}$$

because by hypothesis the restriction of $[\omega]$ to $\mathrm{H}^2(A_n, \mathscr{C}_2)$ is non-zero and the lower orizontal arrow is injective as $\mathrm{H}^1(A_n, \mathscr{C}_{2^{r-1}}) = 0$ for $n \geq 4$. Finally,

$$\lim_{r,\to} \mathrm{H}^2(G_{F,\Sigma}, \mathscr{C}_{2^r}) = \mathrm{H}^2(G_{F,\Sigma}, \mathscr{M}_2) = 0$$

by Theorem 1.2.3, hence the obstruction to the solution of the Galois embedding problem vanishes for $r \gg 0$. $\square$

1.3 On Artin representations

Let K/F be an S_5-quintic extension ramified at a finite set Σ_{ram} of places of F. Suppose the Galois closure J is totally complex and that the subfield of J fixed by A_5 is a totally real quadratic extension L/F. Let Σ be the complement of a finite set Σ_0 disjoint from $\Sigma_{\mathrm{ram}} \cup \Sigma_\infty \cup \Sigma_2$.

The simple group A_5 does not admit an irreducible 2-dimensional representation. However, there are two conjugacy classes of embeddings of A_5 into $\mathrm{PGL}_2(\mathbb{C})$. We fix one such embedding and we consider the projective representation

$$G_{L,\Sigma} \twoheadrightarrow G(J/L) \cong A_5 \subset \mathrm{PGL}_2(\mathbb{C}).$$

We are interested in finding a lift with specific properties. Consider the double cover Ω_5^+ of S_5 where transpositions lift to involutions, and that restricts to the universal central extension of A_5 . By Theorem 1.2.4 there exists a positive integer r and a Galois extension H/F, unramified outside Σ and containing J/F, such that the sequence

$$1 \longrightarrow \mathscr{C}_{2^r} \longrightarrow G(H/F) \longrightarrow G(J/F) \longrightarrow 1$$

is exact. Given our choice of the double cover Ω_5^+, transpositions of $S_5 \cong G(J/F)$ lift to element of order 2 of $G(H/F)$. Moreover, the conjugation action of transpositions of $G(J/F)$ on $\mathscr{C}_{2^r}$ is by inversion: $x \mapsto x^{-1}$. Let $\tilde{A}_5$ denote the universal central extension of $A_5 \cong \tilde{A}_5/\{\pm 1\}$. Complex two-dimensional representations of the group

$$G(H/L) \cong (C_{2^r} \times \tilde{A}_5)/\langle(-1,-1)\rangle$$

are constructed by tensoring a character of C_{2^r} with a 2-dimensional representation of $\tilde{A}_5$ that takes the same value at -1. We consider $\varrho_K : G_{L,\Sigma} \to \mathrm{GL}_2(\mathbb{C})$, a representation obtained by composing the quotient map $G_{L,\Sigma} \twoheadrightarrow G(H/L)$ with any irreducible 2-dimensional representation of $G(H/L)$.

Remark 1.3.1. *Note that since the abelianization of $\tilde{A}_5$ is trivial, there is a dihedral Galois extension D/F such that $\det(\varrho_K)$ factors through the quotient by the subgroup*

$$G(H/D) \cong (C_2 \times \tilde{A}_5)/\langle(-1,-1)\rangle.$$

Therefore, the composition of the determinant with the transfer map, $\det(\varrho_K) \circ V : G_F \longrightarrow \mathbb{C}^\times$, is the trivial character.

Proposition 1.3.2. *The tensor induction $\otimes\text{-}\mathrm{Ind}_L^F(\varrho_K) : G_F \longrightarrow \mathrm{GL}_4(\mathbb{C})$ factors through G_J and induces a faithful representation $\otimes\text{-}\mathrm{Ind}_L^F(\varrho_K) : S_5 \longrightarrow \mathrm{GL}_4(\mathbb{C})$ isomorphic to the standard representation of S_5 on 5 letters.*

Proof. By construction, the action by conjugation of $G(J/F)$ on $G(H/J)$ factors through $G(L/F)$ and sends every element to its inverse. Let $\theta \in G_F$ be an element mapping to a transposition in $G(J/F) \cong S_5$, then

$$\begin{aligned}\ker\left(\otimes\text{-}\mathrm{Ind}_L^F(\varrho_K)\right) \cap G_L &= \ker\left(\varrho_K \otimes (\varrho_K)^\theta\right)\\ &= \{h \in G_L \mid \exists \alpha \in \mathbb{C}^\times \text{ with } \varrho_K(h) = \alpha \mathbb{I}_2,\ \varrho_K^\theta(h) = \alpha^{-1}\mathbb{I}_2\}\\ &= G_J.\end{aligned}$$

Thus, $\otimes$-$\mathrm{Ind}_L^F(\varrho_K)$ induces a 4-dimensional representation $\otimes$-$\mathrm{Ind}_L^F(\varrho_K) : S_5 \to \mathrm{GL}_4(\mathbb{C})$ of S_5. By Proposition 1.1.3, $\otimes$-$\mathrm{Ind}_L^F(\varrho_K)$ has either decomposition type $(3,1)$ or it is irreducible. Hence, it has to be irreducible since S_5 does not have irreducible representations of dimension 3. Finally, S_5 has only two irreducible 4-dimensional representations: the standard representation St_{S_5} on 5 letters and its twist by the sign character $\mathrm{sign} : S_5 \to \{\pm 1\}$. We can distinguish between them by computing the trace of transpositions. Recall that our input was the central extension Ω_5^+ of S_5 with the property that transpositions of S_5 lift to involutions. It follows that $\theta^2 \in G_H$ and $\varrho_K(\theta^2) = \mathbb{I}_2$, hence we can compute that

$$\otimes\text{-}\mathrm{Ind}_L^F(\varrho_K)(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has trace equal to 2. □

Corollary 1.3.3. *Let K/F be an S_5-quintic extension whose Galois closure J is totally complex and contains a totally real quadratic extension L/F. Let Σ be the complement of any finite set Σ_0 of places of F disjoint from $\Sigma_{\mathrm{ram}} \cup \Sigma_\infty \cup \Sigma_2$, then there exists a totally odd 2-dimensional Artin representation $\varrho_K : G_{L,\Sigma} \to GL_2(\mathbb{C})$ such that $\otimes$-$\mathrm{Ind}_L^F(\varrho_K)$ is equivalent to $\mathrm{Ind}_K^F \mathbb{I} - \mathbb{I}$.*

Proof. Thanks to Proposition 1.3.2, we only have to check that the given Artin representation $\varrho_K : G_{L,\Sigma} \to \mathrm{GL}_2(\mathbb{C})$ is totally odd. By assumption the Galois closure J is totally complex, thus the projectivization of ϱ_K is a faithful representation of $G(J/L)$, which contains every complex conjugation of L. □

1.4 Growth of the analytic rank

Let L/F be a quadratic extension of totally real fields, $E_{/F}$ a modular elliptic curve of conductor $\mathfrak{N}$, and g a primitive Hilbert cuspform over L of parallel weight one and level $\mathfrak{Q}$. Attached to this data, there is a unitary cuspidal automorphic representation $\Pi = \Pi_{g,E}$ of the algebraic group $\mathbf{G} = \mathrm{Res}_{L\times F/F}(\mathrm{GL}_{2,L\times F})$. Let $\phi : G_F \to S_3$ be the homomorphism mapping the absolute Galois group of F to the symmetric group over 3 elements associated with the étale cubic algebra $(L \times F)/F$. The L-group ${}^L\mathbf{G}$ is given by the semi-direct product $\mathrm{GL}_2(\mathbb{C})^{\times 3} \rtimes_\phi G_F$ where G_F acts on the first factor through ϕ.

Definition 1.4.1. *The twisted triple product L-function associated with the unitary autormophic representation Π is given by the Euler product*

$$L(s, \Pi, \mathrm{r}) = \prod_v L_v(s, \Pi_v, \mathrm{r})^{-1}$$

where Π_v is the local representation at the finite prime v of F appearing in the restricted tensor product decomposition $\Pi = \bigotimes_v' \Pi_v$, and the representation r gives the action of ${}^L\mathbf{G}$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ which restricts to the natural 8-dimensional representation of $GL_2(\mathbb{C})^{\times 3}$ and for which G_F acts via ϕ permuting the vectors.

Remark 1.4.2. *([PSR87], page 111). When Π_v is ramified, let q_v be the cardinality of the residue field of F_v, then the local L-factor at v of $L(s, \Pi, \mathrm{r})$ is given by*

$$L_v\left(\frac{1+s}{2}, \Pi_v, \mathrm{r}\right) = P_v(q_v^{-s})$$

for a certain polynomial $P_v(X) \in 1 + X\mathbb{C}[X]$. In particular, it is non-vanishing at $s = 1/2$.

Assume the central character ω_Π of Π is trivial when restricted to $\mathbb{A}_F^\times$, then the complex L-function $L(s, \Pi, \mathrm{r})$ has meromorphic continuation to $\mathbb{C}$ with possible poles at $0, \frac{1}{4}, \frac{3}{4}, 1$ and functional equation $L(s, \Pi, \mathrm{r}) = \epsilon(s, \Pi, \mathrm{r})L(1-s, \Pi, \mathrm{r})$ ([PSR87], Theorems 5.1, 5.2, 5.3). When all the primes dividing $\mathfrak{N}$ are unramified in L/F and $(\mathfrak{N}, \mathrm{N}_{L/F}(\mathfrak{Q})) = 1$, the sign of the functional equation can be computed as follows (Theorems B, D, Remark 4.1.1, [Pra92]). Write $\mathfrak{N} = \mathfrak{N}^+\mathfrak{N}^-$, where $\mathfrak{N}^-$ is the square-free part of $\mathfrak{N}$, and suppose that all prime factors of $\mathfrak{N}^+$ are split in L/F, then the sign of the functional equation is determined by the number of prime divisors of $\mathfrak{N}^-$ which are inert in L/F:

$$\epsilon\Big(\frac{1}{2}, \Pi, \mathrm{r}\Big) = \left(\frac{L/F}{\mathfrak{N}^-}\right).$$

Theorem 1.4.3. *Let $E_{/F}$ be a modular elliptic curve of odd conductor $\mathfrak{N}$ and let K/F be an S_5-quintic extension with totally complex Galois closure J. Suppose J is unramified at $\mathfrak{N}$ and contains a totally real quadratic extension L/F, then the ratio of L-functions $L(E/K, s)/L(E/F, s)$ has meromorphic continuation to the whole complex plane and it is holomorphic at $s = 1$. Furthermore, if all prime factors of $\mathfrak{N}^+$ are split in L/F, then*

$$\mathrm{ord}_{s=1}\frac{L(E/K, s)}{L(E/F, s)} \equiv 1 \pmod 2 \qquad \iff \qquad \left(\frac{L/F}{\mathfrak{N}^-}\right) = -1.$$

Proof. Thanks to Corollary 1.3.3 and the modularity of totally odd Artin representations of the absolute Galois group of totally real fields (Theorem 0.3, [PS16]), there is a primitive Hilbert cuspform g of parallel weight one over L, and level $\mathfrak{Q}$ prime to $\mathfrak{N}$, such that $\varrho_g = \varrho_K$. A direct inspection of the Euler product of the twisted triple product L-function $L(s, \Pi, \mathrm{r})$ attached to $\Pi = \Pi_{g,E}$ produces the equality of incomplete L-functions

$$L_S(s, \Pi, \mathrm{r}) = L_S\Big(E, \otimes\text{-}\mathrm{Ind}_L^F(\varrho_g), s + \frac{1}{2}\Big) = \frac{L_S(E/K, s + \frac{1}{2})}{L_S(E/F, s + \frac{1}{2})},$$

where S is any finite set containing the primes dividing $\mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{Q})$ and the primes that ramify in M/F. Remark 1.3.1 ensures the triviality of the central character ω_Π when restricted to $\mathbb{A}_F^\times$, hence, meromorphic continuation, holomorphicity at the center and the criterion for the parity of the order of vanishing at the center of $L(E/K, s)/L(E/F, s)$ follow. □

Corollary 1.4.4. *Let $E_{/F}$ be an elliptic curve of odd conductor $\mathfrak{N}$ and at least one prime of multiplicative reduction. We denote by $G_5(E_{/F}; X)$ the number of quintic extensions K of F such that the norm of the relative discriminant is at most X and the analytic rank of E grows over K, i.e., $r_{\mathrm{an}}(E/K) > r_{\mathrm{an}}(E/F)$. Then $G_5(E; X) \asymp_{+\infty} X$.*

Proof. By Theorem 1.4.3, $G_5(E_{/F}; X)$ contains all S_5-quintic extension K/F with totally complex Galois closure J containing a totally real quadratic extension in which the prime divisors of $\mathfrak{N}$ are unramified and have certain splitting behaviour. Then ([BSW15], Theorem 2) gives $G_5(E_{/F}; X) \gg_{+\infty} X$, while ([BSW15], Theorem 1) provides $X \gg_{+\infty} G_5(E_{/F}; X)$. □

Chapter 2

Twisted triple product L-functions and Hirzebruch-Zagier cycles

This chapter is part of the program pioneered by Darmon and Rotger in [DR14], [DR17a] devoted to studying the p-adic variation of arithmetic invariants for automorphic representations on higher rank groups, with the aim of shedding some light on the relation between p-adic L-functions and Euler systems with applications to the equivariant BSD-conjecture.

Given a totally real number field F, the starting point of the program is to find a reductive group G having $\mathrm{GL}_{2,F}$ as a direct factor together with an automorphic L-function for which there is an explicit formula for the central L-value. The expectation is that there exists a transcendental period for which the ratio between the special value and the period becomes a meaningful algebraic number varying p-adically. More precisely, these modified central L-values should determine a rigid-analytic meromorphic function by interpolation. In the present work, we consider the group $G_{L\times F} = \mathrm{Res}_{L\times F/F}(\mathrm{GL}_{2,L\times F})$ for L/F a quadratic extension of totally real number fields. Piatetski-Shapiro and Rallis [PSR87] studied the analytic properties of the twisted triple product L-function attached to cuspidal representations of $G_{L\times F}$ and Ichino [Ich08] proved a formula for its central value, generalizing earlier work of Harris-Kudla [HK91]. The first part of the paper is devoted to the construction of a p-adic L-function, called the *twisted triple product p-adic L-function*.

Several far-reaching conjectures suggest a strong link between automorphic L-functions and algebraic cycles: relevant cycles should live on a Kuga-Sato variety whose étale cohomology realizes the Galois representation (conjecturally) attached to the automorphic representation of G, out of which one constructs the L-function. Furthermore, as the central L-values should vary p-adically after a modification by an appropriate period, by tinkering with these cycles it should be possible to produce Galois cohomology classes that p-adically interpolate into a *big cohomology class*, giving rise to the p-adic L-function via Perrin-Riou's machinery. We remark that such p-adic L-function and big cohomology class are defined using completely different inputs, an automorphic and a geometric one; the fact that in certain cases it is possible to prove these approaches produce the same object is in itself an amazing confirmation of the power of the existing conjectures.

The relation between p-adic L-functions and algebraic cycles, as we just sketched it, can be very hard to prove since it requires, among various things, a deep understanding of the cohomology of semistable models of Shimura varieties. Therefore, we decided to dedicate the second part of this work to the more humble goal of showing that the p-adic L-function, built using the automorphic input, encodes geometric information of some kind. More precisely, we compute some values of the p-adic L-function in terms of the syntomic Abel-Jacobi image of *generalized Hirzebruch-Zagier cycles*. Our result is evidence that the twisted triple product p-adic L-function and the generalized Hirzebruch-Zagier cycles are the right objects to consider in the framework determined by $G_{L\times F}$ and the twisted triple product L-function.

In the remainder of the introduction we present our results in more detail. We fix, once and for all, a p-adic embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ for every rational prime p, and a complex embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Given a number field $E/\mathbb{Q}$ we let I_E be the set of field embeddings of E into $\overline{\mathbb{Q}}$ and $t_E = \sum_{\tau\in\mathrm{I}_E}\tau \in \mathbb{Z}[\mathrm{I}_E]$. For $k,k' \in \mathbb{Z}[\mathrm{I}_F]$ we write $k \geq k'$ if $k_\tau \geq k'_\tau$ for all $\tau \in \mathrm{I}_F$ and $k > k'$ if $k \geq k'$ and $\exists\tau_\circ$ with $k_{\tau_\circ} > k'_{\tau_\circ}$.

The p-adic L-function

Let L/F be a quadratic extension of totally real number fields, $\mathfrak{Q} \lhd \mathcal{O}_L$ and $\mathfrak{N} \lhd \mathcal{O}_F$ ideals. Consider primitive eigenforms $\mathrm{g}_\circ \in S_{\ell_\circ,x_\circ}(\mathfrak{Q};L;\overline{\mathbb{Q}})$ and $\mathrm{f}_\circ \in S_{k_\circ,w_\circ}(\mathfrak{N};F;\overline{\mathbb{Q}})$, whose weights satisfy $n_\circ t_L = \ell_\circ - 2x_\circ$ and $m_\circ t_F = k_\circ - 2w_\circ$ for $n_\circ, m_\circ \in \mathbb{Z}$, generating irreducible cuspidal automorphic representations π,σ of $G_L(\mathbb{A}), G_F(\mathbb{A})$ respectively. We denote by π^u, σ^u their unitarizations and define a representation of $\mathrm{GL}_2(\mathbb{A}_{L\times F})$ by $\Pi = \pi^u \otimes \sigma^u$. Let $\rho : \Gamma_F \to S_3$ be the homomorphism mapping the absolute Galois group of F to the symmetric group over 3 elements associated with the étale cubic algebra $(L\times F)/F$. The L-group ${}^L(G_{L\times F})$ is given by the semi-direct product $\widehat{G} \rtimes \Gamma_F$ where Γ_F acts on $\widehat{G} = \mathrm{GL}_2(\mathbb{C})^{\times 3}$ through ρ. One can define the twisted triple product L-function $L(s,\Pi,\mathrm{r})$ of Π via the representation r of ${}^L(G_{L\times F})$ on $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, which restricts to the natural 8-dimensional representation of $\widehat{G}$ and through which Γ_F acts via ρ permuting the vectors. We assume the central character ω_Π of Π satisfies $\omega_{\Pi|\mathbb{A}_F^\times} \equiv 1$, so that the twisted triple product L-function has a functional equation and we can talk about its central value.

Definition 2.0.1. *We say that weights $(\ell,x) \in \mathbb{Z}[\mathrm{I}_L]^2$, $(k,w) \in \mathbb{Z}[\mathrm{I}_F]^2$ are* F-dominated *if there exists $r \in \mathbb{Z}[\mathrm{I}_L]$, $r \geq 0$, with $k = (\ell+2r)_{|F}$ and $w = (x+r)_{|F}$. In particular, F-dominated weights satisfy $k-2w = (\ell-2x)_{|F}$.*

Let $\eta : \mathbb{A}_F^\times \to \mathbb{C}^\times$ be the idele character attached to the quadratic extension L/F by class field theory. Suppose that the weights of $\mathrm{g}_\circ$ and $\mathrm{f}_\circ$ are F-dominated and that the local ϵ-factors satisfy

$$\epsilon_v\left(\frac{1}{2},\Pi_v,\mathrm{r}_v\right)\eta_v(-1) = +1 \qquad \forall v \text{ finite place of } F.$$

Building on Ichino's formula [Ich08] and the proof of the Jacquet conjecture [PSP08], Theorem 2.3.4 and Lemma 2.4.1 show that the non-vanishing of the central L-value $L(\frac{1}{2},\Pi,\mathrm{r})$ is equivalent to the existence of a *test vector* $\breve{\mathrm{g}}_\circ$ in π of some level $V_{11}(\mathfrak{A})$ such that the prime factors of $\mathfrak{A}$ are among those dividing $\mathfrak{N}\cdot\mathrm{N}_{L/F}(\mathfrak{Q})\cdot d_{L/F}$. More precisely, $\breve{\mathrm{g}}_\circ$ is a cuspform such that the Petersson inner product

$$\mathrm{I}(\phi) = \left\langle \zeta^*\left(\delta^r\breve{\mathrm{g}}_\circ\right), \mathrm{f}_\circ^*\right\rangle, \tag{2.1}$$

for some $r \in \mathbb{N}[\mathrm{I}_L]$, does not vanish. Therefore, we can take (2.1) as an avatar of the central L-value and use it to construct the p-adic L-function.

Remark 2.0.2. *The assumption on local ϵ-factors at the finite places of F can be satisfied by requiring the ideals $\mathrm{N}_{L/F}(\mathfrak{Q})\cdot d_{L/F}$ and $\mathfrak{N}$ to be coprime and by asking all prime ideals dividing $\mathfrak{N}$ to split in L/F.*

Definition 2.0.3. *Let* $(\ell,x)\in\mathbb{Z}[\mathrm{I}_L]^2$, $(k,w)\in\mathbb{Z}[\mathrm{I}_F]^2$ *be weights and* $\theta\in\mathbb{Z}[\mathrm{I}_L]$ *be an element satisfying* $\theta_{|F}=0\cdot t_F$. *If* $\theta\equiv_2 w$ *holds, i.e.* $\theta_\mu\equiv_2 w_{\mu_{|F}}$ *for all* $\mu\in\mathrm{I}_L$, *we define*

$$r(\theta)=\sum_{\mu\in\mathrm{I}_L}\left[\frac{w_{\mu_{|F}}+\theta_\mu}{2}-x_\mu\right]\cdot\mu\in\mathbb{Z}[\mathrm{I}_L].$$

Let p be a rational prime unramified in L, coprime to the levels $\mathfrak{Q},\mathfrak{N}$. We write $\mathcal{P}$ (resp. $\mathcal{Q}$) for the set of prime $\mathcal{O}_L$-ideals (resp. $\mathcal{O}_F$-ideals) dividing p. We choose an element $\theta\in\mathbb{Z}[\mathrm{I}_L]$ such that $\theta_{|F}=0\cdot t_F$ and $\theta\equiv_2 w_\circ$, and we let $\bar{r}=\sum_{\mu\in\mathrm{I}_L}\bar{r}_\mu\cdot\mu$, with $\bar{r}_\mu\in\mathbb{Z}/(q_{\mathfrak{p}_\mu}-1)\mathbb{Z}$, denote the reduction of $r_\circ=r_\circ(\theta)$ defined using the weights of $\mathrm{g}_\circ$ and $\mathrm{f}_\circ$. We suppose $\mathrm{g}_\circ$, $\mathrm{f}_\circ$ are p-nearly ordinary and we denote by $\mathscr{G}\in\overline{\mathbf{S}}_L^{\text{n.o.}}(\mathfrak{Q},\chi;\mathbf{I}_\mathscr{G})$ and $\mathscr{F}\in\overline{\mathbf{S}}_F^{\text{n.o.}}(\mathfrak{N},\psi;\mathbf{I}_\mathscr{F})$ the Hida families passing through nearly ordinary p-stabilizations $\mathrm{g}_\circ^{(p)}$ and $\mathrm{f}_\circ^{(p)}$. We have

$$\chi_{|Z_L(\mathfrak{Q})_{\text{tor}}}=\chi_\circ\mathcal{N}_L^{n_\circ}\quad\text{and}\quad\psi_{|Z_F(\mathfrak{N})_{\text{tor}}}=\psi_\circ\mathcal{N}_F^{m_\circ}$$

for characters $\chi_\circ:\mathrm{cl}_L^+(\mathfrak{Q})\to\mathbb{C}^\times$, $\psi_\circ:\mathrm{cl}_F^+(\mathfrak{N})\to\mathbb{C}^\times$ and we suppose that $\chi_{\circ|F}\cdot\psi_\circ\equiv 1$. We let $\mathscr{F}^*\in\overline{\mathbf{S}}_F^{\text{n.o.}}(\mathfrak{A},\psi_\circ^{-2}\psi;\mathbf{I}_{\mathscr{F}^*})$ ([Hid91], Section 7F) be the twisted Hida family, where $\mathbf{I}_{\mathscr{F}^*}\cong\mathbf{I}_\mathscr{F}(\psi_\circ^{-2})$ as an $\mathbf{\Lambda}_{F,\psi_\circ^{-2}\psi}$-algebra.

Definition 2.0.4. *Let* $\mathcal{W}=\mathcal{W}_{\mathscr{G},\mathscr{F}^*}$ *be the rigid analytic space* $\mathrm{Spf}(\mathbf{I}_\mathscr{G}\widehat{\otimes}_O\mathbf{I}_{\mathscr{F}^*})^{\text{rig}}$. *The subset of F-*dominated crystalline points *with respect to* $(\theta,\bar{r})$, *denoted* $\mathcal{C}_F^{\theta,\bar{r}}$, *is the subset of arithmetic points* $(\mathrm{P},\mathrm{Q})\in\mathcal{W}$ *whose weights are F-dominated,* $r(\theta)\in\mathbb{Z}[\mathrm{I}_L]$ *is a lift of* $\bar{r}$, *and such that the specialization of the Hida families are old at p; that is, they are the p-stabilization of eigenforms of prime-to-p level:* $\mathscr{G}_\mathrm{P}=\mathrm{g}_\mathrm{P}^{(p)}$ *and* $\mathscr{F}_\mathrm{Q}=\mathrm{f}_\mathrm{Q}^{(p)}$.

Set $\mathbf{K}_{\mathscr{G},\mathscr{F}^*}=(\mathbf{I}_\mathscr{G}\widehat{\otimes}_O\mathbf{I}_{\mathscr{F}^*})\otimes\mathbb{Q}$ and $\mathbf{K}_\mathscr{G}=\mathbf{I}_\mathscr{G}\otimes\mathbb{Q}$. We define a $\mathbf{K}_\mathscr{G}$-adic cuspform $\breve{\mathscr{G}}$ passing through the nearly ordinary p-stabilization of the test vector $\breve{\mathrm{g}}_\circ$ as in [DR14] Section 2.6. Then Lemma 2.4.4 ensures the existence of a meromorphic rigid-analytic function ${}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}},\mathscr{F}):\mathcal{W}\longrightarrow\mathbb{C}_p$ whose value at crystalline points $(\mathrm{P},\mathrm{Q})\in\mathcal{W}$, with $r(\theta)\in\mathbb{Z}[\mathrm{I}_L]$ a lift of $\bar{r}$, is

$${}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}},\mathscr{F})(\mathrm{P},\mathrm{Q})=\frac{1}{\mathsf{E}(\mathrm{f}_\mathrm{Q}^*)}\frac{\left\langle e_{\text{n.o.}}\zeta^*\left(d^{r(\theta)}\breve{\mathrm{g}}_\mathrm{P}^{[\mathcal{P}]}\right),\mathrm{f}_\mathrm{Q}^*\right\rangle}{\left\langle\mathrm{f}_\mathrm{Q}^*,\mathrm{f}_\mathrm{Q}^*\right\rangle}.$$

The number $\mathsf{E}(\mathrm{f}_\mathrm{Q}^*)$ is defined in (2.16). We are justified in calling ${}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}},\mathscr{F})$ a p-adic L-function because it interpolates the algebraic avatar (2.1) of central L-values $L(\frac{1}{2},\Pi_{\mathrm{P},\mathrm{Q}},\mathrm{r})$ at points $(\mathrm{P},\mathrm{Q})\in\mathcal{C}_F^{\theta,\bar{r}}$, as the next theorem shows.

Theorem 2.0.5. *Consider the partition* $\mathcal{Q}_{inert}\coprod\mathcal{Q}_{split}$ *of the set of* $\mathcal{O}_F$*-prime ideals above p determined by the splitting behavior of the primes in the quadratic extension L/F. The value of the twisted triple product p-adic L-function* ${}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}},\mathscr{F}):\mathcal{W}\longrightarrow\mathbb{C}_p$ *at any* $(\mathrm{P},\mathrm{Q})\in\mathcal{C}_F^{\theta,\bar{r}}$ *satisfies*

$$\begin{aligned}{}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}},\mathscr{F})(\mathrm{P},\mathrm{Q})=\pm\frac{1}{\mathsf{E}(\mathrm{f}_\mathrm{Q}^*)}&\left(\prod_{\wp\in\mathcal{Q}_{inert}}\mathcal{E}_\wp^{\text{in}}(\mathrm{g}_\mathrm{P},\mathrm{f}_\mathrm{Q}^*)\prod_{\wp\in\mathcal{Q}_{split}}\frac{\mathcal{E}_\wp^{\text{sp}}(\mathrm{g}_\mathrm{P},\mathrm{f}_\mathrm{Q}^*)}{\mathcal{E}_{1,\wp}(\mathrm{f}_\mathrm{Q}^*)}\right)\\&\times\frac{\left\langle\zeta^*\left(\delta^{s(w-x_{|F})}\breve{\mathrm{g}}_\mathrm{P}\right),\mathrm{f}_\mathrm{Q}^*\right\rangle}{\left\langle\mathrm{f}_\mathrm{Q}^*,\mathrm{f}_\mathrm{Q}^*\right\rangle},\end{aligned}$$

where $s:\mathrm{I}_F\to\mathrm{I}_L$ *is any section of the restriction* $\mathrm{I}_L\to\mathrm{I}_F$, $\mu\mapsto\mu_{|F}$, *and the Euler factors appearing in the formula are defined in Lemmas* 2.4.6 *and* 2.4.8.

A p-adic Gross-Zagier formula.

The second part of the paper deals with the evaluation of the p-adic L-function outside the range of interpolation. From now on, we assume $L/\mathbb{Q}$ to be a real quadratic number field.

Definition 2.0.6. *A triple of integers $(a,b,c) \in \mathbb{Z}^3$, is said to be balanced if none among a,b,c is greater or equal than the sum of the other two. We say that the weights $(\ell,x) \in \mathbb{Z}[\mathrm{I}_L]^2$, $(k,w) \in \mathbb{Z}[\mathrm{I}_\mathbb{Q}]^2$ are* balanced *if there exists $r \in \mathbb{Z}[\mathrm{I}_L]$, $r > 0$, such that $k = |\ell - 2r| := (\ell - 2r)_{|\mathbb{Q}}$, $w = |x - r| := (x - r)_{|\mathbb{Q}}$ and the triple of integers (ℓ_1, ℓ_2, k) is balanced.*

Definition 2.0.7. *The set of* balanced crystalline points *with respect to $(\theta, \bar{r})$, denoted $\mathcal{C}^{\theta,\bar{r}}_{\mathrm{bal}}$, is the subset of arithmetic points $(P,Q) \in \mathcal{W}$, whose weights are balanced, $r(\theta) \in \mathbb{Z}[\mathrm{I}_L]$ is a lift of $\bar{r}$, and such that the specialization of the Hida families are old at p. This set is a disjoint union, indexed by balanced triples (ℓ,k), of subsets $\mathcal{C}^{\theta,\bar{r}}_{\mathrm{bal}}(\ell,k)$ consisting of points whose weights have the form $(\ell,x) \in \mathbb{Z}[\mathrm{I}_L]^2$, $(k,w) \in \mathbb{Z}[\mathrm{I}_\mathbb{Q}]^2$.*

For a balanced crystalline point $(\mathrm{P},\mathrm{Q}) \in \mathcal{C}^{\theta,\bar{r}}_{\mathrm{bal}}$, the global sign of the functional equation of $L(s, \Pi_{\mathrm{P},\mathrm{Q}}, \mathrm{r})$ is -1. This forces the vanishing of the central value, which one expects to be accounted for by the family of generalized Hirzebruch-Zagier cycles. Interestingly, the twisted triple product p-adic L-function is not forced to vanish on $\mathcal{C}^{\theta,\bar{r}}_{\mathrm{bal}}$ and we can try to compute its values there. Let (ℓ,k) be a balanced triple such that either ℓ is not parallel or $(\ell,k) = (2t_L, 2)$. Let $\mathcal{A} \to \mathbf{Sh}_K(G^*_L)$ be the universal abelian surface over the Shimura variety for G^*_L and let $\mathcal{E} \to \mathbf{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})$ be the universal elliptic curve over the modular curve, both defined over some open subset of $\mathrm{Spec}(\mathcal{O}_E)$, where $E/\mathbb{Q}$ is a large enough finite Galois extension. For all but finitely many primes p, let $\wp \lhd \mathcal{O}_E$ be the prime above p induced by the fixed p-adic embedding ι_p, and consider $\mathscr{U}_{\ell-4} \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_{k-2}$ a smooth and proper compactification of $\mathcal{A}^{|\ell|-4} \times \mathcal{E}^{k-2}$. The generalized Hirzebruch-Zagier cycle of weight (ℓ,k) is a De Rham null-homologous cycle

$$\Delta_{\ell,k} \in \mathrm{CH}_{\gamma+2}(\mathscr{U}_{\ell-4} \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_{k-2})_0 \otimes_\mathbb{Z} L$$

of dimension $\gamma + 2 = \frac{|\ell|+k-2}{2}$. Given a pair of eigenforms $\breve{\mathrm{g}}_\mathrm{P} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; E)$ and $\mathrm{f}_\mathrm{Q} \in S_{k,w}(V_1(\mathfrak{N}); E)$ we can produce cohomology classes ω_P and η_Q, as in Definition 2.7.5, such that $\pi_1^*\omega_\mathrm{P} \cup \pi_2^*\eta_\mathrm{Q} \in \mathrm{F}^{|\ell|-2-s}H^{|\ell|+k-3}_{\mathrm{dR}}(U_{\ell-4} \times_{E_\wp} W_{k-2})$ where $s = \frac{|\ell|-k-2}{2}$; that is, the cohomology class $\pi_1^*\omega_\mathrm{P} \cup \pi_2^*\eta_\mathrm{Q}$ lives in the domain of the syntomic Abel-Jacobi image of $\Delta_{\ell,k}$,

$$\mathrm{AJ}_p(\Delta_{\ell,k}) : \mathrm{F}^{|\ell|-2-s}H^{|\ell|+k-3}_{\mathrm{dR}}(U_{\ell-4} \times_{E_\wp} W_{k-2}) \longrightarrow E_\wp,$$

and we can compute the number $\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega_\mathrm{P} \cup \pi_2^*\eta_\mathrm{Q})$ as follows.

Theorem 2.0.8. *Let $L/\mathbb{Q}$ be a real quadratic field and (ℓ,k) a balanced triple. Let p be a prime splitting in L for which the generalized Hirzebruch-Zagier cycle $\Delta_{\ell,k}$ is defined. Then for all $(\mathrm{P},\mathrm{Q}) \in \mathcal{C}^{\theta,\bar{r}}_{\mathrm{bal}}(\ell,k)$ we have*

$${}_{\bar{r}}\mathscr{L}^\theta_p(\breve{\mathscr{G}},\mathscr{F})(\mathrm{P},\mathrm{Q}) = \frac{\pm 1}{s!\mathrm{E}(\mathrm{f}^*_\mathrm{Q})} \frac{\mathcal{E}^{\mathrm{sp}}_p(\mathrm{g}_\mathrm{P},\mathrm{f}^*_\mathrm{Q})}{\mathcal{E}_{1,p}(\mathrm{f}^*_\mathrm{Q})} AJ_p(\Delta_{\ell,k})(\pi_1^*\omega_\mathrm{P} \cup \pi_2^*\eta_\mathrm{Q}).$$

Remark 2.0.9. *The assumption on the splitting behaviour of p in $L/\mathbb{Q}$ should not be necessary. It could be dispensed with by showing the overconvergence of the p-adic cuspform $d_\mu^{1-\ell_\mu}(\breve{\mathrm{g}}^{[p]}_\mathrm{P})$ for $\mu \in \mathrm{I}_L$. It seems reasonable to believe that by generalizing the recent work of Andreatta and Iovita [AI17] one could prove such a result.*

Let A be an elliptic curve over L of conductor $\mathfrak{Q}$ and B a rational elliptic curve of conductor $\mathfrak{N}$, both without complex multiplication over $\overline{\mathbb{Q}}$. We denote by $(\mathrm{M}_{A,B})_p$ the Galois representation $\mathrm{AsV}_p(A)(-1)\otimes_{\mathbb{Q}_p}\mathrm{V}_p(B)$ of the absolute Galois group of $\mathbb{Q}$. We can use Theorem 2.0.8 to give a criterion for the Bloch-Kato Selmer group $H^1_f(\mathbb{Q},(\mathrm{M}_{A,B})_p)$ to be of dimension one in terms of the non-vanishing of a value of one of our twisted triple product p-adic L-functions. We build on the recent work of Liu [Liu16], where he computes the dimension of $H^1_f(\mathbb{Q},(\mathrm{M}_{A,B})_p)$ assuming the non-vanishing of the étale Abel-Jacobi map of certain cycle closely related to our Hirzebruch-Zagier cycle of weight $(2t_L,2)$. Let $\mathsf{g}_A\in S_{2t_L,t_L}(V_1(\mathfrak{Q});L;\mathbb{Q})$, $\mathsf{f}_B\in S_{2,1}(V_1(\mathfrak{N});\mathbb{Q})$ be the newforms attached to A and B by modularity and p a rational prime coprime to $\mathfrak{N}\cdot\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})\cdot d_{L/F}$. If $\mathsf{g}_A,\mathsf{f}_B$ are p-nearly ordinary, we denote by $\mathscr{G},\mathscr{F}$ the Hida families passing through the p-nearly ordinary stabilizations $\mathscr{G}_{\mathrm{P}_A}=\mathsf{g}_A^{(p)}$ and $\mathscr{F}_{\mathrm{Q}_B}=\mathsf{f}_B^{(p)}$, respectively.

Corollary 2.0.10. *Suppose that $\mathfrak{N}$ and $N_{L/\mathbb{Q}}(\mathfrak{Q})\cdot d_{L/\mathbb{Q}}$ are coprime ideals and that all the primes dividing $\mathfrak{N}$ split in L. For all but finitely many primes p that are split in L and such that $\mathsf{g}_A,\mathsf{f}_B$ are p-nearly ordinary we have*

$$ {}_{\bar{r}}\mathscr{L}_p^{\theta}(\breve{\mathscr{G}},\mathscr{F})(\mathrm{P}_A,\mathrm{Q}_B)\neq 0 \quad\Longrightarrow\quad \dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q},(M_{A,B})_p)=1, $$

where $\theta=-\mu+\mu'\in\mathbb{Z}[\mathrm{I}_L]$, $\bar{r}=-\mu$.

The setting of this Chapter had been considered by several independent groups: [BCS17], [For17] and [Ish17]. Ivan Blanco and Ignacio Sols computed syntomic Abel-Jacobi images of some Hirzebruch-Zagier cycles in terms of p-adic modular forms, while Ishikawa constructed twisted triple product p-adic L-functions over $\mathbb{Q}$ following the refined approach of Hsieh [Hsi17]. Given the similarities between the computations of syntomic Abel-Jacobi images in the work of Blanco-Sols and M.F., the two groups agreed to publish together [BCF].

2.1 Automorphic forms

2.1.1 Adelic Hilbert modular forms

Let $F/\mathbb{Q}$ be a totally real number field and let I_F be the set of field embeddings of F into $\overline{\mathbb{Q}}$. We denote by G_F the algebraic group $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$. We choose a square root $i\in\mathbb{C}$ of -1 which allows us to define the Poincaré half-plane $\mathfrak{H}$, we consider the complex manifold $\mathfrak{H}^{\mathrm{I}_F}$ which is endowed with a transitive action of $G_F(\mathbb{R})^+\cong\prod_{\mathrm{I}_F}\mathrm{GL}_2(\mathbb{R})^+$ and contains the point $\mathbf{i}=(i,\dots,i)$. For any $K\leq G_F(\mathbb{A}^\infty)$ compact open subgroup we denote by $S_{k,w}(K;F;\mathbb{C})$, or simply $S_{k,w}(K;\mathbb{C})$ when there is no risk of confusion, the space of holomorphic Hilbert cuspforms of weight $(k,w)\in\mathbb{Z}[\mathrm{I}_F]^2$, $k-2w=mt_F$ for some $m\in\mathbb{Z}$, and level K. It is defined as the space of functions $\mathsf{f}:G_F(\mathbb{A})\to\mathbb{C}$ that satisfy the following list of properties:

- $\mathsf{f}(\alpha xu)=\mathsf{f}(x)j_{k,w}(u_\infty,\mathbf{i})^{-1}$ where $\alpha\in G_F(\mathbb{Q})$, $u\in K\cdot C_\infty^+$ for C_∞^+ the stabilizer of $\mathbf{i}$ in $G_F(\mathbb{R})^+$ and the automorphy factor is $j_{k,w}\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},z\right)=(ad-bc)^{-w}(cz+d)^k$ for $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G_F(\mathbb{R})$, $z\in\mathfrak{H}^{\mathrm{I}_F}$;
- for every finite adelic point $x\in G_F(\mathbb{A}^\infty)$ the well-defined function $\mathsf{f}_x:\mathfrak{H}^{\mathrm{I}_F}\to\mathbb{C}$ given by $\mathsf{f}_x(z)=\mathsf{f}(xu_\infty)j_{k,w}(u_\infty,\mathbf{i})$ is holomorphic, where for each $z\in\mathfrak{H}^{\mathrm{I}_F}$ we choose $u_\infty\in G_F(\mathbb{R})^+$ such that $u_\infty\mathbf{i}=z$.

- for all adelic points $x \in G_F(\mathbb{A})$ and for all additive measures on $F\backslash\mathbb{A}_F$ we have

$$\int_{F\backslash\mathbb{A}_F} \mathsf{f}\left(\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} x\right) da = 0.$$

- If the totally real field is just the field of rational numbers, $F = \mathbb{Q}$, we need to impose the extra condition that for all finite adelic point $x \in G_{\mathbb{Q}}(\mathbb{A}^\infty)$ the function $|\mathrm{Im}(z)^{\frac{k}{2}}\mathsf{f}_x(z)|$ is uniformly bounded on $\mathfrak{H}$.

Definition 2.1.1. *We denote by G_F^* the algebraic group $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F} \times_{\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}} \mathbb{G}_m$. By replacing G_F by G_F^* in the previous definition, we define $S^*_{k,\nu}(K;\mathbb{C})$ to be the space of cuspforms for G_F^* of weight $(k,\nu) \in \mathbb{Z}[\mathrm{I}_F]\times\mathbb{Z}$ and level K, for any $K \leq G_F^*(\mathbb{Q})$ compact open subgroup.*

Note that for all pairs of weights $(k,\nu),(k,\nu') \in \mathbb{Z}[\mathrm{I}_F]\times\mathbb{Z}$ there is a natural isomorphism

$$\Psi_{\nu,\nu'} : S^*_{k,\nu}(K;\mathbb{C}) \xrightarrow{\sim} S^*_{k,\nu'}(K;\mathbb{C}) \tag{2.2}$$

given by $\mathsf{f}(x) \mapsto \mathsf{f}(x)|\det(x)|^{\nu'-\nu}_{\mathbb{A}_{\mathbb{Q}}}$.

Each irreducible automorphic representation π spanned by some form in $S_{k,w}(K;\mathbb{C})$ has central character equal to $|-|^{-m}_{\mathbb{A}_F}$ up to finite order characters. The twist $\pi^u := \pi\otimes|-|^{\frac{m}{2}}_{\mathbb{A}_F}$ is called the unitarization of π. Note that there is an isomorphism of function spaces (not of $G_F(\mathbb{A})$-modules)

$$\begin{aligned}\pi &\xrightarrow{\sim} \pi^u\\ \mathsf{f} &\mapsto \mathsf{f}^u\end{aligned} \qquad \text{where} \qquad \mathsf{f}^u(x) = \mathsf{f}(x)|\det(x)|^{\frac{m}{2}}_{\mathbb{A}_F}. \tag{2.3}$$

Let $\mathrm{d}x$ be the Tamagawa measure on $[G_F(\mathbb{A})] = \mathbb{A}_F^\times G_F(\mathbb{Q})\backslash G_F(\mathbb{A})$, for any two cuspforms $\mathsf{f}_1,\mathsf{f}_2 \in S_{k,w}(K;\mathbb{C})$, with $k-2w = mt_F$, we define their Petersson inner product to be

$$\langle \mathsf{f}_1,\mathsf{f}_2\rangle = \int_{[G_F(\mathbb{A})]} \mathsf{f}_1(x)\overline{\mathsf{f}_2(x)}|\det(x)|^m_{\mathbb{A}_F}\mathrm{d}x = \langle \mathsf{f}_1^u,\mathsf{f}_2^u\rangle. \tag{2.4}$$

For an $\mathcal{O}_F$-ideal $\mathfrak{N}$ we consider the following compact open sugroups of $G_F(\widehat{\mathbb{Z}})$:

- $U_0(\mathfrak{N}) = \left\{ \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in G_F(\widehat{\mathbb{Z}}) \middle| \ c \in \mathfrak{N}\widehat{\mathcal{O}_F}\right\}$,
- $V_1(\mathfrak{N}) = \left\{ \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in U_0(\mathfrak{N}) \middle| \ d \equiv 1 \pmod{\mathfrak{N}\widehat{\mathcal{O}_F}}\right\}$,
- $V_{11}(\mathfrak{N}) = \left\{ \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in V_1(\mathfrak{N}) \middle| \ a \equiv 1 \pmod{\mathfrak{N}\widehat{\mathcal{O}_F}}\right\}$,
- $U(\mathfrak{N}) = \left\{ \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in V_{11}(\mathfrak{N}) \middle| \ b \equiv 0 \pmod{\mathfrak{N}\widehat{\mathcal{O}_F}}\right\}$

For any prime p coprime to $\mathfrak{N}$ and any compact open subgroups satisfying $V_1(\mathfrak{N}) \leq K \leq U_0(\mathfrak{N})$, we set $K(p^\alpha) = K\cap V_{11}(p^\alpha)$ and $Z_F(K) = \mathbb{A}_F^\times/\overline{F^\times\det K(p^\infty)F^\times_{\infty,+}}$. One can decompose the ideles of F as

$$\mathbb{A}_F^\times = \coprod_{i=1}^{h_F^+(\mathfrak{N})} F^\times a_i \det V_{11}(\mathfrak{N})F^\times_{\infty,+}$$

where $a_i \in \mathbb{A}_F^{\infty,\times}$ and $h_F^+(\mathfrak{N})$ is the cardinality of $\mathrm{cl}_F^+(\mathfrak{N}) := F_+^\times \backslash \mathbb{A}_F^{\infty,\times} / \det V_{11}(\mathfrak{N})$. The ideles decomposition induces a decomposition of the adelic points of G_F

$$G_F(\mathbb{A}) = \coprod_{i=1}^{h_F^+(\mathfrak{N})} G_F(\mathbb{Q}) t_i V_{11}(\mathfrak{N}) G_F(\mathbb{R})^+ \qquad \text{for} \qquad t_i = \begin{pmatrix} a_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Adelic q-expansion

The Shimura variety $\mathrm{Sh}_K(G_F)$, determined by G_F and a compact open subgroup K, is not compact, therefore there is a notion of q-expansion for Hilbert modular forms. Even more, Shimura found a way to package the q-expansions of each connected component of $\mathrm{Sh}_K(G_F)$ into a unique adelic q-expansion. Fix $\mathsf{d}_F \in \mathbb{A}_F^{\infty,\times}$ such that $\mathsf{d}_F \mathcal{O}_F = \mathfrak{d}_F$ is the absolute different ideal of F. Let F^{Gal} be the Galois closure of F in $\overline{\mathbb{Q}}$ and write $\mathcal{V}$ for the ring of integers or a valuation ring of a finite extension F_0 of F^{Gal} such that for every ideal $\mathfrak{a}$ of $\mathcal{O}_F$, for all $\tau \in \mathrm{I}_F$, the ideal $\mathfrak{a}^\tau \mathcal{V}$ is principal. Choose a generator $\{\mathfrak{q}^\tau\} \in \mathcal{V}$ of $\mathfrak{q}^\tau \mathcal{V}$ for each prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$ and by multiplicativity define $\{\mathfrak{a}^v\} \in \mathcal{V}$ for each fractional ideal $\mathfrak{a}$ of F and each $v \in \mathbb{Z}[\mathrm{I}_F]$. Given a Hilbert cuspform $\mathsf{f} \in S_{k,w}(V_{11}(\mathfrak{N}); \mathbb{C})$, one can consider for every index $i \in \{1, \ldots, h_F^+(\mathfrak{N})\}$, the holomorphic function $\mathsf{f}_i : \mathfrak{H}^{\mathrm{I}_F} \to \mathbb{C}$

$$\mathsf{f}_i(z) = y_\infty^{-w} \mathsf{f}\left(t_i \begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix} \right) = \sum_{\xi \in (\mathfrak{a}_i \mathfrak{d}_F^{-1})_+} a(\xi, \mathsf{f}_i) e_F(\xi z)$$

for $z = x_\infty + \mathbf{i} y_\infty$, $\mathfrak{a}_i = a_i \mathcal{O}_F$ and $e_F(\xi z) = \exp\big(2\pi i \sum_{\tau \in \mathrm{I}_F} \tau(\xi) z_\tau\big)$. Every idele y in $\mathbb{A}_{F,+}^\times := \mathbb{A}_F^{\infty,\times} F_{\infty,+}^\times$ can be written as $y = \xi a_i^{-1} \mathsf{d} u$ for $\xi \in F_+^\times$ and $u \in \det U(\mathfrak{N}) F_{\infty,+}^\times$; the following functions

$$\mathsf{a}(-, \mathsf{f}) : \mathbb{A}_{F,+}^\times \longrightarrow \mathbb{C}, \qquad \mathsf{a}_p(-, \mathsf{f}) : \mathbb{A}_{F,+}^\times \longrightarrow \overline{\mathbb{Q}}_p$$

are defined by

$$\mathsf{a}(y, \mathsf{f}) := a(\xi, \mathsf{f}_i) \{y^{w - t_F}\} \xi^{t_F - w} |a_i|_{\mathbb{A}_F} \quad \text{and} \quad \mathsf{a}_p(y, \mathsf{f}) := a(\xi, \mathsf{f}_i) y_p^{w - t_F} \xi^{t_F - w} \mathcal{N}_F(a_i)^{-1}$$

if $y \in \widehat{\mathcal{O}}_F F_{\infty,+}^\times$ and zero otherwise. Here $\mathcal{N}_F : Z_F(1) \to \overline{\mathbb{Q}}_p^\times$ is the p-adic cyclotomic character given by $y \mapsto y_p^{-t_F} |y^\infty|_{\mathbb{A}_F}^{-1}$. Clearly, the function $\mathsf{a}_p(-, f)$ makes sense only if the coefficients $a(\xi, f_i) \in \overline{\mathbb{Q}}$ are algebraic $\forall \xi, i$. For each $\mathcal{V}$-algebra A contained in $\mathbb{C}$ we denote by $S_{k,w}(K; A)$ the A-module $\big\{\mathsf{f} \in S_{k,w}(K; \mathbb{C}) \big|\ \mathsf{a}(y, \mathsf{f}) \in A\ \forall y \in \mathbb{A}_{F,+}^\times\big\}$.

Theorem 2.1.2. *([Hid91], Theorem 1.1) Consider the map $e_F : \mathbb{C}^{\mathrm{I}_F} \longrightarrow \mathbb{C}^\times$ defined by $e_F(z) = \exp\big(2\pi i \sum_{\tau \in \mathrm{I}_F} z_\tau\big)$ and the additive character of the ideles $\chi_F : \mathbb{A}_F / F \longrightarrow \mathbb{C}^\times$ which satisfies $\chi_F(x_\infty) = e_F(x_\infty)$. Each cuspform $\mathsf{f} \in S_{k,w}(V_{11}(\mathfrak{N}); \mathbb{C})$ has an adelic q-expansion of the form*

$$\mathsf{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = |y|_{\mathbb{A}_F} \sum_{\xi \in F_+} \mathsf{a}(\xi y \mathsf{d}_F, \mathsf{f}) \{(\xi y \mathsf{d}_F)^{t_F - w}\} (\xi y_\infty)^{w - t_F} e_F(\mathbf{i} \xi y_\infty) \chi_F(\xi x)$$

for $y \in \mathbb{A}_{F,+}^\times$, $x \in \mathbb{A}_F^\times$, where $\mathsf{a}(-, \mathsf{f}) : \mathbb{A}_{F,+}^\times \longrightarrow \mathbb{C}$ vanishes outside $\widehat{\mathcal{O}}_F F_{\infty,+}^\times$ and depends only on the coset $y^\infty \det V_{11}(\mathfrak{N})$.

Nearly holomorphic cuspforms

For K compact open subgroup satisfying $V_{11}(\mathfrak{N}) \leq K \leq G_F(\mathbb{A}^\infty)$ we denote by

$$N_{k,w,q}(K;F;\mathbb{C}) = N_{k,w,q}(K;\mathbb{C})$$

the space of nearly holomorphic cuspforms of weight $(k,w) \in \mathbb{Z}[\mathrm{I}_F]^2$ and order less or equal to $q \in \mathbb{N}[\mathrm{I}_F]$ with respect to K. It is the space of functions $\mathsf{f} : G_F(\mathbb{A}) \to \mathbb{C}$ that satisfy the following list of properties:

- $\mathsf{f}(\alpha x u) = \mathsf{f}(x) j_{k,w}(u_\infty, \mathbf{i})^{-1}$ where $\alpha \in G_F(\mathbb{Q})$, $u \in K \cdot C_\infty^+$;
- for each $x \in G_F(\mathbb{A}^\infty)$ the well-defined function $\mathsf{f}_x(z) = \mathsf{f}(xu_\infty) j_{k,w}(u_\infty, \mathbf{i})$ can be written as
$$\mathsf{f}_x(z) = \sum_{\xi \in L(x)_+} a(\xi, \mathsf{f}_x)((4\pi y)^{-1}) e_F(\xi z)$$
for polynomials $a(\xi, \mathsf{f}_x)(Y)$ in the variables $(Y_\tau)_{\tau \in I}$ of degree less than q_τ in Y_τ for each $\tau \in \mathrm{I}_F$ and for $L(x)$ a lattice of F.

As before f_i stands for f_{t_i} and we consider adelic Fourier coefficients

$$\mathsf{a}(y,\mathsf{f})(Y) = \{y^{w-t_F}\}\xi^{t_F-w}|a_i|_{\mathbb{A}_F} a(\xi,\mathsf{f}_i)(Y), \;\; \mathsf{a}_p(y,\mathsf{f})(Y) = y_p^{w-t}\xi^{t_F-w}\mathcal{N}_F(a_i)^{-1}a(\xi,\mathsf{f}_i)(Y)$$

if $y = \xi a_i^{-1} \mathsf{d}_F u \in \widehat{\mathcal{O}_F} F_{\infty,+}^\times$ and zero otherwise. The adelic Fourier expansion of a nearly holomorphic cuspform f is given by

$$\mathsf{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbb{A}_F} \sum_{\xi \in F_+} \mathsf{a}(\xi y \mathsf{d}_F, \mathsf{f})(Y)\{(\xi y \mathsf{d}_F)^{t_F - w}\}(\xi y_\infty)^{w-t_F} e_F(\mathbf{i}\xi y_\infty)\chi_F(\xi x)$$

for $Y = (4\pi y_\infty)^{-1}$ and for A a subring of $\mathbb{C}$ one can consider the A-module $N_{k,w,q}(K;A)$ defined by $\{\mathsf{f} \in N_{k,w,q}(K;\mathbb{C}) |\ \mathsf{a}(y,\mathsf{f}) \in A[Y]\ \forall y \in \mathbb{A}_{F,+}^\times\}$.

There are Maass-Shimura differential operators for $r \in \mathbb{N}[\mathrm{I}_F]$, $k \in \mathbb{Z}[\mathrm{I}_F]$ defined as

$$\delta_k^r = \prod_{\tau \in \mathrm{I}_F} (\delta_{k_\tau + 2r_\tau - 2}^\tau \circ \cdots \circ \delta_{k_\tau}^\tau), \qquad \text{where} \qquad \delta_\lambda^\tau = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy_\tau} + \frac{\partial}{\partial z_\tau}\right). \tag{2.5}$$

They act on a nearly holomorphic cuspform $\mathsf{f} \in N_{k,w,q}(K;\mathbb{C})$ via the expression

$$\mathsf{a}(y, \delta_k^r \mathsf{f})(Y) = \{y^{w-t_F+r}\}\xi^{t_F-w-r}|a_i|_{\mathbb{A}_F} a(\xi, \delta_k^r \mathsf{f}_i)(Y).$$

Suppose that $\mathbb{Q} \subset A$, then Hida showed ([Hid91], Proposition 1.2) the differential operator δ_k^r maps $N_{k,w,q}(K;A)$ to $N_{k+2r,w+r,q+r}(K;A)$ and if $k_\tau > 2q_\tau\ \forall \tau \in \mathrm{I}_F$, then there is a holomorphic projector $\Pi^{\mathrm{hol}} : N_{k,w,q}(K;A) \longrightarrow S_{k,w}(K;A)$.

2.1.2 Hecke theory

Consider a compact open subgroup $K \leq G_F(\mathbb{A}^\infty)$ of the finite adelic points of G_F that satisfies $V_{11}(\mathfrak{N}) \leq K \leq U_0(\mathfrak{N})$. Suppose that $\mathcal{V}$ is the valuation ring corresponding to the fixed embedding $\iota_p : F^{\mathrm{Gal}} \hookrightarrow \overline{\mathbb{Q}}_p$, so that we may assume $\{y^{t_F - w}\} = 1$ whenever the ideal $y\mathcal{O}_F$ generated by y is prime to $p\mathcal{O}_F$. Let ϖ be a uniformizer of the completion $\mathcal{O}_{F,\mathfrak{q}}$ of $\mathcal{O}_F$ at a prime $\mathfrak{q}$. We are interested in Hecke operators defined by the following double cosets

$$T_0(\varpi) = \{\varpi^{w-t_F}\}\left[V_{11}(\mathfrak{N})\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}V_{11}(\mathfrak{N})\right] \qquad \text{if } \mathfrak{q}\nmid\mathfrak{N},$$

$$U_0(\varpi) = \{\varpi^{w-t_F}\}\left[V_{11}(\mathfrak{N})\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}V_{11}(\mathfrak{N})\right] \qquad \text{if } \mathfrak{q}\mid\mathfrak{N},$$

and for $a \in \mathcal{O}_{F,\mathfrak{N}}^\times := \prod_{\mathfrak{q}|\mathfrak{N}} \mathcal{O}_{F,\mathfrak{q}}^\times$ the double coset

$$T(a,1) = \left[V_{11}(\mathfrak{N})\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}V_{11}(\mathfrak{N})\right].$$

If the prime $\mathfrak{q}$ is coprime to the level, then the Hecke operator $T_0(\varpi)$ acting on modular forms is independent of the choice of the uniformizer ϖ and we simply denote it $T_0(\mathfrak{q})$. For any finite adelic point $z \in Z_G(\mathbb{A}^\infty)$ of the center of G_F we define the diamond operator associated to it by $\mathsf{f}_{|\langle z\rangle}(x) = \mathsf{f}(xz)$, for any modular form f. For a prime ideal $\mathfrak{q}$ such that $\mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}}) \subset K$, we write $\langle\mathfrak{q}\rangle$ for the operator $\langle\varpi\rangle$, where ϖ is a uniformizer of $\mathcal{O}_{F,\mathfrak{q}}$. The action of the operators on adelic q-expansion is given by the following formulas. If $\mathfrak{q}\nmid\mathfrak{N}$ one can compute

$$\mathsf{a}_p(y,\mathsf{f}_{|T_0(\mathfrak{q})}) = \mathsf{a}_p(y\varpi,\mathsf{f})\{\varpi^{w-t_F}\}\varpi_p^{t_F-w} + \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})\{\mathfrak{q}^{2(w-t_F)}\}\mathsf{a}_p(y\varpi^{-1},\mathsf{f}_{|\langle\mathfrak{q}\rangle})\{\varpi^{t_F-w}\}\varpi_p^{w-t_F}$$

and

$$\mathsf{a}(y,\mathsf{f}_{|T_0(\mathfrak{q})}) = \mathsf{a}(y\varpi,\mathsf{f}) + \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})\{\mathfrak{q}^{2(w-t_F)}\}\mathsf{a}(y\varpi^{-1},\mathsf{f}_{|\langle\mathfrak{q}\rangle}).$$

If $\mathfrak{q}\mid\mathfrak{N}$ one can compute

$$\mathsf{a}_p(y,\mathsf{f}_{|U_0(\varpi)}) = \mathsf{a}_p(y\varpi,\mathsf{f})\{\varpi^{w-t_F}\}\varpi_p^{t_F-w}$$

and

$$\mathsf{a}(y,\mathsf{f}_{|U_0(\varpi)}) = \mathsf{a}(y\varpi,\mathsf{f}).$$

Finally, for $a \in \mathcal{O}_{F,\mathfrak{N}}^\times$ one finds $\mathsf{a}_p(y,\mathsf{f}_{|T(a,1)}) = \mathsf{a}_p(ya,\mathsf{f})a_p^{t_F-w}$. It follows that if $\varpi \in \mathcal{O}_{F,\mathfrak{q}}$ is a uniformizer and $a \in \mathcal{O}_{F,\mathfrak{q}}^\times$ then $U_0(a\varpi) = T(a,1)U_0(\varpi)$. The Hecke algebra $\mathsf{h}_{k,w}(K;\mathcal{V})$ is defined to be the $\mathcal{V}$-subalgebra of $\mathrm{End}_\mathbb{C}(S_{k,w}(K;\mathbb{C}))$ generated by the Hecke operators $T_0(\mathfrak{q})$'s for primes outside the level $\mathfrak{q}\nmid\mathfrak{N}$, $U_0(\varpi)$'s for primes dividing the level $\mathfrak{q}\mid\mathfrak{N}$, $T(a,1)$'s for $a \in \mathcal{O}_{F,\mathfrak{N}}^\times$ and the diamond operators. For each $\mathcal{V}$-algebra A contained in $\mathbb{C}$ one defines $\mathsf{h}_{k,w}(K;A) = \mathsf{h}_{k,w}(K;\mathcal{V}) \otimes_\mathcal{V} A$.

Theorem 2.1.3. *([Hid91], Theorem 2.2) For any finite field extension L/F^{Gal} and any $\mathcal{V}$-subalgebra A of L, there is a natural isomorphism $S_{k,w}(K;L) \cong S_{k,w}(K;A)\otimes_A L$. Moreover, if A an integrally closed domain containing $\mathcal{V}$, finite flat over either $\mathcal{V}$ or $\mathbb{Z}_p$, then $S_{k,w}(K;A)$ is stable under $\mathsf{h}_{k,w}(K;A)$ and the pairing $(\ ,\ ) : S_{k,w}(K;A)\times\mathsf{h}_{k,w}(K;A) \to A$ given by $(f,h) = \mathsf{a}(1,f_{|h})$ induces isomorphisms of A-modules*

$$\mathsf{h}_{k,w}(K;A) \cong S_{k,w}(K;A)^* \ \ and \ \ S_{k,w}(K;A) \cong \mathsf{h}_{k,w}(K;A)^*,$$

where $(-)^*$ denotes the A-linear dual $\mathrm{Hom}_A(-,A)$.

Every idele $y \in \widehat{\mathcal{O}_F} \cap \mathbb{A}_F^\times$ can be written as $y = a \prod_{\mathfrak{q}} \varpi_{\mathfrak{q}}^{e(\mathfrak{q})} u$ with $u \in \det V_{11}(\mathfrak{N})$ and $a \in \mathcal{O}_{F,\mathfrak{N}}^\times$. Write $\mathfrak{n}$ for the ideal $\left(\prod_{\mathfrak{q} \nmid \mathfrak{N}} \varpi_{\mathfrak{q}}^{e(\mathfrak{q})}\right)\mathcal{O}_F$, then the Hecke operator

$$T_0(y) = T(a,1)T_0(\mathfrak{n}) \prod_{\mathfrak{q}|\mathfrak{N}} U_0(\varpi_{\mathfrak{q}}^{e(\mathfrak{q})}) \tag{2.6}$$

depends only on the idele y. A cuspform that is an eigenvector for all the Hecke operators is called an eigenform and it is normalized when $\mathsf{a}(1,\mathsf{f}) = 1$. Shimura proved ([Shi78], Proposition 2.2) that the eigenvalues for the Hecke operators are algebraic numbers, hence a normalized eigenform $\mathsf{f} \in S_{k,w}(K;\mathbb{C})$ is an element of $S_{k,w}(K;\overline{\mathbb{Q}})$ since the $T_0(y)$-eigenvalue is $\mathsf{a}(y,\mathsf{f})$ for every idele y. For an idele $y \in \widehat{\mathcal{O}_F} \cap \mathbb{A}_F^\times$, let $T(y) = T_0(y)\{y^{t_F - w}\}$.

Definition 2.1.4. *Let $\mathfrak{p} \mid p$ be a prime of $\mathcal{O}_F$ coprime to the level K and $(k,w) \in \mathbb{Z}[\mathrm{I}_F]$ with $k \geq 2t_F$. A normalized eigenform $\mathsf{f} \in S_{k,w}(K;\overline{\mathbb{Q}})$ is nearly ordinary at $\mathfrak{p}$ if the $T_0(\mathfrak{p})$-eigenvalue is a p-adic unit with respect to the specified embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. If f is nearly ordinary at $\mathfrak{p}$ for all $\mathfrak{p} \mid p$ we say that f is p-nearly ordinary.*

Definition 2.1.5. *For every idele $b \in \mathbb{A}_F^\times$ there is an operator $V(b)$ on cuspforms defined by*

$$\mathsf{f}_{|V(b)}(x) = N_{F/\mathbb{Q}}(b\mathcal{O}_F)\mathsf{f}\left(x \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

that acts on p-adic q-expansions as $\mathsf{a}_p(y,\mathsf{f}_{|V(b)}) = b_p^{w-t_F}\mathsf{a}_p(yb^{-1},\mathsf{f})$ (this operator is denoted $[b]$ in [Hid91], Section 7B). Its normalization $[b] = \{b^{t_F-w}\}V(b)$ acts on q-expansions by $\mathsf{a}(y,\mathsf{f}_{|[b]}) = \mathsf{a}(yb^{-1},\mathsf{f})$.

Remark 2.1.6. *We have $U_0(\varpi) \circ [\varpi] = U(\varpi) \circ V(\varpi) = 1$.*

Let $\mathsf{f} \in S_{k,w}(K,\overline{\mathbb{Q}})$ be a normalized eigenform of level prime to $\mathfrak{p}$. Set

$$\langle \mathfrak{p} \rangle_0 := \{\varpi_{\mathfrak{p}}^{2(w-t_F)}\}\langle \mathfrak{p} \rangle,$$

then the $\langle \mathfrak{p} \rangle_0$-eigenvalue of f is $\psi_{\mathsf{f},0}(\mathfrak{p}) = \{\varpi_{\mathfrak{p}}^{2(w-t_F)}\}\psi_{\mathsf{f}}(\mathfrak{p})$ for $\psi_{\mathsf{f}}(\mathfrak{p})$ the $\langle \mathfrak{p} \rangle$-eigenvalue of f. The $T_0(\mathfrak{p})$-Hecke polynomial for f is given by

$$1 - \mathsf{a}(\mathfrak{p},\mathsf{f})X + \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})\psi_{\mathsf{f},0}(\mathfrak{p})X^2 = (1 - \alpha_{0,\mathfrak{p}}X)(1 - \beta_{0,\mathfrak{p}}X).$$

If f is nearly ordinary at $\mathfrak{p}$, $\mathsf{a}(\mathfrak{p},\mathsf{f})$ is a p-adic unit and we can assume that $\alpha_{0,\mathfrak{p}}$ is a p-adic unit too. The nearly ordinary $\mathfrak{p}$-stabilization of f is the cuspform $\mathsf{f}^{(\mathfrak{p})} = (1 - \beta_{0,\mathfrak{p}}[\varpi_{\mathfrak{p}}])\mathsf{f}$ that has the same Hecke eigenvalues of f away from $\mathfrak{p}$ and whose $U_0(\varpi_{\mathfrak{p}})$-eigenvalue is $\alpha_{0,\mathfrak{p}}$. For $\mathcal{S}$ a finite set of prime $\mathcal{O}_F$-ideals, the $\mathcal{S}$-depletion of a cuspform f is the cuspform $\mathsf{f}^{[\mathcal{S}]} = \prod_{\mathfrak{p}\in\mathcal{S}} (1 - V(\varpi_{\mathfrak{p}}) \circ U(\varpi_{\mathfrak{p}}))\,\mathsf{f}$ whose Fourier coefficient $\mathsf{a}_p(y,\mathsf{f}^{[\mathcal{S}]})$ equals $\mathsf{a}_p(y,\mathsf{f})$ if $y_{\mathcal{S}} \in \mathcal{O}_{F,\mathcal{S}}^\times$ and 0 otherwise.

Lemma 2.1.7. *For all pairs of weights $(k,\nu),(k,\nu') \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ we have the equality $V(p) \circ \Psi_{\nu,\nu'} = p^{\nu'-\nu}\Psi_{\nu,\nu'} \circ V(p)$ of maps from $S_{k,\nu}^*(K,\mathbb{C})$ to $S_{k,\nu'}^*(K,\mathbb{C})$.*

Proof. Follows directly from the definitions. □

2.2 Hida families

We consider compact open subgroups that satisfy $V_1(\mathfrak{N}) \leq K \leq U_0(\mathfrak{N})$. The group $Z_F(K)$ has a finite torsion, so we can fix a prime p coprime to $\mathfrak{N}$ and the order of $\mathrm{Cl}_F^+(\mathfrak{N})$. Let O be a valuation ring in $\overline{\mathbb{Q}}_p$ finite flat over $\mathbb{Z}_p$ containing $\iota_p(\mathcal{V})$. Consider the space of p-adic cuspforms

$$S_{k,w}(K(p^\infty); O) = \varinjlim_{\alpha,\rightarrow} S_{k,w}(K(p^\alpha); O)$$

on which the p-adic Hecke algebra

$$\mathsf{h}_{k,w}(K(p^\infty); O) = \varprojlim_{\leftarrow,\alpha} \mathsf{h}_{k,w}(K(p^\alpha); O)$$

naturally acts. The Hecke operators defined by $\mathbf{T}(y) = \varprojlim T(y) y_p^{w-t_F}$ play an important role in the theory. There is a p-adic norm on the space of p-adic cuspforms $S_{k,w}(K(p^\infty); O)$ defined by $|\mathsf{f}|_p = \sup_y\{|\mathsf{a}_p(y,\mathsf{f})|_p\}$; the resulting completed space is denoted by

$$\overline{S}_{k,w}(K(p^\infty); O)$$

and it has a natural perfect O-pairing with the p-adic Hecke algebra ([Hid91], Theorem 3.1). Each element $\mathsf{f} \in \overline{S}_{k,w}(K(p^\infty); O)$ induces a continuous function $\mathsf{f} : \mathfrak{I} \longrightarrow O$, defined by $y \mapsto \mathsf{a}_p(y, \mathsf{f})$, on the topological semigroup

$$\mathfrak{I} = \widehat{\mathcal{O}_F}^\times F_{\infty,+}^\times / \det V_{11}(p^\infty) F_{\infty,+}^\times,$$

isomorphic to $\mathcal{O}_{F,p}^\times \times \mathscr{I}_F$ for $\mathscr{I}_F$ the free semigroup of integral ideals of F. Hence, there is a continuous embedding $\overline{S}_{k,w}(K(p^\infty); O) \hookrightarrow \mathcal{C}(\mathfrak{I}; O)$ of the completed space of p-adic cuspforms into the continuous functions from $\mathfrak{I}$ to O. The image of the embedding, denoted $\overline{\mathbf{S}}_F(K; O)$, is independent of the weight (k, w) since there exists a canonical algebra isomorphism

$$\mathsf{h}_{k,w}\,(K(p^\infty); O) \cong \mathsf{h}_{2t_F,t_F}\,(K(p^\infty); O)$$

which takes $\mathbf{T}(y)$ to $\mathbf{T}(y)$ ([Hid89b], Theorem 2.3). Hence, we are justified in choosing the notation $\mathbf{h}_F(K; O)$ for $\mathsf{h}_{k,w}(K(p^\infty); O)$ which is independ of the weights up to a canonical isomorphism. From now on, $\overline{\mathbf{S}}_F(\mathfrak{N}, O)$ and $\mathbf{h}_F(\mathfrak{N}; O)$ stand respectively for $\overline{\mathbf{S}}_F(V_1(\mathfrak{N}); O)$ and $\mathbf{h}_F(V_1(\mathfrak{N}); O)$.

Remark 2.2.1. *Nearly holomorphic cuspforms can be seen as p-adic cuspforms ([Hid91], Proposition 7.3). For each nearly holomorphic cuspform* $\mathsf{f} \in N_{k,w,q}(K(p^\alpha); F; O)$ *one can define a p-adic cuspform by setting*

$$c(\mathsf{f}) = \mathcal{N}_F(y)^{-1} \sum_{\xi \in F_+^\times} \mathsf{a}_p(\xi y \mathsf{d}_F, \mathsf{f})(0) q^\xi \in \overline{\mathbf{S}}_F(K; O).$$

It is possible to decompose the compact ring $\mathbf{h}_F(K; O)$ as a direct sum of algebras

$$\mathbf{h}_F(K; O) = \mathbf{h}_F^{\mathrm{n.o.}}(K; O) \oplus \mathbf{h}_F^{\mathrm{ss}}(K; O)$$

in such a way that $\mathbf{T}(p)$ is a unit in $\mathbf{h}_F^{\mathrm{n.o.}}(K; O)$ and it is topologically nilpotent in $\mathbf{h}_F^{\mathrm{ss}}(K; O)$. Furthermore, the idempotent $e_{\mathrm{n.o.}}$ of the nearly ordinary part $\mathbf{h}_F^{\mathrm{n.o.}}(K; O)$ has the familiar expression $e_{\mathrm{n.o.}} = \lim_{n\to\infty} \mathbf{T}(p)^{n!}$. Let $\overline{\mathbf{S}}_F^{\mathrm{n.o.}}(K; O) = e_{\mathrm{n.o.}} \overline{\mathbf{S}}_F(K; O)$ be the space of nearly ordinary p-adic cuspforms.

Consider the topological group $\mathbb{G}_F(K) = Z_F(K) \times \mathcal{O}_{F,p}^{\times}$ equipped with the continuous group homomorphism $\mathbb{G}_F(K) \longrightarrow \mathbf{h}_F^{\text{n.o.}}(K;O)^{\times}$ given by $\langle z, a\rangle \mapsto \langle z\rangle T(a^{-1}, 1)$. As p is prime to the order of $\mathbb{G}_F(K)_{\text{tor}}$, there is a canonical decomposition $\mathbb{G}_F(K) \cong \mathbb{G}_F(K)_{\text{tor}} \times \mathbf{W}_F$ for a $\mathbb{Z}_p$-torsion free subgroup $\mathbf{W}_F$. Then $\mathbf{W}_F \cong \mathbb{Z}_p^r$ for $r = [F : \mathbb{Q}] + 1 + \delta$, where δ is Leopoldt's defect for F, and we denote by $O[\![\mathbf{W}_F]\!] \cong O[\![X_1, \dots, X_r]\!]$ the completed group ring.

Theorem 2.2.2. *([Hid89b], Theorem 2.4) The universal nearly ordinary Hecke algebra $\mathbf{h}_F^{n.o.}(K;O)$ is finite and torsion-free over $\mathbf{\Lambda}_F = O[\![\mathbf{W}_F]\!]$.*

One can write $O[\![\mathbb{G}_F(K)]\!] = \bigoplus_\chi \mathbf{\Lambda}_{F,\chi}$ as a direct sum ranging over all the characters of $\mathbb{G}_F(K)_{\text{tor}}$ where $\mathbf{\Lambda}_{F,\chi} \cong \mathbf{\Lambda}_F$, and obtain a similar decomposition of the universal nearly ordinary Hecke algebra $\mathbf{h}_F^{\text{n.o.}}(K;O) = \bigoplus_\chi \mathbf{h}_F^{\text{n.o.}}(K;O)_\chi$.

Definition 2.2.3. *Let K be a compact open subgroup satisfying $V_1(\mathfrak{N}) \leq K \leq U_0(\mathfrak{N})$ for an $\mathcal{O}_F$-ideal $\mathfrak{N}$ prime to p. Given a character $\chi : \mathbb{G}(K)_{tor} \to O^{\times}$ and a $\mathbf{\Lambda}_{F,\chi}$-algebra $\mathbf{I}$, we define the space of nearly ordinary $\mathbf{I}$-adic cuspforms of tame level K and character χ to be*

$$\overline{\mathbf{S}}_F^{n.o.}(K, \chi; \mathbf{I}) = \mathrm{Hom}_{\mathbf{\Lambda}_{F,\chi}\text{-}mod}(\mathbf{h}_F^{n.o.}(K;O)_\chi, \mathbf{I}).$$

We call Hida families those homomorphisms that are homomorphisms of $\mathbf{\Lambda}_{F,\chi}$-algebras.

Given a pair of weights $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$, with $k - 2w = mt_F$, and finite order characters $\psi : Z_F(K) \to O^{\times}, \psi' : \mathcal{O}_{F,p}^{\times} \to O^{\times}$ one can define a homorphism $\mathbb{G}_F(K) \to O^{\times}$ by

$$(z, a) \mapsto \psi(z)\psi'(a)\mathcal{N}_F(z)^m a^{t_F - w},$$

which determines an O-algebra homomorphism $\mathrm{P}_{k,w,\psi,\psi'} : O[\![\mathbb{G}_F(K)]\!] \to O$. Let's fix an algebraic closure $\overline{\mathbf{L}}$ of the fraction field $\mathbf{L}$ of $\mathbf{\Lambda}_{F,\chi}$ with an embedding $\overline{\mathbb{Q}}_p \hookrightarrow \overline{\mathbf{L}}$. Suppose $\lambda : \mathbf{h}_F^{\text{n.o.}}(K;O)_\chi \to \overline{\mathbf{L}}$ is an $\mathbf{\Lambda}_{F,\chi}$-linear map; since the universal nearly ordinary Hecke algebra is finite over $\mathbf{\Lambda}_{F,\chi}$, the image of λ is contained in the integral closure $\mathbf{I}_\lambda$ of $\mathbf{\Lambda}_{F,\chi}$ in a finite extension $\mathbf{K}_\lambda$ of $\mathbf{L}$.

Definition 2.2.4. *Let $\mathbf{I}$ be a finite integrally closed extension of $\mathbf{\Lambda}_{F,\chi}$. We denote by $\mathcal{A}_\chi(\mathbf{I})$ the set of* arithmetic points, *i.e., the subset of* $\mathrm{Hom}_{O\text{-alg}}(\mathbf{I}, \overline{\mathbb{Q}}_p)$ *consisting of homomorphisms that coincide with some $\mathrm{P}_{k,w,\psi,\psi'}$ (with $k \geq 2t_F, w \leq t_F$) on $\mathbf{\Lambda}_{F,\chi}$.*

If $\mathrm{P} \in \mathcal{A}_\chi(\mathbf{I}_\lambda)$, $\mathrm{P}_{|\mathbf{\Lambda}_{F,\chi}} = \mathrm{P}_{k,w,\psi,\psi'|\mathbf{\Lambda}_{F,\chi}}$, the composite $\lambda_\mathrm{P} = \mathrm{P} \circ \lambda$ induces a $\overline{\mathbb{Q}}_p$-linear map $\lambda_\mathrm{P} : \mathsf{h}_{k,w}^{\text{n.o.}}(K(p^\alpha); \overline{\mathbb{Q}}_p) \longrightarrow \overline{\mathbb{Q}}_p$ for some $\alpha > 0$ ([Hid89b], Theorem 2.4). Therefore, the duality between Hecke algebra and cuspforms produces a unique p-adic cuspform $\mathsf{f}_\mathrm{P} \in S_{k,w}^{\text{n.o.}}(K(p^\alpha); \overline{\mathbb{Q}}_p)$ that satisfies $\mathsf{a}_p(y, \mathsf{f}_\mathrm{P}) = \lambda_\mathrm{P}(\mathbf{T}(y))$ for all integral ideles y. Furthermore, if λ is an algebra homomorphism, each specialization at an arithmetic point is an eigenform and so classical, i.e., an element of $S_{k,w}^{\text{n.o.}}(K(p^\alpha); \overline{\mathbb{Q}})$. On the other hand, if $\mathsf{f} \in S_{k,w}(K(p^\alpha); \overline{\mathbb{Q}})$ is an eigenform for all Hecke operators and its $U_0(p)$-eigenvalue is a p-adic unit with respect to the fixed p-adic embedding ι_p, then there is character χ, a finite integrally closed extension $\mathbf{I}_{\mathscr{F}}$ of $\mathbf{\Lambda}_{F,\chi}$ and a nearly ordinary $\mathbf{I}_{\mathscr{F}}$-adic Hida family $\mathscr{F} : \mathbf{h}_F^{\text{n.o.}}(K;O) \to \mathbf{I}_{\mathscr{F}}$ passing through f ([Hid89b], Theorem 2.4).

Definition 2.2.5. *We define the set of* crystalline *points, $\mathcal{A}_\chi^\circ(\mathbf{I})$, to be the subset of arithmetic points $\mathrm{P} \in \mathcal{A}_\chi(\mathbf{I})$ such that $\mathrm{P}_{|\mathbf{\Lambda}_{F,\chi}} = \mathrm{P}_{k,w,\psi,1|\mathbf{\Lambda}_{F,\chi}}$ for ψ factoring through $\psi : \mathrm{cl}_F^+(\mathfrak{N}) \to O^{\times}$ and the eigenform f_P is p-old.*

Specializations of Hida families with trivial nebentype at p are automatically p-old when $k > 2t_F$ ([Hid89b], Lemma 12.2).

2.2.1 Diagonal restriction

If L/F is an extension of totally real fields, there is a restriction map $\mathrm{I}_L \to \mathrm{I}_F$ which induces a group homomorphism $\mathbb{Z}[\mathrm{I}_L] \to \mathbb{Z}[\mathrm{I}_F]$ denoted by $\ell \mapsto \ell_{|F}$ and satisfies $(t_L)_{|F} = [L:F] \cdot t_F$. Let $\mathfrak{N}$ an ideal of $\mathcal{O}_F$, the natural inclusion $\zeta : \mathrm{GL}_2(\mathbb{A}_F) \hookrightarrow \mathrm{GL}_2(\mathbb{A}_L)$ defines by composition a *diagonal restriction* map

$$\zeta^* : S_{\ell,x}(V_{11}(\mathfrak{N}\mathcal{O}_L); L; \mathbb{C}) \to S_{\ell_{|F},x_{|F}}(V_{11}(\mathfrak{N}); F; \mathbb{C}).$$

Proposition 2.2.6. *Let $b \in \mathbb{A}_F^\times$. For any cuspform $\mathrm{g} \in S_{\ell,x}(V_{11}(\mathfrak{N}\mathcal{O}_L); L; \mathbb{C})$ we have*

$$\zeta^*(\mathrm{g}_{|V(b)}) = N_{F/\mathbb{Q}}(b\mathcal{O}_F)^{1-[L:F]}(\zeta^*\mathrm{g})_{|V(b)}.$$

Proof. Follows directly from the definitions. □

Definition 2.2.7. *Let L/F be an extension of totally real number fields and let $\mathfrak{N}$ be an $\mathcal{O}_F$-ideal. For every prime p coprime to $\mathfrak{N}$ and the orders of $Z_F(V_1(\mathfrak{N}))_{\mathrm{tor}}, Z_L(V_1(\mathfrak{N}\mathcal{O}_L))_{\mathrm{tor}}$, diagonal restriction of cuspforms induces by O-duality a map between universal Hecke algebras*

$$\zeta : \mathbf{h}_F(\mathfrak{N}; O) \longrightarrow \mathbf{h}_L(\mathfrak{N}\mathcal{O}_L; O).$$

The element $\zeta(\mathbf{T}(y))$ is determined by the equality

$$\mathsf{a}_p\big(1, \mathrm{g}_{|\zeta(\mathbf{T}(y))}\big) = \mathsf{a}_p\big(1, (\zeta^*\mathrm{g})_{|\mathbf{T}(y)}\big) \qquad \forall \mathrm{g} \in \overline{\mathbf{S}}_L(\mathfrak{N}\mathcal{O}_L; O).$$

We endow $O[\![\mathbb{G}_L(V_1(\mathfrak{N}\mathcal{O}_L))]\!]$ with the $O[\![\mathbb{G}_F(V_1(\mathfrak{N}))]\!]$-algebra structure $[(z,a)] \mapsto [(z,a)]a^{-t_F}$. The homomorphism ζ is also $O[\![\mathbb{G}_F(V_1(\mathfrak{N}))]\!]$-linear because diamond operators and operators $T(a,1)$ for $a \in \mathcal{O}_{F,p}^\times$ commute with diagonal restriction:

$$(\zeta^*\mathrm{g})_{|\langle z\rangle} = \zeta^*(\mathrm{g}_{|\langle z\rangle}) \quad \textit{and} \quad (\zeta^*\mathrm{g})_{|T(a,1)} = \zeta^*(\mathrm{g}_{|T(a,1)}).$$

2.2.2 On differential operators

For each $\mu \in \mathrm{I}_L$ there is an operator on p-adic cuspforms $d_\mu : \overline{\mathbf{S}}_L(\mathfrak{Q}; O) \to \overline{\mathbf{S}}_L(\mathfrak{Q}; O)$ given on q-expansions by $\mathsf{a}_p(y, d_\mu \mathrm{g}) = y_p^\mu \mathsf{a}_p(y, \mathrm{g})$. The definition can be extended to all $r \in \mathbb{N}[\mathrm{I}_L]$ by setting $d^r = \prod_{\mu \in \mathrm{I}_L} d_\mu^{r_\mu}$ ([Hid91], Section 6G).

Lemma 2.2.8. *Let $r \in \mathbb{N}[\mathrm{I}_L]$ and let $\mathrm{g} \in S_{\ell,x}(V_1(\mathfrak{Q}p^\alpha); L; O)$ be a cuspform, then*

$$e_{n.o.}\Pi^{hol}\zeta^*(\delta_\ell^r \mathrm{g}) = e_{n.o.}\zeta^*(d^r \mathrm{g}),$$

where δ_ℓ^r is the Maass-Shimura differential operator (2.5).

Proof. Proposition 7.3 of [Hid91] gives the equality $e_{\mathrm{n.o.}}\Pi^{\mathrm{hol}}\zeta^*(\delta_\ell^r \mathrm{g}) = e_{\mathrm{n.o.}}c(\zeta^*(\delta_\ell^r \mathrm{g}))$. Since $c(\zeta^*(\delta_\ell^r \mathrm{g})) = \zeta^* c(\delta_\ell^r \mathrm{g})$, we conclude by showing that $c(\delta_\ell^r \mathrm{g}) = d^r c(\mathrm{g})$. Indeed,

$$\begin{aligned} \mathsf{a}_p(y, c(\delta_\ell^r \mathrm{g})) = \mathsf{a}_p(y, \delta_\ell^r \mathrm{g})(0) &= y_p^{x-t_L+r}\mathcal{N}_L(a_i)^{-1}\xi^{t_L-x-r}a(\xi, \delta_\ell^r \mathrm{g}_i)(0) \\ &= y_p^{x-t_L+r}\mathcal{N}_L(a_i)^{-1}\xi^{t_L-x}a(\xi, \mathrm{g}_i) = \mathsf{a}_p(y, d^r c(\mathrm{g})). \end{aligned}$$

□

2.3 Twisted triple product L-functions

2.3.1 Complex L-functions

Let L/F be a quadratic extension of totally real number fields, $\mathfrak{L} \triangleleft \mathcal{O}_L$ and $\mathfrak{N} \triangleleft \mathcal{O}_F$ ideals. Two primitive eigenforms $\mathrm{g} \in S_{\ell,x}(V_1(\mathfrak{L}); L; \overline{\mathbb{Q}})$ and $\mathrm{f} \in S_{k,w}(V_1(\mathfrak{N}); F; \overline{\mathbb{Q}})$ generate irreducible cuspidal automorphic representations π, σ of $G_L(\mathbb{A}), G_F(\mathbb{A})$ respectively.
Let $\pi^u = \pi \otimes |\bullet|_{\mathbb{A}_L}^{n/2}$, $\sigma^u = \sigma \otimes |\bullet|_{\mathbb{A}_F}^{m/2}$ be their unitarizations, where n, m are the integers satisfying $n \cdot t_L = \ell - 2x$, $m \cdot t_F = k - 2w$. One can define a unitary representation of $G_{L\times F} = \mathrm{Res}_{L\times F/F}(\mathrm{GL}_{2,L\times F})$ by $\Pi = \pi^u \otimes \sigma^u$. Let $\rho : \Gamma_F \to S_3$ be the homomorphism mapping the absolute Galois group of F to the symmetric group over 3 elements associated with the étale cubic algebra $(L \times F)/F$. The L-group ${}^L(G_{L\times F})$ is given by the semi-direct product $\widehat{G} \rtimes \Gamma_F$ where Γ_F acts on $\widehat{G} = \mathrm{GL}_2(\mathbb{C})^{\times 3}$ through ρ.

Definition 2.3.1. *The twisted triple product L-function associated with the unitary autormophic representation Π is given by the Euler product*

$$L(s, \Pi, \mathrm{r}) = \prod_v L_v(s, \Pi_v, \mathrm{r})^{-1}$$

where Π_v is the local representation at the finite prime v of F appearing in the restricted tensor product decomposition $\Pi = \bigotimes_v' \Pi_v$ and representation r gives the action of the L-group of $G_{L\times F}$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ which restricts to the natural 8-dimensional representation of $\widehat{G}$ and for which Γ_F acts via ρ permuting the vectors.

Remark 2.3.2. *([PSR87], page 111). When Π_v is ramified, let q_v be the cardinality of the residue field of F_v, then the local L-factor at v of $L(s, \Pi, \mathrm{r})$ is given by*

$$L_v\left(\frac{1+s}{2}, \Pi_v, \mathrm{r}\right) = P_v(q_v^{-s})$$

for a certain polynomial $P_v(X) \in 1 + X\mathbb{C}[X]$. In particular, it is non-vanishing at $s = 1/2$.

Let v be a prime of F unramified in L for which Π_v is an unramified principal series, i.e., $v \nmid \mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{L}) \cdot d_{L/F}$. We write ϖ_v for a uniformizer of F_v and q_v for the cardinality of the residue field of F_v. If $v = \mathscr{V} \cdot \overline{\mathscr{V}}$ splits in L, the $\mathrm{GL}_2(F_v)^{\times 3}$-representation Π_v can be written as $\Pi_v = \pi(\chi_{1,\mathscr{V}}, \chi_{2,\mathscr{V}}) \otimes \pi(\chi_{1,\overline{\mathscr{V}}}, \chi_{2,\overline{\mathscr{V}}}) \otimes \pi(\psi_{1,v}, \psi_{2,v})$ and the local Euler factor is given by

$$L_v(s, \Pi_v, \mathrm{r}) = \prod_{i,j,k} \left(1 - \chi_{i,\mathscr{V}}(\varpi_v)\chi_{j,\overline{\mathscr{V}}}(\varpi_v)\psi_{k,v}(\varpi_v)q_v^{-s}\right). \tag{2.7}$$

When the prime v is inert in L, the $\mathrm{GL}_2(L_v) \times \mathrm{GL}_2(F_v)$-representation Π_v can be written as $\Pi_v = \pi(\chi_{1,v}, \chi_{2,v}) \otimes \pi(\psi_{1,v}, \psi_{2,v})$ and the local Euler factor is given by

$$L_v(s, \Pi_v, \mathrm{r}) = \prod_{i,j} \left(1 - \chi_{i,v}(\varpi_v)\psi_{j,v}(\varpi_v)q_v^{-s}\right) \times \prod_k \left(1 - \chi_{1,v}(\varpi_v)\chi_{2,v}(\varpi_v)\psi_{k,v}^2(\varpi_v)q_v^{-2s}\right). \tag{2.8}$$

Assume the central character ω_Π of Π is trivial when restricted to $\mathbb{A}_F^\times$, then the complex L-function $L(s, \Pi, \mathrm{r})$ has meromorphic continuation to $\mathbb{C}$ with possible poles at $0, \frac{1}{4}, \frac{3}{4}, 1$ and functional equation $L(s, \Pi, \mathrm{r}) = \epsilon(s, \Pi, \mathrm{r})L(1 - s, \Pi, \mathrm{r})$ ([PSR87], Theorems 5.1, 5.2, 5.3).

Remark 2.3.3. *The relation between Satake parameters of π^u, σ^u and Hecke eigenvalues of the primitive eigenforms $\mathrm{g}^u, \mathrm{f}^u$ can be given explicitly as follows. Suppose $v \nmid \mathfrak{Q}$ and $v = \mathscr{V}\overline{\mathscr{V}}$ splits in L, then*

$$\mathrm{g}^u{}_{|T(\mathscr{V})} = q_v^{1/2}\left(\chi_{1,\mathscr{V}}(\varpi_v) + \chi_{2,\mathscr{V}}(\varpi_v)\right)\mathrm{g}^u, \quad \mathrm{g}^u{}_{|T(\overline{\mathscr{V}})} = q_v^{1/2}\left(\chi_{1,\overline{\mathscr{V}}}(\varpi_v) + \chi_{2,\overline{\mathscr{V}}}(\varpi_v)\right)\mathrm{g}^u. \tag{2.9}$$

Moreover, if $v \nmid \mathfrak{Q}$ and v is inert in L then

$$\mathrm{g}^u{}_{|T(v\mathcal{O}_L)} = q_v\left(\chi_{1,v}(\varpi_v) + \chi_{2,v}(\varpi_v)\right)\mathrm{g}^u. \tag{2.10}$$

Finally, if $v \nmid \mathfrak{N}$ a finite place of F we have

$$\mathrm{f}^u{}_{|T(v)} = q_v^{1/2}\left(\psi_{1,v}(\varpi_v) + \psi_{2,v}(\varpi_v)\right)\mathrm{f}^u. \tag{2.11}$$

2.3.2 Central L-values and period integrals

Let $D_{/F}$ be a quaternion algebra. We denote be Π^D the irreducible unitary cuspidal automorphic representation of $D^\times(\mathbb{A}_{L\times F})$ associated with Π by the Jacquet-Langlands correpondence when it exists. For a vector $\phi \in \Pi^D$ one defines its period integral as

$$\mathrm{I}^D(\phi) = \int_{[D^\times(\mathbb{A}_F)]} \phi(x)\mathrm{d}x$$

where $[D^\times(\mathbb{A}_F)] = \mathbb{A}_F^\times D^\times(F)\backslash D^\times(\mathbb{A}_F)$. To simplify the notation we write $\mathrm{I}(\phi)$ to denote the period integral for the quaternion algebra $\mathrm{M}_2(F)$.

Theorem 2.3.4. *Let $\eta : \mathbb{A}_F^\times \to \mathbb{C}^\times$ be the quadratic character attached to L/F by class field theory. Then the following are equivalent:*

(1) The central L-value $L(\frac{1}{2}, \Pi, \mathrm{r})$ does not vanish, and for every place v of F the local ϵ-factor satisfies $\epsilon_v(\frac{1}{2}, \Pi_v, \mathrm{r}) \cdot \eta_v(-1) = 1$.

(2) There exists a vector $\phi \in \Pi$, called a test vector, whose period integral $I(\phi)$ does not vanish.

Proof. (1) $\implies$ (2). By the Jacquet conjecture, as proved in ([PSP08], Theorem 1.1), the non-vanishing of the central value implies that there exists a quaternion algebra D/F and a vector $\phi \in \Pi^D$ such that its period integral is non-zero, i.e., $\mathrm{I}^D(\phi) \neq 0$. We want to show that the assumption on local ϵ-factors forces the quaternion algebra to be split everywhere. Ichino's formula ([Ich08], Theorem 1.1) gives an equality, up to non-zero constants,

$$\mathrm{I}^D \cdot \mathrm{I}^{\widetilde{D}} \doteq L(\frac{1}{2}, \Pi, \mathrm{r}) \cdot \prod_v \mathrm{I}_v^D$$

of linear forms in $\mathrm{Hom}_{D^\times(\mathbb{A}_F)\times D^\times(\mathbb{A}_F)}\left(\Pi^D \otimes \widetilde{\Pi}^D, \mathbb{C}\right)$ where $\widetilde{\Pi}^D$ is the contragredient representation and the I_v^D's are local linear forms in

$$\mathrm{Hom}_{D^\times(F_v)\times D^\times(F_v)}\left(\Pi_v^D \otimes (\widetilde{\Pi}^D)_v, \mathbb{C}\right).$$

Suppose v is a place of F at which the quaternion algebra D ramifies, i.e. $v \mid \mathrm{disc}D$. Requiring the value of the expression $\epsilon_v(\frac{1}{2}, \Pi_v, \mathrm{r}) \cdot \eta_v(-1)$ to be equal to 1 forces the local Hom-space $\mathrm{Hom}_{D^\times(F_v)\times D^\times(F_v)}\left(\Pi_v^D \otimes (\widetilde{\Pi}^D)_v, \mathbb{C}\right)$ to be trivial ([Gan08], Theorem 1.2); in particular it forces the local linear form I_v^D to be trivial. This produces a contradiction because the LHS of Ichino's formula is non-trivial. Indeed, choosing the complex conjugate $\bar{\phi} \in \overline{\Pi^D} \cong \widetilde{\Pi}^D$ of the test vector ϕ we compute that

$$\mathrm{I}^D \cdot \widetilde{\mathrm{I}^D}(\phi \otimes \bar{\phi}) = \left|\mathrm{I}^D(\phi)\right|^2 \neq 0.$$

Hence, the discriminant of D has to be trivial, i.e., $D = M_2(F)$.
$(2) \implies (1)$. The existence of a test vector $\phi \in \Pi$ implies the non-vanishing of the central value $L(\frac{1}{2}, \Pi, \mathrm{r})$ by Jacquet conjecture. Moreover, Ichino's formula provides us with non-trivial local linear forms, the I_v's, in the local Hom-spaces

$$\mathrm{Hom}_{\mathrm{GL}_2(F_v)\times \mathrm{GL}_2(F_v)}\left(\Pi_v \otimes (\widetilde{\Pi})_v, \mathbb{C}\right)$$

which force the equality $\epsilon_v(\frac{1}{2}, \Pi_v, \mathrm{r}) \cdot \eta_v(-1) = 1$ for every place v of F ([Gan08], Theorem 1.1). □

Remark 2.3.5. *We can give sufficient conditions on the eigenforms* $\mathrm{g} \in S_{\ell,x}(V_1(\mathfrak{Q}); L; \overline{\mathbb{Q}})$ *and* $\mathrm{f} \in S_{k,w}(V_1(\mathfrak{N}); F; \overline{\mathbb{Q}})$ *such that the local* ϵ*-factors of the automorphic representation* Π *satisfy the hypothesis of Theorem* 2.3.4. *The local* ϵ*-factor at the archimedean places of* F *satisfy the hypothesis of the theorem if the weights of* g *and* f *are F-dominated (Definition* 2.0.1*). Moreover, the same is true for the* ϵ*-factors at the finite places if we assume that* $N_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$ *and* $\mathfrak{N}$ *are coprime and that every finite prime* v *dividing* $\mathfrak{N}$ *splits in* L *([Pra92], Theorems B, D and Remark 4.1.1).*

Proposition 2.3.6. *For all finite places* v *of* F *away from the level of* Π *and unramified in* L/F*, a newvector in* Π_v *is a choice of test-vector for Ichino's local linear functional.*

Proof. If v is a place splitting in L, the claim follows from ([Pra90], Theorem 5.10). We show that the proof given by Prasad can be adapted to deal with the inert case as follows. Our claim is that the image of the spherical vector under the non-trivial linear functional Y : $(\pi^u)_v \to (\widetilde{\sigma^u})_v$, unique up to scaling, is non-zero. As in ([Pra92], Section 4) we can assume that $(\pi^u)_v$ is the principal series V_χ for the character of the Borel

$$\chi\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} = \alpha(a)\beta(d)^{-1}, \quad \text{for unramified characters } \alpha, \beta : L_v^\times \to \mathbb{C}^\times,$$

so that the representation V_χ can be realized in the space of functions over $\mathbb{P}^1_{L_v}$ and the spherical vector corresponds to the constant function $1_{\mathbb{P}^1_{L_v}}$. The projective line $\mathbb{P}^1_{L_v}$ can be decomposed into an open and a closed orbit for the action of $\mathrm{GL}_2(F_v)$,

$$\mathbb{P}^1_{L_v} = \left(\mathbb{P}^1_{L_v} \setminus \mathbb{P}^1_{F_v}\right) \coprod \mathbb{P}^1_{F_v},$$

which produces an exact sequence of $\mathrm{GL}_2(F_v)$-modules

$$0 \longrightarrow \mathrm{ind}_{L_v^\times}^{\mathrm{GL}_2(F_v)}(\chi') \longrightarrow V_\chi \longrightarrow \mathrm{Ind}_{\mathrm{B}(F_v)}^{\mathrm{GL}_2(F_v)}(\chi\delta_{F_v}^{1/2}) \longrightarrow 0 \tag{2.12}$$

for $\chi' : L_v^\times \to \mathbb{C}^\times$ the character defined by $\chi'(x) = \alpha(x)\beta(\overline{x})$. If $\mathrm{Ind}_{\mathrm{B}(F_v)}^{\mathrm{GL}_2(F_v)}(\chi\delta_{F_v}^{1/2})$ is isomorphic to the contragradient representation $(\widetilde{\sigma^u})_v$ then we are done, because $1_{\mathbb{P}^1_{L_v}} \mapsto 1_{\mathbb{P}^1_{F_v}} \neq 0$.

Otherwise, suppose $\Upsilon(1_{\mathbb{P}^1_{L_v}}) = 0$. Let T_v be the Hecke operator given by the double coset

$$T_v = \left[\mathrm{GL}_2(\mathcal{O}_{F_v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_v})\right],$$

then the function

$$\frac{1}{(q_v+1)\chi\delta^{1/2}(\varpi_v)} \left(T_v(1_{\mathbb{P}^1_{L_v}}) - q_v\chi\delta^{1/2}(\varpi_v) - \chi\delta^{1/2}(1/\varpi_v)\right) \tag{2.13}$$

is the constant function 1 on the $\mathrm{GL}_2(\mathcal{O}_{F_v})$-orbit of $\mathbb{P}^1_{L_v}$ consisting of those points that reduce to a point in $\mathbb{P}^1(\mathcal{O}_{L_v}/\varpi_v) \setminus \mathbb{P}^1(\mathcal{O}_{F_v}/\varpi_v)$, and the constant function zero everywhere else. Therefore, the function (2.13) is an element of $\mathrm{ind}_{L_v^\times}^{\mathrm{GL}_2(F_v)}(\chi')$ because of the short exact sequence (2.12). The function (2.13) is sent to zero by Υ by $\mathrm{GL}_2(F_v)$-equivariance, but at the same time that is not possible because we can explicitly describe the elements of $\mathrm{Hom}_{\mathrm{GL}_2(F_v)}\left(\mathrm{ind}_{L_v^\times}^{\mathrm{GL}_2(F_v)}(\chi'), (\widetilde{\sigma^u})_v\right)$ in terms of integration over $\mathrm{GL}_2(\mathcal{O}_{F_v})$-orbits of $\mathbb{P}^1_{L_v}$ giving a contradiction.

□

2.4 p-adic L-functions

Let $\mathsf{g} \in S_{\ell,x}(V_1(\mathfrak{Q}); L; E)$, $\mathsf{f} \in S_{k,w}(V_1(\mathfrak{N}); F; E)$ be primitive eigenforms defined over a number field E whose weights are F-dominated. We assume the central character ω_Π of Π to be trivial when restricted to $\mathbb{A}_F^\times$, that the central L-value $L(\frac{1}{2}, \Pi, \mathrm{r})$ does not vanish, and that for every place v of F we have the condition $\epsilon_v(\frac{1}{2}, \Pi_v, \mathrm{r})\eta_v(-1) = 1$ on local ϵ-factors satisfied. Then there exists a vector $\phi \in \Pi$ such that the period integral $\mathrm{I}(\phi)$ is non-zero (Theorem 2.3.4). Let $\mathfrak{J}$ be the element

$$\mathfrak{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{\mathrm{I}_F} \in \mathrm{GL}_2(\mathbb{R})^{\mathrm{I}_F}.$$

For any $\mathsf{h} \in \sigma^u$ we define $\mathsf{h}^{\mathfrak{J}} \in \sigma^u$ to be the vector obtained by right translation $\mathsf{h}^{\mathfrak{J}}(g) = \mathsf{h}(g\mathfrak{J})$. If h has weight $k \in \mathbb{Z}[\mathrm{I}_F]$ then $\mathsf{h}^{\mathfrak{J}}(h)$ has weight $-k$.

Lemma 2.4.1. *Let $r \in \mathbb{N}[\mathrm{I}_L]$ be such that $k = (\ell + 2r)_{|F}$ and $w = (x+r)_{|F}$. Then there is an $\mathcal{O}_F$-ideal $\mathfrak{A}$ supported on a subset of the prime factors of $\mathfrak{N} \cdot N_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$ such that a test vector ϕ can be chosen to be of the form $\phi = (\delta^r \breve{\mathsf{g}})^u \otimes (\mathsf{f}^{\mathfrak{J}})^u$ for $\breve{\mathsf{g}} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; E)$ eigenform for all Hecke operators outside $\mathfrak{N} \cdot N_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$ with the same Hecke eigenvalues of* g.

Proof. By linearity of the period integral we can assume ϕ to be a simple tensor. We can also assume $\phi = \delta^r \vartheta \otimes \nu^{\mathfrak{J}} \in \Pi$ because the archimedean linear functional appearing in Ichino's formula is non-zero if and only if the sum of the weights of the local vectors is zero.

Proposition 2.3.6 allows us to take ϑ_v and ν_v newvectors for all finite places that do not divide $\mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$. Moreover, by direct inspection of Ichino's local functionals – expressed in terms of matrix coefficients – one can see that we are allowed to choose the newvector ν_v for all finite places. Note that newvectors are mapped to newvectors by the isomorphism $\pi \otimes \sigma \xrightarrow{\sim} \pi^u \otimes \sigma^u$ as in (2.3). Therefore we can write $\phi = \delta^r \vartheta \otimes \nu^{\mathfrak{J}}$ as $(\delta^r \breve{\mathsf{g}})^u \otimes (\mathsf{f}^{\mathfrak{J}})^u$, for $\breve{\mathsf{g}} \in \pi$ of level $U(\mathfrak{B}\mathcal{O}_L)$ for some $\mathcal{O}_F$-ideal $\mathfrak{B}$ supported on a subset of the places dividing $\mathfrak{N} \cdot \mathrm{N}_{L/F}(\mathfrak{Q}) \cdot d_{L/F}$. We conclude by showing that we can assume that

$\breve{g} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; E)$ for $\mathfrak{A} = \mathfrak{B}^2$. Indeed, right translation by

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \qquad b\mathcal{O}_F = \mathfrak{B},$$

induces an injection $S_{\ell,x}(U(\mathfrak{B}\mathcal{O}_L); L; E) \hookrightarrow S_{\ell,x}(V_{11}(\mathfrak{B}^2\mathcal{O}_L); L; E)$ equivariant for the action of Hecke operators away from the level and that change the period by a non-zero constant. □

When the test-vector ϕ is as in Lemma 2.4.1, we can rewrite the period integral $\mathrm{I}(\phi)$ as a Petersson inner product

$$\mathrm{I}(\phi) = \int_{[\mathrm{GL}_2(\mathbb{A}_F)]} (\delta^r \breve{g})^u \otimes (\mathrm{f}^{\mathfrak{I}})^u \mathrm{d}^\times x = \langle \zeta^* (\delta^r \breve{g}), \mathrm{f}^* \rangle \tag{2.14}$$

where $\mathrm{f}^* = \overline{(\mathrm{f}^{\mathfrak{I}})}$ is the cuspform in $S_{k,w}(V_1(\mathfrak{N}); F; E)$ whose Fourier coefficients are complex conjugates of those of f. We conclude the section with a proposition showing that a good transcendental period for the central L-value of the twisted triple product L-function is the Petersson norm of the eigenform f^*.

Proposition 2.4.2. *Let E be a number field and let $\mathrm{f} \in S_{k,w}(V_1(\mathfrak{N}); F; E)$ be a primitive cuspform spanning an irreducible cuspidal automorphic representation σ. Then for any $\varphi \in S_{k,w}(V_{11}(\mathfrak{A}); F; E)$ the Petersson inner product $\langle \varphi, \mathrm{f} \rangle$ is a E-rational multiple of $\langle \mathrm{f}, \mathrm{f} \rangle$.*

Proof. We follow the argument of ([DR14], Lemma 2.12). Note that the Petersson inner product $\langle \varphi, \mathrm{f} \rangle$ depends only on the projection $e_{\mathrm{f}}\varphi$ of φ to σ. In fact, by the orthogonality of the character eigenspaces for the action of $V_1(\mathfrak{A})/V_{11}(\mathfrak{A})$, the only relevant part is the projection to $e_{\mathrm{f}} S_{k,w}(V_1(\mathfrak{A}); F; E)$. This E-vector space is spanned by the cuspforms

$$\{\mathrm{f}_{\mathfrak{a}} |\, \mathrm{f}_{\mathfrak{a}}(x) = \mathrm{f}(x s_{\mathfrak{a}})\}_{\mathfrak{a} | \frac{\mathfrak{A}}{\mathfrak{N}}}, \qquad s_{\mathfrak{a}} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \; a\mathcal{O}_F = \mathfrak{a}$$

for all ideals $\mathfrak{a}$ dividing $\mathfrak{A}/\mathfrak{N}$ ([Miy71], Proposition 6 and [Shi78], Proposition 2.3). Thus, it suffices to prove the statement for $\mathrm{f}_{\mathfrak{a}}$ when $\mathfrak{a} \mid \mathfrak{A}/\mathfrak{N}$. We prove the claim by induction on the prime divisors of $\mathfrak{a}$. If $\mathfrak{a} = \mathcal{O}_F$ then the claim is clear. Let $\mathfrak{q}$ be a prime dividing $\mathfrak{a}$. If $\mathfrak{q} \mid \mathfrak{N}$, we compute

$$\mathrm{a}_p(\varpi_{\mathfrak{q}}, \mathrm{f})\langle \mathrm{f}, \mathrm{f}_{\mathfrak{a}/\mathfrak{q}} \rangle = \langle U_0(\varpi_{\mathfrak{q}})\mathrm{f}, \mathrm{f}_{\mathfrak{a}/\mathfrak{q}} \rangle = \frac{\chi_{\mathrm{f}}(\varpi_{\mathfrak{q}})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^{m-1}} \langle \mathrm{f}, \mathrm{f}_{\mathfrak{a}} \rangle,$$

while, if $\mathfrak{q} \nmid \mathfrak{N}$,

$$\mathrm{a}_p(\varpi_{\mathfrak{q}}, \mathrm{f})\langle \mathrm{f}, \mathrm{f}_{\mathfrak{a}/\mathfrak{q}} \rangle = \begin{cases} \frac{(\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})+1)\chi_{\mathrm{f}}(\varpi_{\mathfrak{q}})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^m} \langle \mathrm{f}, \mathrm{f}_{\mathfrak{a}} \rangle & \text{if } \mathfrak{q}^2 \nmid \mathfrak{a} \\ \frac{\chi_{\mathrm{f}}(\varpi_{\mathfrak{q}})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^{m-1}} \langle \mathrm{f}, \mathrm{f}_{\mathfrak{a}} \rangle + \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^m \langle \mathrm{f}, \mathrm{f}_{\mathfrak{a}/\mathfrak{q}^2} \rangle & \text{if } \mathfrak{q}^2 \mid \mathfrak{a}, \end{cases}$$

concluding the inductive step. □

2.4.1 Construction

Given primitive eigenforms

$$\mathrm{g}_\circ \in S_{\ell_\circ, x_\circ}(V_1(\mathfrak{Q}); L; \overline{\mathbb{Q}}) \quad \text{and} \quad \mathrm{f}_\circ \in S_{k_\circ, w_\circ}(V_1(\mathfrak{N}); F; \overline{\mathbb{Q}})$$

with $n_\circ t_L = \ell_\circ - 2x_\circ$, $m_\circ t_F = k_\circ - 2w_\circ$ for $n_\circ, m_\circ \in \mathbb{Z}$, we choose an element $\theta \in \mathbb{Z}[\mathrm{I}_L]$ such that $\theta_{|F} = 0 \cdot t_F$, $\theta \equiv_2 w_\circ$ and set

$$r_\circ = r_\circ(\theta) = \sum_{\mu \in \mathrm{I}_L} \left[\frac{(w_\circ)_{\mu_{|F}} + \theta_\mu}{2} - (x_\circ)_\mu \right] \cdot \mu \in \mathbb{Z}[\mathrm{I}_L].$$

Let p be a rational prime unramified in L, coprime to the levels $\mathfrak{Q}, \mathfrak{N}$. We write $\mathcal{P}$ (resp. $\mathcal{Q}$) for the set of prime $\mathcal{O}_L$-ideals (resp. $\mathcal{O}_F$-ideals) dividing p. We suppose $\mathrm{g}_\circ$, $\mathrm{f}_\circ$ are p-nearly ordinary and we denote by $\mathscr{G} \in \overline{\mathbf{S}}_L^{\text{n.o.}}(\mathfrak{Q}, \chi; \mathbf{I}_{\mathscr{G}})$ and $\mathscr{F} \in \overline{\mathbf{S}}_F^{\text{n.o.}}(\mathfrak{N}, \psi; \mathbf{I}_{\mathscr{F}})$ the Hida families passing through nearly ordinary p-stabilizations $\mathrm{g}_\circ^{(p)}$ and $\mathrm{f}_\circ^{(p)}$. We have

$$\chi_{|Z_L(\mathfrak{Q})_{\mathrm{tor}}} = \chi_\circ \mathcal{N}_L^{n_\circ} \quad \text{and} \quad \psi_{|Z_F(\mathfrak{N})_{\mathrm{tor}}} = \psi_\circ \mathcal{N}_F^{m_\circ}$$

for characters $\chi_\circ : \mathrm{cl}_L^+(\mathfrak{Q}) \to \mathbb{C}^\times$, $\psi_\circ : \mathrm{cl}_F^+(\mathfrak{N}) \to \mathbb{C}^\times$ and we suppose that $\chi_{\circ|F} \cdot \psi_\circ \equiv 1$. Let $\mathscr{F}^* \in \overline{\mathbf{S}}_F^{\text{n.o.}}(\mathfrak{A}, \psi_\circ^{-2}\psi; \mathbf{I}_{\mathscr{F}^*})$ ([Hid91], Section 7F) be the twisted Hida family, where $\mathbf{I}_{\mathscr{F}^*} \cong \mathbf{I}_{\mathscr{F}}(\psi_\circ^{-2})$ as an $\boldsymbol{\Lambda}_{F,\psi_\circ^{-2}\psi}$-algebra.

Set $\mathbf{K}_{\mathscr{G},\mathscr{F}^*} = (\mathbf{I}_{\mathscr{G}} \widehat{\otimes}_O \mathbf{I}_{\mathscr{F}^*}) \otimes \mathbb{Q}$ and $\mathbf{K}_{\mathscr{G}} = \mathbf{I}_{\mathscr{G}} \otimes \mathbb{Q}$. We define a $\mathbf{K}_{\mathscr{G}}$-adic cuspform $\breve{\mathscr{G}}$ passing through the nearly ordinary p-stabilization of the test vector $\breve{\mathrm{g}}_\circ$ as in [DR14] Section 2.6. Let $\bar{r} = \sum_{\mu \in \mathrm{I}_L} \bar{r}_\mu \cdot \mu$, with $\bar{r}_\mu \in \mathbb{Z}/(q_{\mathfrak{p}_\mu} - 1)\mathbb{Z}$, denote the reduction of $r_\circ$. We define a homomorphism of $O[\![\mathbb{G}_L(V_1(\mathfrak{A}\mathcal{O}_L))]\!]$-modules ${}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]} : \mathbf{h}_L(\mathfrak{A}\mathcal{O}_L; O) \longrightarrow \mathbf{K}_{\mathscr{G},\mathscr{F}^*}$ by

$${}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\left(\langle z \rangle \mathbf{T}(y)\right) = \begin{cases} \breve{\mathscr{G}}\left(\langle z \rangle \mathbf{T}(y)\right) [\langle y_p \rangle] \otimes [\mathrm{N}_{L/F}\langle y_p \rangle^{-1/2}] \langle y_p \rangle^{\frac{\theta - t_L}{2}} \omega(y_p)^{\bar{r}} & \text{if } y_p \in \mathcal{O}_{L,p}^\times \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{K}_{\mathscr{G},\mathscr{F}^*}$ is given the $O[\![\mathbb{G}_L(V_1(\mathfrak{A}\mathcal{O}_L))]\!]$-algebra structure

$$[(z,a)] \mapsto \breve{\mathscr{G}}(\langle z \rangle T(a^{-1}, 1))[\langle a \rangle^{-1}] \otimes [\mathrm{N}_{L/F}\langle a \rangle^{1/2}] \langle a \rangle^{\frac{-\theta + t_L}{2}} \omega(a)^{-\bar{r}}$$

and $\langle\ \rangle : \mathcal{O}_{L,p}^\times \to (\mathcal{O}_{L,p}^\times)_{\text{pro-}p}$, $\omega : \mathcal{O}_{L,p}^\times \to (\mathcal{O}_{L,p}^\times)_{\mathrm{tor}}$ are the canonical projections. The composition of the natural maps $\mathbf{h}_F^{\text{n.o.}}(\mathfrak{A}; O) \to \mathbf{h}_F(\mathfrak{A}; O) \to \mathbf{h}_L(\mathfrak{A}\mathcal{O}_L; O)$ with the homomorphism ${}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]} : \mathbf{h}_L(\mathfrak{A}\mathcal{O}_L; O) \to \mathbf{K}_{\mathscr{G},\mathscr{F}^*}$ defines a nearly ordinary $\mathbf{K}_{\mathscr{G},\mathscr{F}^*}$-adic cuspform $e_{\text{n.o.}}\zeta^*\left({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\right) \in \overline{\mathbf{S}}_F^{\text{n.o.}}(\mathfrak{A}, \psi_\circ^{-2}\psi; \mathbf{K}_{\mathscr{G},\mathscr{F}^*})$.

Proposition 2.4.3. *Let $s : \mathrm{I}_F \to \mathrm{I}_L$ be any section of the restriction $\mathrm{I}_L \to \mathrm{I}_F$, $\mu \mapsto \mu_{|F}$. For any crystalline point $(\mathrm{P}, \mathrm{Q}) \in \boldsymbol{\mathcal{W}}$, with $r(\theta)$ a lift of $\bar{r}$, we have*

$$e_{n.o.}\zeta^*\left({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\right)(\mathrm{P}, \mathrm{Q}) = e_{n.o.}\zeta^*\left(d^{r(\theta)} \breve{\mathrm{g}}_{\mathrm{P}}^{[\mathcal{P}]}\right) = \pm e_{n.o.}\zeta^*\left(d^{s(w - x_{|F})} \breve{\mathrm{g}}_{\mathrm{P}}^{[\mathcal{P}]}\right).$$

Proof. For a crystalline point $(\mathrm{P}, \mathrm{Q}) \in \boldsymbol{\mathcal{W}}$, with $r(\theta)$ a lift of $\bar{r}$, the explicit description of ${}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}$ produces the equality of modular forms

$$e_{\text{n.o.}}\zeta^*\left({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\right)(\mathrm{P}, \mathrm{Q}) = e_{\text{n.o.}}\zeta^*\left(d^{r(\theta)} \breve{\mathrm{g}}_{\mathrm{P}}^{[\mathcal{P}]}\right).$$

Let now $\mu, \mu' \in \mathrm{I}_L$, $\mu \neq \mu'$, be such that $\tau = \mu_{|F} = \mu'_{|F}$. A direct computation shows that

$$0 = e_{\text{n.o.}} d_\tau \zeta^* \mathrm{g} = e_{\text{n.o.}}\zeta^*(d_\mu + d_{\mu'})\mathrm{g}$$

for any g, which implies

$$e_{\text{n.o.}}\zeta^*(d_\mu^\alpha \mathrm{g}) = (-1)^\alpha e_{\text{n.o.}}\zeta^*(d_{\mu'}^\alpha \mathrm{g})$$

for any $\alpha \in \mathbb{N}$. When $\mathrm{g} = \mathrm{g}^{[\mathcal{P}]}$ is $\mathcal{P}$-depleted, we also have

$$e_{\text{n.o.}}\zeta^*(d_\mu^\alpha \mathrm{g}^{[\mathcal{P}]}) = (-1)^\alpha e_{\text{n.o.}}\zeta^*(d_{\mu'}^\alpha \mathrm{g}^{[\mathcal{P}]})$$

for any $\alpha \in \varprojlim_{n} \mathbb{Z}/p^n(q_{\mathfrak{p}_\mu} - 1)$ by taking p-adic limits. Thus, the second equality follows. $\square$

Lemma 2.4.4. *There is an element $J(e_{n.o.}\zeta^*({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}), \mathscr{F}^*) \in \mathbf{I}_{\mathscr{G}} \widehat{\otimes}_O \mathrm{Frac}(\mathbf{I}_{\mathscr{F}^*})$ such that for any crystalline point $(\mathrm{P}, \mathrm{Q}) \in \boldsymbol{\mathcal{W}}$, with $r(\theta) \in \mathbb{Z}[\mathrm{I}_L]$ a lift of $\bar{r}$, we have*

$$J\big(e_{n.o.}\zeta^*\big({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big), \mathscr{F}^*\big)(\mathrm{P}, \mathrm{Q}) = \frac{\big\langle e_{n.o.}\zeta^*\big(d^{r(\theta)} \breve{\mathrm{g}}_{\mathrm{P}}^{[\mathcal{P}]}\big), \mathrm{f}_{\mathrm{Q}}^{*(p)}\big\rangle}{\big\langle \mathrm{f}_{\mathrm{Q}}^{*(p)}, \mathrm{f}_{\mathrm{Q}}^{*(p)}\big\rangle}. \tag{2.15}$$

Proof. We follow the argument of ([DR14], Lemma 2.19). The relevant part of the $\mathscr{F}^*$-isotypic projection $e_{\mathscr{F}^*}\zeta^*\big({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big)$ is a $\mathbf{I}_{\mathscr{G}} \widehat{\otimes}_O \mathrm{Frac}(\mathbf{I}_{\mathscr{F}^*})$-linear combination of the $\mathbf{I}_{\mathscr{F}^*}$-adic cuspforms $\mathscr{F}_{\mathfrak{a}}^*$ for $\mathfrak{a} \mid \mathfrak{A}/\mathfrak{N}$. Hence, the element J exists because we can interpolate expressions of the form

$$\big\langle \mathrm{f}_{\mathfrak{a},\mathrm{Q}}^{*(p)}, \mathrm{f}_{\mathrm{Q}}^{*(p)}\big\rangle / \big\langle \mathrm{f}_{\mathrm{Q}}^{*(p)}, \mathrm{f}_{\mathrm{Q}}^{*(p)}\big\rangle$$

for $\mathrm{Q} \in \mathcal{A}^\circ(\mathbf{I}_{\mathscr{F}^*})$ using the explicit computations in the proof of Proposition 2.4.2 and the fact that $\mathfrak{A}$ is prime to p. $\square$

Definition 2.4.5. *The twisted triple product p-adic L-function attached to $\big(\breve{\mathscr{G}}, \mathscr{F}, \theta, \bar{r}\big)$ is the meromorphic rigid-analytic function*

$${}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}}, \mathscr{F}) : \boldsymbol{\mathcal{W}}_{\mathscr{G},\mathscr{F}^*} \longrightarrow \mathbb{C}_p$$

determined by $J\big(e_{n.o.}\zeta^*\big({}_{\bar{r}}d_\theta^\bullet \breve{\mathscr{G}}^{[\mathcal{P}]}\big), \mathscr{F}^*\big) \in \mathbf{I}_{\mathscr{G}} \widehat{\otimes}_O \mathrm{Frac}(\mathbf{I}_{\mathscr{F}^*})$.

For every $\mathcal{O}_F$-prime $\mathfrak{p}$ above p, let $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ be the inverses of the roots of the $T(\mathfrak{p})$-Hecke polynomial of $\mathrm{f}_{\mathrm{Q}}^*$. Let $\mathrm{h}_{\mathrm{P,Q}} = e_{\mathrm{f}_{\mathrm{Q}}^*,\text{n.o.}}\zeta^*(d^{r(\theta)} \breve{\mathrm{g}}_{\mathrm{P}}^{[\mathcal{P}]})$ where $e_{\mathrm{f}_{\mathrm{Q}}^*,\text{n.o.}} = e_{\mathrm{f}_{\mathrm{Q}}^*} e_{\text{n.o.}}$ denotes the composition of the $\mathrm{f}_{\mathrm{Q}}^*$-isotypic projection with the nearly ordinary projector, and suppose we named the roots of the Hecke polynomials such that $\mathrm{h}_{\mathrm{P,Q}}^{(p)} = \prod_{\mathfrak{p}|p}(1 - \beta_{\mathfrak{p}} V(\mathfrak{p}))\mathrm{h}_{\mathrm{P,Q}}$ is the nearly ordinary p-stabilization of $\mathrm{h}_{\mathrm{P,Q}}$. By definition $e_{\text{n.o.}}\mathrm{h}_{\mathrm{P,Q}}^{(p)} = \mathrm{h}_{\mathrm{P,Q}}^{(p)}$ and we can compute that

$$\mathrm{h}_{\mathrm{P,Q}}^{(p)} = e_{\text{n.o.}}\mathrm{h}_{\mathrm{P,Q}}^{(p)} = \prod_{\mathfrak{p}|p}\big(1 - \beta_{\mathfrak{p}}\alpha_{\mathfrak{p}}^{-1}\big) \cdot e_{\text{n.o.}}\mathrm{h}_{\mathrm{P,Q}}.$$

More explicitly, if we set $\mathsf{E}(\mathrm{f}_{\mathrm{Q}}^*) = \prod_{\mathfrak{p}|p}\big(1 - \beta_{\mathfrak{p}}\alpha_{\mathfrak{p}}^{-1}\big)$, then

$$\Big(e_{\mathrm{f}_{\mathrm{Q}}^*,\text{n.o.}}\zeta^*(d^{r(\theta)} \breve{\mathrm{g}}_{\mathrm{P}}^{[\mathcal{P}]})\Big)^{(p)} = \mathsf{E}(\mathrm{f}_{\mathrm{Q}}^*) \cdot e_{\mathrm{f}_{\mathrm{Q}}^*,\text{n.o.}}\zeta^*(d^{r(\theta)} \breve{\mathrm{g}}_{\mathrm{P}}^{[\mathcal{P}]}).$$

Therefore, we can rewrite the values of the p-adic L-function at every crystalline point $(\mathrm{P}, \mathrm{Q}) \in \boldsymbol{\mathcal{W}}$, with $r(\theta) \in \mathbb{Z}[\mathrm{I}_L]$ a lift of $\bar{r}$, as

$${}_{\bar{r}}\mathscr{L}_p^\theta(\breve{\mathscr{G}}, \mathscr{F})(\mathrm{P}, \mathrm{Q}) = \frac{1}{\mathsf{E}(\mathrm{f}_{\mathrm{Q}}^*)} \frac{\big\langle e_{\text{n.o.}}\zeta^*\big(d^{r(\theta)} \breve{\mathrm{g}}_{\mathrm{P}}^{[\mathcal{P}]}\big), \mathrm{f}_{\mathrm{Q}}^*\big\rangle}{\big\langle \mathrm{f}_{\mathrm{Q}}^*, \mathrm{f}_{\mathrm{Q}}^*\big\rangle}. \tag{2.16}$$

2.4.2 Interpolation formulas

The interpolation formulas satisfied by the twisted triple product p-adic L-function include Euler factors that depend on whether the primes in $\mathcal{P}$ are above a prime of F that is split or inert in the extension L/F. We partition the set of primes of F above p accordingly to the splitting behavior in L/F, $\mathcal{Q} = \mathcal{Q}_{\text{inert}} \coprod \mathcal{Q}_{\text{split}}$. For every prime $\mathcal{O}_F$-ideal $\wp \in \mathcal{Q}$ we denote by $q_\wp$ the cardinality of its residue field.

Inert case. For a prime ideal $\mathfrak{p} \in \mathcal{P}$ with $\wp = \mathfrak{p} \cap \mathcal{O}_F \in \mathcal{Q}_{\text{inert}}$, we write $\mathfrak{p} = \wp\mathcal{O}_L$.

Lemma 2.4.6. *Let* $\mathsf{g} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; E)$ *be a* $T(\mathfrak{p})$*-eigenvector,* $\mathsf{f} \in S_{k,w}(V_1(\mathfrak{N}); F; E)$ *a* p*-nearly ordinary eigenform. Suppose* $\mathfrak{N} \mid \mathfrak{A}$ *and that the weights of* g, f *are* F*-dominated, then we have*

$$e_{\mathsf{f},n.o.}\zeta^*\left(d^r\mathsf{g}^{[\mathfrak{p}]}\right) = \mathcal{E}^{\text{in}}_\wp(\mathsf{g},\mathsf{f})e_{\mathsf{f},n.o.}\zeta^*\left(d^r\mathsf{g}\right)$$

for

$$\mathcal{E}^{\text{in}}_\wp(\mathsf{g},\mathsf{f}) = \left(1-\alpha_\mathsf{g}\alpha_\mathsf{f}^{-1}q_\wp^{-1}\right)\left(1-\beta_\mathsf{g}\alpha_\mathsf{f}^{-1}q_\wp^{-1}\right)$$

where $\alpha_\mathsf{g}, \beta_\mathsf{g}$ *are the inverses of the roots of the* $T(\mathfrak{p})$*-Hecke polynomial of* g *and* α_f *is determined by* $(e_{n.o.}\mathsf{f})_{|U(\varpi_\wp)} = \alpha_\mathsf{f} \cdot e_{n.o.}\mathsf{f}$.

Proof. Let $\mathsf{g}_\alpha^{(\mathfrak{p})} = (1-\beta_\mathsf{g}V(\varpi_\mathfrak{p}))\mathsf{g}$, $\mathsf{g}_\beta^{(\mathfrak{p})} = (1-\alpha_\mathsf{g}V(\varpi_\mathfrak{p}))\mathsf{g}$ be the two $\mathfrak{p}$-stabilizations of g, they satisfy $U(\varpi_\mathfrak{p})\mathsf{g}_\bullet^{(\mathfrak{p})} = (\bullet)\mathsf{g}_\bullet^{(\mathfrak{p})}$ and $\mathsf{g} = 1/(\alpha_\mathsf{g}-\beta_\mathsf{g})\left(\alpha_\mathsf{g}\mathsf{g}_\alpha^{(\mathfrak{p})} - \beta_\mathsf{g}\mathsf{g}_\beta^{(\mathfrak{p})}\right)$. Using Proposition 2.2.6, we compute

$$\begin{aligned}
e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r\left(1-V(\varpi_\mathfrak{p})\circ U(\varpi_\mathfrak{p})\right)\mathsf{g}_\bullet^{(\mathfrak{p})}\right] &= e_{\mathsf{f},\text{n.o.}}\zeta^*\left[\left(1-(\bullet)V(\varpi_\mathfrak{p})\right)d^r\mathsf{g}_\bullet^{(\mathfrak{p})}\right]\\
&= e_{\mathsf{f},\text{n.o.}}\left(1-(\bullet)q_\wp^{-1}V(\varpi_\wp)\right)\zeta^*(d^r\mathsf{g}_\bullet^{(\mathfrak{p})})\\
&= \left(1-(\bullet)q_\wp^{-1}\alpha_\mathsf{f}^{-1}\right)e_{\mathsf{f},\text{n.o.}}\zeta^*(d^r\mathsf{g}_\bullet^{(\mathfrak{p})}).
\end{aligned}$$

Noting that the $\mathfrak{p}$-depletions of the $\mathfrak{p}$-stabilizations are equal, $(\mathsf{g}_\alpha^{(\mathfrak{p})})^{[\mathfrak{p}]} = (\mathsf{g}_\beta^{(\mathfrak{p})})^{[\mathfrak{p}]} = \mathsf{g}^{[\mathfrak{p}]}$, we deduce the claim:

$$\begin{aligned}
e_{\mathsf{f},\text{n.o.}}\zeta^*(d^r\mathsf{g}) &= \frac{1}{\alpha_\mathsf{g}-\beta_\mathsf{g}}\left(\alpha_\mathsf{g}e_{\mathsf{f},\text{n.o.}}\zeta^*(d^r\mathsf{g}_\alpha^{(\mathfrak{p})}) - \beta_\mathsf{g}e_{\mathsf{f},\text{n.o.}}\zeta^*(d^r\mathsf{g}_\beta^{(\mathfrak{p})})\right)\\
&= \frac{1}{\left(1-\alpha_\mathsf{g}\alpha_\mathsf{f}{}^{-1}q_\wp^{-1}\right)\left(1-\beta_\mathsf{g}\alpha_\mathsf{f}{}^{-1}q_\wp^{-1}\right)}e_{\mathsf{f},\text{n.o.}}\zeta^*(d^r\mathsf{g}^{[\mathfrak{p}]}).
\end{aligned}$$

□

Split case. For a prime ideal $\mathfrak{p} \in \mathcal{P}$ with $\wp = \mathfrak{p} \cap \mathcal{O}_F \in \mathcal{Q}_{\text{split}}$, we write $\wp\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$.

Lemma 2.4.7. *Let* $\mathfrak{Q}$ *be any* $\mathcal{O}_L$*-ideal and* $\mathsf{g} \in S_{\ell,x}(V_{11}(\mathfrak{Q}); L; E)$ *a cuspform. If the indexes* $i, j \in \{1,2\}$ *are different, then*

$$U(p)\zeta^*\left((\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})}\right) = 0,$$

which implies

$$e_{n.o.}\zeta^*(\mathsf{g}_{|V(\varpi_{\mathfrak{p}_i})}) = e_{n.o.}\zeta^*((U(\varpi_{\mathfrak{p}_j})\mathsf{g})_{|V(\varpi_\wp)}).$$

In particular, $e_{\text{n.o.}}\zeta^*\left(\mathsf{g}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\right) = e_{\text{n.o.}}\zeta^*(\mathsf{g}^{[\mathfrak{p}_1]}) = e_{\text{n.o.}}\zeta^*(\mathsf{g}^{[\mathfrak{p}_2]})$.

Proof. For any $y \in \widehat{\mathcal{O}_F}F_{\infty,+}^\times$ we can compute that

$$\begin{aligned}
\mathsf{a}_p\left(y, U(p)[\zeta^*\left((\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})}\right)]\right) &= p_p^{t_F-x_{|F}}\mathsf{a}_p\left(py, \zeta^*\left((\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})}\right)\right)\\
&= C\sum_{\text{Tr}_{L/F}(\xi)=p}\mathsf{a}_p(\xi y\mathsf{d}_F^{-1}\mathsf{d}_L, (\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})})((\xi\mathsf{d}_L)_p\xi^{-1})^{t_L-x},
\end{aligned}$$

where C is a non-zero explicit constant. Suppose that $\mathsf{a}_p(\xi y \mathsf{d}_F^{-1}\mathsf{d}_L, (\mathsf{g}^{[\mathfrak{p}_j]})_{|V(\varpi_{\mathfrak{p}_i})}) \neq 0$ for some $\xi \in L_+^\times$ with $\mathrm{Tr}_{L/F}(\xi) = p$, then $\xi y \mathsf{d}_F^{-1}\mathsf{d}_L \in \widehat{\mathcal{O}}_L L_{\infty,+}^\times$, $\varpi_{\mathfrak{p}_i} \mid (\xi y \mathsf{d}_F^{-1}\mathsf{d}_L)_{\mathfrak{p}_i}$ and $\varpi_{\mathfrak{p}_j} \nmid (\xi y \mathsf{d}_F^{-1}\mathsf{d}_L)_{\mathfrak{p}_j}$. Since p is unramified in L, that is equivalent to $\varpi_{\mathfrak{p}_i} \mid (\xi y)_{\mathfrak{p}_i}$ and $\varpi_{\mathfrak{p}_j} \nmid (\xi y)_{\mathfrak{p}_j}$ which implies that $\varpi_{\mathfrak{p}_i} \nmid (\mathrm{Tr}_{L/F}(\xi) y)_{\mathfrak{p}_i} = (py)_{\mathfrak{p}_i}$. This is absurd.

Regarding the second claim, for any $\mathfrak{p}_i$-stabilizations $\mathsf{g}_\bullet^{[\mathfrak{p}_j],(\mathfrak{p}_i)}$ we have that

$$\begin{aligned} e_{\text{n.o.}}\zeta^*\left(\mathsf{g}^{[\mathfrak{p}_1,\mathfrak{p}_2]}\right) = e_{\text{n.o.}}\zeta^*\left((1 - V(\varpi_{\mathfrak{p}_i})U(\varpi_{\mathfrak{p}_i}))\mathsf{g}_\bullet^{[\mathfrak{p}_j],(\mathfrak{p}_i)}\right) &= e_{\text{n.o.}}\zeta^*\left((1 - (\bullet)V(\varpi_{\mathfrak{p}_i}))\mathsf{g}_\bullet^{[\mathfrak{p}_j],(\mathfrak{p}_i)}\right) \\ &= e_{\text{n.o.}}\zeta^*\left(\mathsf{g}_\bullet^{[\mathfrak{p}_j],(\mathfrak{p}_i)}\right). \end{aligned}$$

Taking the appropriate linear combination we prove the statement. □

Lemma 2.4.8. *Let* $\mathsf{g} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; E)$ *be an eigenvector for the Hecke operators* $T(\mathfrak{p}_1)$ *and* $T(\mathfrak{p}_2)$, $\mathsf{f} \in S_{k,w}(V_1(\mathfrak{N}); F; E)$ *a p-nearly ordinary eigenform. Suppose* $\mathfrak{N} \mid \mathfrak{A}$ *and that the weights of* g, f *are F-dominated. For* α_i, β_i *the inverses of the roots of the* $T(\mathfrak{p}_i)$*-Hecke polynomial for* g, $i = 1, 2$, *and* $(e_{n.o.}\mathsf{f})_{|U(\varpi_\wp)} = \alpha_\mathsf{f} \cdot e_{n.o.}\mathsf{f}$ *we have*

$$e_{\mathsf{f},n.o.}\zeta^*\left(d^r \mathsf{g}^{[\mathfrak{p}_i]}\right) = \frac{\mathcal{E}_\wp^{\mathrm{sp}}(\mathsf{g},\mathsf{f})}{\mathcal{E}_{1,\wp}(\mathsf{g},\mathsf{f})} e_{\mathsf{f},n.o.}\zeta^*(d^r \mathsf{g}),$$

where

$$\mathcal{E}_\wp^{\mathrm{sp}}(\mathsf{g},\mathsf{f}) = \prod_{\bullet,\star \in \{\alpha,\beta\}} \left(1 - \bullet_1 \star_2 \alpha_\mathsf{f}^{-1} q_\wp^{-1}\right), \qquad \mathcal{E}_{1,\wp}(\mathsf{g},\mathsf{f}) = 1 - \alpha_1\beta_1\alpha_2\beta_2(\alpha_\mathsf{f}^{-1}q_\wp^{-1})^2.$$

Proof. Let $\mathsf{g}_{\alpha_i}^{(\mathfrak{p}_i)} = (1 - \beta_i V(\varpi_{\mathfrak{p}_i}))\mathsf{g}$, $\mathsf{g}_{\beta_i}^{(\mathfrak{p}_i)} = (1 - \alpha_i V(\varpi_{\mathfrak{p}_i}))\mathsf{g}$ be the two $\mathfrak{p}_i$-stabilizations of g. They satisfy $U(\varpi_{\mathfrak{p}_i})\mathsf{g}_\bullet^{(\mathfrak{p}_i)} = (\bullet)\mathsf{g}_\bullet^{(\mathfrak{p}_i)}$ and $\mathsf{g} = 1/(\alpha_i - \beta_i)\left(\alpha_i \mathsf{g}_{\alpha_i}^{(\mathfrak{p}_i)} - \beta_i \mathsf{g}_{\beta_i}^{(\mathfrak{p}_i)}\right)$. Using Lemma 2.4.7 we compute

$$\begin{aligned} e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r\left(\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right)^{[\mathfrak{p}_i]}\right] &= e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r(1 - (\bullet)V(\varpi_{\mathfrak{p}_i}))\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right] \\ &= e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r \mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right] - (\bullet)e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r(U(\varpi_{\mathfrak{p}_j})\mathsf{g}_\bullet^{(\mathfrak{p}_i)})_{|V(\varpi_\wp)}\right] \\ &= e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r \mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right] - (\bullet)\alpha_\mathsf{f}^{-1}q_\wp^{-1}e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r(T(\mathfrak{p}_j) - \alpha_j\beta_j V(\varpi_{\mathfrak{p}_j}))\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right]. \end{aligned} \tag{2.17}$$

Recall that $\mathsf{g}_\bullet^{(\mathfrak{p}_i)}$ is an eigenform for the operator $T(\mathfrak{p}_j)$ of eigenvalue $\alpha_j + \beta_j$. The chain of identities in (2.17) continues as:

$$\begin{aligned} e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r\left(\mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right)^{[\mathfrak{p}_i]}\right] &= e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r \mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right] - (\bullet)\alpha_\mathsf{f}^{-1}q_\wp^{-1}\Big[(\alpha_j + \beta_j)e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r \mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right] + \\ &\qquad - \alpha_j\beta_j e_{\mathsf{f},\text{n.o.}}\zeta^*\left[d^r(U(\varpi_{\mathfrak{p}_i})\mathsf{g}_\bullet^{(\mathfrak{p}_i)})_{|V(\varpi_\wp)}\right]\Big] \\ &= \left(1 - (\bullet)\alpha_\mathsf{f}^{-1}q_\wp^{-1}(\alpha_j + \beta_j) + \alpha_j\beta_j\left[(\bullet)\alpha_\mathsf{f}^{-1}q_\wp^{-1}\right]^2\right) e_{\mathsf{f},\text{n.o.}}\zeta^*\left(d^r \mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right) \\ &= \left(1 - (\bullet)\alpha_j\alpha_\mathsf{f}^{-1}q_\wp^{-1}\right)\left(1 - (\bullet)\beta_j\alpha_\mathsf{f}^{-1}q_\wp^{-1}\right) e_{\mathsf{f},\text{n.o.}}\zeta^*\left(d^r \mathsf{g}_\bullet^{(\mathfrak{p}_i)}\right). \end{aligned}$$

Finally, noting that $\big(\mathrm{g}_{\alpha_i}^{(\mathfrak{p}_i)}\big)^{[\mathfrak{p}_i]} = \big(\mathrm{g}_{\beta_i}^{(\mathfrak{p}_i)}\big)^{[\mathfrak{p}_i]} = \mathrm{g}^{[\mathfrak{p}_i]}$, we can put together the previous identities to prove the claim:

$$\begin{aligned} e_{\mathrm{f,n.o.}}\zeta^*(d^r\mathrm{g}) =& \frac{1}{\alpha_i-\beta_i}\left(\alpha_i e_{\mathrm{f,n.o.}}\zeta^*(d^r\mathrm{g}_{\alpha_i}^{(\mathfrak{p}_i)}) - \beta_i e_{\mathrm{f,n.o.}}\zeta^*(d^r\mathrm{g}_{\beta_i}^{(\mathfrak{p}_i)})\right)\\ =& \frac{1-\alpha_i\beta_i\alpha_j\beta_j(\alpha_{\mathrm{f}}^{-1}q_{\wp}^{-1})^2}{\prod\limits_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_i\star_j\,\alpha_{\mathrm{f}}^{-1}q_{\wp}^{-1})} e_{\mathrm{f,n.o.}}\zeta^*(d^r\mathrm{g}^{[\mathfrak{p}_i]}). \end{aligned}$$

□

Remark 2.4.9. *When* $(\mathrm{P},\mathrm{Q})\in\mathcal{C}_F^{\theta,\bar{r}}$ *we write* $\mathcal{E}_{1,\wp}(\mathrm{f}_{\mathrm{Q}}^*)$ *for* $\mathcal{E}_{1,\wp}(\mathrm{g}_{\mathrm{P}},\mathrm{f}_{\mathrm{Q}}^*)$ *since the quantity depends only on* $\mathrm{f}_{\mathrm{Q}}^*$ *because of the condition on central characters.*

Theorem 2.4.10. *The value of the twisted triple product p-adic L-function* ${}_{\bar{r}}\mathscr{L}_p^{\theta}(\mathscr{G},\mathscr{F}):\boldsymbol{\mathcal{W}}\to\mathbb{C}_p$ *at all* $(\mathrm{P},\mathrm{Q})\in\mathcal{C}_F^{\theta,\bar{r}}$ *satisfies*

$$\begin{aligned} {}_{\bar{r}}\mathscr{L}_p^{\theta}(\breve{\mathscr{G}},\mathscr{F})(\mathrm{P},\mathrm{Q}) = \pm\frac{1}{\mathsf{E}(\mathrm{f}_{\mathrm{Q}}^*)}&\left(\prod_{\wp\in\mathcal{Q}_{inert}}\mathcal{E}_{\wp}^{\mathrm{in}}(\mathrm{g}_{\mathrm{P}},\mathrm{f}_{\mathrm{Q}}^*)\prod_{\wp\in\mathcal{Q}_{split}}\frac{\mathcal{E}_{\wp}^{\mathrm{sp}}(\mathrm{g}_{\mathrm{P}},\mathrm{f}_{\mathrm{Q}}^*)}{\mathcal{E}_{1,\wp}(\mathrm{f}_{\mathrm{Q}}^*)}\right)\\ &\times\frac{\left\langle\zeta^*\left(\delta^{s(w-x_{|F})}\breve{\mathrm{g}}_{\mathrm{P}}\right),\mathrm{f}_{\mathrm{Q}}^*\right\rangle}{\left\langle\mathrm{f}_{\mathrm{Q}}^*,\mathrm{f}_{\mathrm{Q}}^*\right\rangle}, \end{aligned}$$

where $s:\mathrm{I}_F\to\mathrm{I}_L$ *is any section of the restriction* $\mathrm{I}_L\to\mathrm{I}_F$, $\mu\mapsto\mu_{|F}$ *and the Euler factors appearing in the formula are defined in Lemmas* 2.4.6 *and* 2.4.8.

Proof. We use (2.16) and Proposition 2.4.3 to obtain an explicit expression for the value of the p-adic L-function at a point $(\mathrm{P},\mathrm{Q})\in\mathcal{C}_F^{\theta,\bar{r}}$. Then Lemmas 2.4.6, 2.4.8 give us

$$\begin{aligned} {}_{\bar{r}}\mathscr{L}_p^{\theta}(\breve{\mathscr{G}},\mathscr{F})(\mathrm{P},\mathrm{Q}) &= \pm\frac{1}{\mathsf{E}(\mathrm{f}_{\mathrm{Q}}^*)}\frac{\left\langle e_{\mathrm{n.o.}}\zeta^*\big(d^{s(w-x_{|F})}\breve{\mathrm{g}}_{\mathrm{P}}^{[\mathcal{P}]}\big),\mathrm{f}_{\mathrm{Q}}^*\right\rangle}{\left\langle\mathrm{f}_{\mathrm{Q}}^*,\mathrm{f}_{\mathrm{Q}}^*\right\rangle}\\ &= \pm\frac{1}{\mathsf{E}(\mathrm{f}_{\mathrm{Q}}^*)}\left(\prod_{\wp\in\mathcal{Q}_{\mathrm{inert}}}\mathcal{E}_{\wp}^{\mathrm{in}}(\mathrm{g}_{\mathrm{P}},\mathrm{f}_{\mathrm{Q}}^*)\prod_{\wp\in\mathcal{Q}_{\mathrm{split}}}\frac{\mathcal{E}_{\wp}^{\mathrm{sp}}(\mathrm{g}_{\mathrm{P}},\mathrm{f}_{\mathrm{Q}}^*)}{\mathcal{E}_{1,\wp}(\mathrm{f}_{\mathrm{Q}}^*)}\right)\\ &\qquad\times\frac{\left\langle e_{\mathrm{n.o}}\zeta^*\big(d^{s(w-x_{|F})}\breve{\mathrm{g}}_{\mathrm{P}}\big),\mathrm{f}_{\mathrm{Q}}^*\right\rangle}{\left\langle\mathrm{f}_{\mathrm{Q}}^*,\mathrm{f}_{\mathrm{Q}}^*\right\rangle}. \end{aligned}$$

We conclude the proof applying Lemma 2.2.8 to compare p-adic and real analytic differential operators on cuspforms: $e_{\mathrm{n.o}}\zeta^*\big(d^{s(w-x_{|F})}\breve{\mathrm{g}}_{\mathrm{P}}\big) = e_{\mathrm{n.o}}\Pi^{\mathrm{hol}}\zeta^*\big(\delta^{s(w-x_{|F})}\breve{\mathrm{g}}_{\mathrm{P}}\big)$. □

Remark 2.4.11. *Recall that for every* $(\mathrm{P},\mathrm{Q})\in\mathcal{C}_F^{\theta,\bar{r}}$ *there is a unitary automorphic representation* $\Pi_{\mathrm{P,Q}}$ *of prime-to-p level. The Euler factors in Theorem* 2.4.10 *also appear the expression for the local L-factor* $L_{\wp}(\frac{1}{2},\Pi_{\mathrm{P,Q}},\mathrm{r})$. *Indeed, if* $\wp\in\mathcal{Q}_{inert}$ *by using* (2.11), (2.10) *we compute*

$$\begin{aligned} \mathcal{E}_{\wp}^{\mathrm{in}}(\mathrm{g},\mathrm{f}^*) &= \left(1-\alpha_{\mathrm{g}}\alpha_{\mathrm{f}^*}^{-1}q_{\wp}^{-1}\right)\left(1-\beta_{\mathrm{g}}\alpha_{\mathrm{f}^*}^{-1}q_{\wp}^{-1}\right)\\ &= \left(1-\chi_{1,\mathfrak{p}}(\varpi_{\wp})\psi_{i,\wp}(\varpi_{\wp})q_{\wp}^{-1/2}\right)\left(1-\chi_{2,\mathfrak{p}}(\varpi_{\wp})\psi_{i,\wp}(\varpi_{\wp})q_{\wp}^{-1/2}\right). \end{aligned}$$

Similarly if $\wp\in\mathcal{Q}_{split}$ *by using* (2.11), (2.9) *we obtain*

$$\mathcal{E}_{\wp}^{\mathrm{sp}}(\mathrm{g},\mathrm{f}^*) = \prod_{\bullet,\star\in\{\alpha,\beta\}}\left(1-\bullet_1\star_2\,\alpha_{\mathrm{f}^*}^{-1}q_{\wp}^{-1}\right) = \prod_{i,j}\left(1-\chi_{i,\mathfrak{p}_1}(\varpi_{\wp})\chi_{j,\mathfrak{p}_2}(\varpi_{\wp})\psi_{k,\wp}(\varpi_{\wp})q_{\wp}^{-1/2}\right).$$

2.5 Geometric theory

2.5.1 Geometric Hilbert modular forms

Let F be a totally real number field and $G_F = \mathrm{Res}_{L/\mathbb{Q}}(\mathrm{GL}_{2,F})$. For any open compact subgroup $K \leq G_F(\mathbb{A}^\infty)$ we consider the Shimura variety

$$\mathrm{Sh}_K(G_F)(\mathbb{C}) = G_F(\mathbb{Q})\backslash(\mathfrak{H}^\pm)^{\mathrm{I}_F} \times G_F(\mathbb{A}^\infty)/K$$

where $\gamma \in G_F(\mathbb{Q}) = \mathrm{GL}_2(F)$ acts on $z = (z_\tau)_\tau \in (\mathfrak{H}^\pm)^{\mathrm{I}_F}$ via Moebius transformations $\gamma \cdot z = (\tau(\gamma)z_\tau)_\tau$. The complex manifold $\mathrm{Sh}_K(G_F)(\mathbb{C})$ has a canonical structure of quasi-projective variety over its reflex field $\mathbb{Q}$ ([Mil90], Chapter II, Theorem 5.5). Let ω be the dual of the tautological quotient bundle on $\mathbb{P}^1_{\mathbb{C}}$ with $p : \omega \to \mathbb{P}^1_{\mathbb{C}}$ the natural projection. The group $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathbb{P}^1_{\mathbb{C}}$ via Moebius transformations and there is a natural way to define a $\mathrm{GL}_2(\mathbb{C})$-action on ω such that the projection p is equivariant. For any weight $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$ such that $k - 2w = mt_F$, one can define a line bundle

$$\underline{\omega}^{(k,w)} = \bigotimes_{\tau\in\mathrm{I}_F} \mathrm{pr}_\tau^*\left(\omega^{\otimes k_\tau} \otimes \det^{\frac{m+k_\tau}{2}}\right) \tag{2.18}$$

on $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{I}_F}$ with $G_F(\mathbb{C})$-action given as follows. For each $\tau \in \mathrm{I}_F$, the action of $G_F(\mathbb{C})$ on $\mathrm{pr}_\tau^*(\omega^{\otimes k_\tau} \otimes \det^{\frac{m+k_\tau}{2}})$ factors through the τ-copy of $\mathrm{GL}_2(\mathbb{C})$, which in turn acts as the product of $\det^{\frac{m+k_\tau}{2}}$ and the k_τ-th power of the natural action on ω. One has to twist the action by such a power of the determinant because it allows the line bundle to descend to the Galois closure F^{Gal} of F over $\mathbb{Q}$. Indeed, consider the subgroup $Z_s = \mathrm{Ker}\big(\mathrm{N}_{F/\mathbb{Q}} : \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m) \to \mathbb{G}_m\big)$ of the center $Z = \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ of G_F and denote by G_F^c the quotient of G_F by Z_s. The action of $G_F(\mathbb{C})$ on $\underline{\omega}^{(k,w)}$ factors through $G_F^c(\mathbb{C})$, thus $\underline{\omega}^{(k,w)}$ descends to an algebraic invertible sheaf on $\mathrm{Sh}_K(G_F)_{\mathbb{C}}$ if K is sufficiently small by ([Mil90], Chapter III, Proposition 2.1), and it has a canonical model over F^{Gal} by ([Mil90], Chapter III, Theorem 5.1).

Suppose $F \neq \mathbb{Q}$, then for every field E, $F^{\mathrm{Gal}} \subset E \subset \mathbb{C}$, and sufficiently small compact open subgroup $K \leq G_F(\mathbb{A}^\infty)$, one can give a geometric interpretation of Hilbert modular forms of weight (k, w), level K, defined over E as $M_{k,w}(K; E) = H^0(\mathrm{Sh}_K(G_F)_E, \underline{\omega}^{(k,w)})$. To deal with cuspforms and treat the case $F = \mathbb{Q}$, one has to consider compactifications of the Shimura variety $\mathrm{Sh}_K(G_F)_{\mathbb{Q}}$, which we discuss in Section 2.6.

2.5.2 Integral models

Fix p a rational prime unramified in F and consider a level structure of type $K = K^pK_p$, where K^p is an open compact subgroup of $G_F(\widehat{\mathcal{O}_F}^p)$ and $K_p = \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$. The determinant map $\det : G_F \to \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ induces a bijection between the set of geometric connected components of $\mathrm{Sh}_K(G_F)$ and $\mathrm{cl}_F^+(K)$, the strict class group of K, $\mathrm{cl}_F^+(K) = F_+^\times\backslash\mathbb{A}_F^{\infty,\times} / \det(K)$. Since $\det(K) \subseteq \widehat{\mathcal{O}_F}^\times$, there is a surjection $\mathrm{cl}_F^+(K) \twoheadrightarrow \mathrm{cl}_F^+$ to the strict ideal class group of F, which one uses to label the geometric components of the Shimura variety $\mathrm{Sh}_K(G_F)$. Fix fractional ideals $\mathfrak{c}_1, \ldots, \mathfrak{c}_{h_F^+}$, coprime to p, forming a set of representatives of cl_F^+. Then by strong approximation there is a decomposition

$$\mathrm{Sh}_K(G_F)(\mathbb{C}) = G(\mathbb{Q})^+\backslash\mathfrak{H}^{\mathrm{I}_F} \times G_F(\mathbb{A}^\infty)/K = \coprod_{[\mathfrak{c}]\in\mathrm{cl}_F^+} \mathrm{Sh}_K^{\mathfrak{c}}(G_F)(\mathbb{C}),$$

where each $\mathrm{Sh}^{\mathfrak{c}}_K(G_F)(\mathbb{C})$ is the disjoint union of quotients of $\mathfrak{H}^{\mathrm{I}_F}$ by groups of the form $\Gamma(g,K) = gKg^{-1} \cap G(\mathbb{Q})^+$. A different choice $\mathfrak{c}'$ of fractional ideal representing $[\mathfrak{c}] \in \mathrm{cl}_F^+$ produces a canonically isomorphic manifold $\mathrm{Sh}^{\mathfrak{c}'}_K(G_F)(\mathbb{C}) \cong \mathrm{Sh}^{\mathfrak{c}}_K(G_F)(\mathbb{C})$ ([TX16], Remark 2.8). Suppose K^p is sufficiently small so there exists a smooth, quasi-projective $\mathbb{Z}_{(p)}$-scheme $\mathcal{M}^{\mathfrak{c}}_K$ representing the moduli problem of isomorphism classes of quadruples $(A, \iota, \lambda, \alpha_{K^p})_{/S}$ where (A,ι) is a Hilbert-Blumenthal abelian variety over S of dimension $g = [F : \mathbb{Q}]$, λ a $\mathfrak{c}$-polarization and α_{K^p} a level-K^p structure, ([TX16], Section 2.3).

The group of totally positive units $\mathcal{O}^\times_{F,+}$ acts on $\mathcal{M}^{\mathfrak{c}}_K$ by modifying the $\mathfrak{c}$-polarization. The subgroup $(K \cap \mathcal{O}_F^\times)^2$ of $\mathcal{O}^\times_{F,+}$ acts trivially, where by $K \cap \mathcal{O}_F^\times$ we mean the intersection of K and $\mathcal{O}_F^\times \hookrightarrow Z(\mathbb{A}^\infty)$ in $G_F(\mathbb{A}^\infty)$. Therefore, the finite group $\mathcal{O}^\times_{F,+}/(K \cap \mathcal{O}_F^\times)^2$ acts on the moduli scheme $\mathcal{M}^{\mathfrak{c}}_K$ and the stabilizer of each geometric connected component is $\big(\det(K) \cap \mathcal{O}^\times_{F,+}\big)/(K \cap \mathcal{O}_F^\times)^2$.

Proposition 2.5.1. *There is an isomorphism between the quotient of $\mathcal{M}^{\mathfrak{c}\mathfrak{d}^{-1}}_K(\mathbb{C})$ by the finite group $\mathcal{O}^\times_{F,+}/(K \cap \mathcal{O}_F^\times)^2$ and $Sh^{\mathfrak{c}}_K(G_F)(\mathbb{C})$. Moreover, if $\det(K) \cap \mathcal{O}^\times_{F,+} = (K \cap \mathcal{O}_F^\times)^2$, then the quotient map $\mathcal{M}^{\mathfrak{c}\mathfrak{d}^{-1}}_K(\mathbb{C}) \to Sh^{\mathfrak{c}}_K(G_F)(\mathbb{C})$ induces an isomorphism between any geometric connected component of $\mathcal{M}^{\mathfrak{c}\mathfrak{d}^{-1}}_K(\mathbb{C})$ and its image.*

Proof. This is ([TX16], Proposition 2.4) with a shift in the indices by the absolute different. It is necessary for the conventions for the complex uniformization used in ([Hid04], Section 4.1.3). ☐

Definition 2.5.2. *Let p be a prime unramified in F, $K = K^pK_p$ a compact open subgroup of $G_F(\mathbb{A}^\infty)$ such that K^p is sufficiently small, $K_p = GL_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$ and $\det(K) \cap \mathcal{O}^\times_{F,+} = (K \cap \mathcal{O}_F^\times)^2$. The integral model of the Shimura variety $Sh_K(G_F)$ over $\mathbb{Z}_{(p)}$ is the quotient of $\mathcal{M}_{K,F} = \coprod_{[\mathfrak{c}]\in cl_F^+} \mathcal{M}^{\mathfrak{c}}_K$ by $\mathcal{O}^\times_{F,+}/(K \cap \mathcal{O}_F)^2$, which we denote $\mathbf{Sh}_K(G_F)$.*

Note that the assumptions on the level K in the definition are always satisfied up to replacing K^p by an open compact subgroup ([TX16], Lemma 2.5). Moreover, by Proposition 2.5.1, the scheme $\mathbf{Sh}_K(G_F)$ is smooth quasi-projective over $\mathbb{Z}_{(p)}$ and has an abelian scheme with real multiplication over it.

Remark 2.5.3. *An integral model $\mathbf{Sh}_K(G^*_F)$ of the Shimura variety for the algebraic group G^*_F of level $K^* = K \cap G^*_F(\mathbb{A}^\infty)$ is the subscheme of $\mathcal{M}^{\mathfrak{d}^{-1}}_K$ whose components are indexed by the image of $\mathrm{cl}^+_{\mathbb{Q}}(K^*)$ in $\mathrm{cl}^+_F(K)$ [Rap78]. We let $\xi : \mathbf{Sh}_K(G^*_F) \to \mathbf{Sh}_K(G_F)$ be the natural morphism.*

2.5.3 Diagonal morphism

Let L/F be an extension of totally real fields with $[F : \mathbb{Q}] = g$. Consider the map of algebraic groups $\zeta : G_F \longrightarrow G_L$ defined by the natural inclusion $\zeta(B) : \mathrm{GL}_2(B \otimes_{\mathbb{Q}} F) \to \mathrm{GL}_2(B \otimes_{\mathbb{Q}} L)$ of groups for any $\mathbb{Q}$-algebra B. For compact open subgroups $K \le G_L(\mathbb{A}^\infty)$ and $K' \le K \cap G_F(\mathbb{A}^\infty)$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}_{K'}(G_F)(\mathbb{C}) \arrow[r, "\zeta"] \arrow[d, "\det"'] & \mathrm{Sh}_K(G_L)(\mathbb{C}) \arrow[d, "\det"] \\
\mathrm{cl}^+_F(K') \arrow[r, "\zeta"] \arrow[d] & \mathrm{cl}^+_L(K) \arrow[d] \\
\mathrm{cl}^+_F \arrow[r, "\zeta"] & \mathrm{cl}^+_L,
\end{tikzcd} \tag{2.19}$$

hence for every fractional ideal $\mathfrak{c}$ of F there is an induced map

$$\zeta : \mathrm{Sh}^{\mathfrak{c}}_{K'}(G_F)(\mathbb{C}) \to \mathrm{Sh}^{\mathfrak{c}}_{K}(G_L)(\mathbb{C}).$$

Suppose that $K \leq G_L(\mathbb{A}^\infty)$ and $K' \leq K \cap G_F(\mathbb{A}^\infty)$ satisfy the assumptions in Definition 2.5.2. There is a morphism of $\mathbb{Z}_{(p)}$-schemes $\zeta : \mathbf{Sh}_{K'}(G_F) \longrightarrow \mathbf{Sh}_K(G_L)$ induced by morphisms $\tilde{\zeta} : \mathcal{M}_{K',F} \to \mathcal{M}_{K,L}$ that maps any quadruple $[A, \iota, \lambda, \alpha_{(K')^p}]_{/S}$ over a $\mathbb{Z}_{(p)}$-scheme S to the quadruple

$$\tilde{\zeta}\left([A, \iota, \lambda, \alpha_{(K')^p}]\right) = [A', \iota', \lambda', \alpha'_{K^p}]_{/S}$$

over S defined as follows. First, the abelian scheme A' is $A \otimes_{\mathcal{O}_F} \mathcal{O}_L$, then we can describe the $\mathcal{O}_L$-action on $\mathcal{O}_L$ via a ring homomorphism $\bar{\iota} : \mathcal{O}_L \to \mathrm{M}_g(\mathcal{O}_F)$ by choosing an $\mathcal{O}_F$-basis of $\mathcal{O}_L$; the choice of basis induces an identification between $A \otimes_{\mathcal{O}_F} \mathcal{O}_L$ and A^g. Thus, the ring homomorphism $\iota' : \mathcal{O}_L \to \mathrm{End}_S(A')$ is defined as the arrow that makes the following diagram commute

$$\begin{tikzcd} \mathcal{O}_L \arrow[r, "\bar{\iota}"] \arrow[dr, dotted, "\iota'"'] & \mathrm{M}_g(\mathcal{O}_F) \arrow[d] \\ & \mathrm{M}_g(\mathrm{End}_S(A)) \cong \mathrm{End}_S(A'). \end{tikzcd}$$

By ([BBGK07], Lemma 5.11), one can compute the dual abelian scheme $(A')^\vee \cong A^\vee \otimes_{\mathcal{O}_F} \mathfrak{d}^{-1}_{L/F}$ and realize that if $\lambda : (\mathfrak{c}, \mathfrak{c}^+) \xrightarrow{\sim} (\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_F}(A, A^\vee), \mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_F}(A, A^\vee)^+)$ is a $\mathfrak{c}$-polarization of A then $\lambda' = \lambda \otimes \mathrm{id}$ is a $\mathfrak{c} \otimes_{\mathcal{O}_F} \mathfrak{d}^{-1}_{L/F}$-polarization of $A' = A \otimes_{\mathcal{O}_F} \mathcal{O}_L$. Finally, it is enough to define $\tilde{\zeta}$ for principal $\mathfrak{N}$-level structures, for $\mathfrak{N}$ an $\mathcal{O}_F$-ideal. A principal $\mathfrak{N}$-level structure is an $\mathcal{O}_F$-linear isomorphism of group schemes $(\mathcal{O}_F/\mathfrak{N})^2 \xrightarrow{\sim} A[\mathfrak{N}]$, thus by tensoring such an isomorphism with $\mathcal{O}_L$ over $\mathcal{O}_F$ we obtain a principal $\mathfrak{N}$-level structure on A'.

Remark 2.5.4. *For any fractional ideal $\mathfrak{c}$ of F there is a commutative diagram*

$$\begin{tikzcd} \mathcal{M}^{\mathfrak{c}\mathfrak{d}_F^{-1}}_{K',F} \arrow[r, "\tilde{\zeta}"] \arrow[d] & \mathcal{M}^{\mathfrak{c}\mathfrak{d}_L^{-1}}_{K,L} \arrow[d] \\ \mathbf{Sh}^{\mathfrak{c}}_{K'}(G_F) \arrow[r, "\zeta"] & \mathbf{Sh}^{\mathfrak{c}}_{K}(G_L) \end{tikzcd}$$

giving, when $F = \mathbb{Q}$,

$$\begin{tikzcd} \mathbf{Sh}_{K'}(GL_{2,\mathbb{Q}}) \arrow[r, "\tilde{\zeta}"] \arrow[dr, dotted, "\zeta"'] & \mathcal{M}^{\mathfrak{d}_L^{-1}}_{K,L} \arrow[d, "\xi"] \\ & \mathbf{Sh}_K(G_L). \end{tikzcd}$$

2.6 Compactifications and p-adic theory

Sometimes we drop part of the decorations from the symbols denoting Shimura varieties when we believe it does not cause confusion, both to simplify the notation and to state facts that hold for both groups G and G^*. We denote by $\mathbf{Sh}^*_K$ the minimal compactification of $\mathbf{Sh}_K$ which is normal and projective. By choosing some auxiliary data Σ, one can construct an arithmetic toroidal compactification $\mathbf{Sh}^{\mathrm{tor}}_{K,\Sigma}$ smooth and projective over $\mathbb{Z}_{(p)}$. It comes equipped with a natural map $\oint : \mathbf{Sh}_K^{\mathrm{tor}} \longrightarrow \mathbf{Sh}_K^{*}$ and an open immersion $\mathbf{Sh}_K \hookrightarrow \mathbf{Sh}_K^{\mathrm{tor}}$ such that the boundary $\mathrm{D} = \mathbf{Sh}_K^{\mathrm{tor}} \setminus \mathbf{Sh}_K$ is a relative simple normal crossing Cartier divisor. The Hilbert-Blumenthal abelian scheme $\mathcal{A}$ over $\mathbf{Sh}_K$ extends to a semi-abelian scheme $\mathcal{A}^{\mathrm{sa}} \to \mathbf{Sh}_K^{\mathrm{tor}}$ with an $\mathcal{O}_F$-action, a K-level structure and a zero section $e : \mathbf{Sh}_K^{\mathrm{tor}} \to \mathcal{A}^{\mathrm{sa}}$ ([Rap78]; [Lan13], Chapter VI). There is a canonical way to extend the rank 2 vector bundle of relative de Rham cohomology $\mathbb{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathbf{Sh}_{K,R})$ to an $(\mathcal{O}_{\mathbf{Sh}_K^{\mathrm{tor}}} \otimes_{\mathbb{Z}} \mathcal{O}_F)$-module $\mathbb{H}^1$ locally free of rank 2 over $\mathbf{Sh}_{K,R}^{\mathrm{tor}}$ together with a logarithmic Gauss-Manin connection and Kodaira-Spencer isomorphism. If $\underline{\omega} = e^*\big(\Omega^1_{\mathcal{A}^{\mathrm{sa}}/\mathbf{Sh}_K^{\mathrm{tor}}}\big)$ is the cotangent space at the origin of the universal semi-abelian scheme, the vector bundle $\mathbb{H}^1$ has an $\mathcal{O}_F$-equivariant Hodge filtration

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathbb{H}^1 \longrightarrow \mathrm{Lie}((\mathcal{A}^{\mathrm{sa}})^\vee) \longrightarrow 0.$$

Let R be an $\mathcal{O}_{F^{\mathrm{Gal}},(p)}$-algebra in which the discriminant $d_{F/\mathbb{Q}}$ is invertible. For a coherent $(\mathcal{O}_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}} \otimes_{\mathbb{Z}} \mathcal{O}_F)$-module M, we denote by $M = \bigoplus_{\tau \in \mathrm{I}_F} M_\tau$ its canonical decomposition for the $\mathcal{O}_F$-action ([Kat78], Lemma 2.0.8): M_τ is the direct summand of M on which $\mathcal{O}_F$ acts via $\tau : \mathcal{O}_F \to R \to \mathcal{O}_{\mathbf{Sh}_{K,R}^{\mathrm{tor}}}$. Then the τ-component of the Hodge filtration is

$$0 \longrightarrow \underline{\omega}_\tau \longrightarrow \mathbb{H}^1_\tau \longrightarrow \textstyle\bigwedge^2(\mathbb{H}^1_\tau) \otimes \underline{\omega}_\tau^{-1} \longrightarrow 0\ .$$

For a weight $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$ with $k - 2w = mt_F$, we define the integral model of the line bundle (2.18) by

$$\underline{\omega}_G^{(k,w)} := \bigotimes_{\tau \in \mathrm{I}_F} \Big((\wedge^2 \mathbb{H}^1_\tau)^{-\frac{m+k_\tau}{2}} \otimes \underline{\omega}_\tau^{k_\tau} \Big)$$

as a sheaf over $\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G)$. Hilbert cuspforms can be identified with its global sections

$$S_{k,w}(K; R) = H^0\big(\mathbf{Sh}_K^{\mathrm{tor}}(G)_R, \underline{\omega}_G^{(k,w)}(-\mathrm{D})\big).$$

Remark 2.6.1. *A general compact open subgroup $K \leq G(\mathbb{A}^\infty)$ of prime-to-p level doesn't satisfy the assumptions in Definition* 2.5.2*. Anyway, one can work with modular forms of level K by considering a subgroup K' that does satisfy them and then take K/K'-invariants ([TX16], Section 6.4).*

Definition 2.6.2. *Let R be an $\mathcal{O}_{F^{Gal},(p)}$-algebra and let $(k, \nu) \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ be any weight. We fix one $\tau_\circ \in \mathrm{I}_F$ and set $\wedge^2\mathbb{H}^1_\circ := \wedge^2\mathbb{H}^1_{\tau_\circ}$. We define a line bundle over $\mathbf{Sh}_{K,R}^{tor}(G^*)$ by*

$$\underline{\omega}_{G^*}^{(k,\nu)} := (\wedge^2 \mathbb{H}^1_\circ)^{\nu - |k|} \otimes \bigotimes_{\tau \in \mathrm{I}_F} \underline{\omega}_\tau^{k_\tau}.$$

It provides a geometric incarnation of cuspforms on G^* of weight $(k, \nu) \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ by setting $S^*_{k,\nu}(K; R) = H^0\big(\mathbf{Sh}_K^{tor}(G^*)_R, \underline{\omega}_{G^*}^{(k,\nu)}(-D)\big)$.

According to [TX16], a weight $(k, w) \in \mathbb{Z}[\mathrm{I}_F]^2$, $k - 2w = mt_F$, is cohomological if $2 - m \geq k_\tau \geq 2$ for all $\tau \in \mathrm{I}_F$. For any cohomological weight we define the vector bundle $\mathcal{F}_G^{(k,w)}$ on $\mathbf{Sh}_{K,R}^{\mathrm{tor}}(G)$ by $\mathcal{F}_G^{(k,w)} := \bigotimes_{\tau \in \mathrm{I}_F} \mathcal{F}_\tau^{(k,w)}$ for

$$\mathcal{F}_\tau^{(k,w)} := (\wedge^2 \mathbb{H}^1_\tau)^{\frac{2-m-k_\tau}{2}} \otimes \mathrm{Sym}^{k_\tau - 2} \mathbb{H}^1_\tau.$$

Similarly, a weight $(k,\nu) \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ is cohomological if $k \geq 2t_F$ and $\nu \geq |k - t_F|$. For any cohomological weight we define the vector bundle $\mathcal{F}^{(k,\nu)}_{G^*}$ on $\mathbf{Sh}^{\mathrm{tor}}_{K,R}(G^*)$ by

$$\mathcal{F}^{(k,\nu)}_{G^*} := (\wedge^2 \mathbb{H}^1_\circ)^{\nu+|t_F-k|} \otimes \bigotimes_{\tau \in \mathrm{I}_F} \mathrm{Sym}^{k_\tau - 2}\mathbb{H}^1_\tau.$$

The extended Gauss-Manin connection on $\mathbb{H}^1$ induces by functoriality logarithmic integrable connections

$$\nabla : \mathcal{F}^{(k,w)}_G \to \mathcal{F}^{(k,w)}_G \otimes \Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}(G)}(\log \mathrm{D})$$

and

$$\nabla : \mathcal{F}^{(k,\nu)}_{G^*} \to \mathcal{F}^{(k,\nu)}_{G^*} \otimes \Omega^1_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}(G^*)}(\log \mathrm{D})$$

out of which one can form the complexes

$$\mathrm{DR}^\bullet\big(\mathcal{F}^{(k,w)}_G\big) = \Big[0 \to \mathcal{F}^{(k,w)}_G \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{F}^{(k,w)}_G \otimes \Omega^g_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}(G)}(\log \mathrm{D}) \to 0\Big], \tag{2.20}$$

$$\mathrm{DR}^\bullet\big(\mathcal{F}^{(k,\nu)}_{G^*}\big) = \Big[0 \to \mathcal{F}^{(k,\nu)}_{G^*} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{F}^{(k,\nu)}_{G^*} \otimes \Omega^g_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}(G^*)}(\log \mathrm{D}) \to 0\Big] \tag{2.21}$$

equipped with their natural Hodge filtration. We denote by $\mathrm{DR}^\bullet_c\big(\mathcal{F}^{(k,w)}_G\big)$ (resp. $\mathrm{DR}^\bullet_c\big(\mathcal{F}^{(k,\nu)}_{G^*}\big)$) the complex of sheaves obtained from (2.20) (resp.(2.21)) by tensoring with $\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}(G)}(-\mathrm{D})$ (resp. $\mathcal{O}_{\mathbf{Sh}^{\mathrm{tor}}_{K,R}(G^*)}(-\mathrm{D})$).

One can associate dual BGG complexes to $\mathrm{DR}^\bullet\big(\mathcal{F}^{(k,w)}_G\big)$, $\mathrm{DR}^\bullet\big(\mathcal{F}^{(k,\nu)}_{G^*}\big)$ and their compactly supported versions. We recall the definition of $\mathrm{BGG}(\mathcal{F}^{(k,\nu)}_{G^*})$ and we refer to ([TX16], Section 2.15) for the definition of $\mathrm{BGG}(\mathcal{F}^{(k,w)}_G)$. The compactly supported version is obtained by tensoring with the sheaf of functions vanishing at the boundary divisor. For any subset $J \subset \mathrm{I}_F$, let $s_J \in \{\pm 1\}^{\mathrm{I}_F}$ be the element whose τ-component is -1 if $\tau \notin J$ and 1 if $\tau \in J$. For $0 \leq j \leq g$ we put

$$\mathrm{BGG}^j(\mathcal{F}^{(k,\nu)}_{G^*}) = \bigoplus_{J \subset \mathrm{I}_F,\ \#J = j} \underline{\omega}^{s_J \cdot (k,\nu)}_{G^*} e_J$$

for e_J the Cech symbol and $\underline{\omega}^{s_J \cdot (k,\nu)}_{G^*} = (\wedge^2 \mathbb{H}^1_\circ)^{\nu - |\mathrm{I}_F \setminus J| - \sum_{\tau \in J} k_\tau} \otimes \bigotimes_{\tau \notin J} \underline{\omega}^{2-k_\tau}_\tau \otimes \bigotimes_{\tau \in J} \underline{\omega}^{k_\tau}_\tau$. There are differential operators $d : \mathrm{BGG}^j(\mathcal{F}^{(k,\nu)}_{G^*}) \to \mathrm{BGG}^{j+1}(\mathcal{F}^{(k,\nu)}_{G^*})$ given on local sections by $d : \mathsf{f} e_J \mapsto \sum_{\tau \notin J} \Theta_{\tau, k_{\tau-1}}(\mathsf{f}) e_\tau \wedge e_J$ where

$$\Theta_{\tau, k_\tau - 1}(\mathsf{f}) = \frac{(-1)^{k_\tau - 2}}{(k_\tau - 2)!} \sum_\xi \tau_0(\xi)^{k_\tau - 1}(\xi) a_\xi q^\xi$$

if the local section is written as $\mathsf{f} = \sum_\xi a_\xi q^\xi$.

Theorem 2.6.3. *([TX16], Theorem 2.16; [LP], remark 5.24) Let R be an F^{Gal}-algebra, then for $\mathcal{S} = \mathcal{F}^{(k,w)}_G$ (resp. $\mathcal{F}^{(k,\nu)}_{G^*}$) there are canonical quasi-isomorphic embeddings $BGG^\bullet(\mathcal{S}) \hookrightarrow DR^\bullet(\mathcal{S})$ and $BGG^\bullet_c(\mathcal{S}) \hookrightarrow DR^\bullet_c(\mathcal{S})$ of complexes of abelian sheaves on $\mathbf{Sh}^{tor}_{K,R}(G)$ (resp. $\mathbf{Sh}^{tor}_{K,R}(G^*)$). Moreover, the Hodge spectral sequences for both complexes degenerate at the first page.*

p-Adic theory

Katz's idea for a geometric theory of p-adic modular forms [Kat73] consists in removing from the relevant Shimura variety the preimages, under the specialization map, of those points in the special fiber that correspond to non-ordinary abelian varieties.

Let $E \subset \mathbb{C}$ be a number field containing F^{Gal}. The fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ determines a prime ideal $\wp \mid p$ of E. We denote by $E_\wp$ the completion, $\mathcal{O}_\wp$ the ring of integers and κ the residue field. Let $\mathcal{A}^{\mathrm{sa}}_\kappa$ be the semi-abelian scheme over the special fiber $\mathbf{Sh}^{\mathrm{tor}}_{K,\kappa}$ of the Shimura variety. The determinant of the map, induced by Verschiebung $\mathrm{V} : (\mathcal{A}^{\mathrm{sa}}_\kappa)^{(p)} \to \mathcal{A}^{\mathrm{sa}}_\kappa$ between cotangent spaces at the origin, corresponds to a characteristic p Hilbert modular form $\mathrm{Ha} \in H^0(\mathbf{Sh}^{\mathrm{tor}}_{K,\kappa}, \det(\underline{\omega})^{\otimes(p-1)})$, called the Hasse invariant. The ordinary locus $\mathbf{Sh}^{\mathrm{tor,ord}}_{K,\kappa}$ is the complement of the zero locus of the Hasse invariant. Let $\mathscr{S}^{\mathrm{tor}}_K$ denote the formal completion of $\mathbf{Sh}^{\mathrm{tor}}_{K,\mathcal{O}_\wp}$ along its special fiber and $j : \,]\mathbf{Sh}^{\mathrm{tor,ord}}_{K,\kappa}[\, \hookrightarrow \mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}$ the inverse image of the ordinary locus under the specialization map $\mathrm{sp} : \mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}} \to \mathbf{Sh}^{\mathrm{tor}}_{K,\kappa}$. Let $\mathcal{F}$ be a coherent sheaf on $\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}$; one defines $j^\dagger\mathcal{F}$ to be the sheaf whose sections on an admissible open $U \subset \mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}$ are the direct limit of $\Gamma(V \cap U, \mathcal{F})$ computed over strict neighborhoods V of $]\mathbf{Sh}^{\mathrm{tor,ord}}_{K,\kappa}[$ in $\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}$. For the minimal compactification $\mathbf{Sh}^*_{K,\kappa}$ one can similarly define the ordinary locus $\mathbf{Sh}^{*,\mathrm{ord}}_{K,\kappa}$ of the special fiber, which is an affine scheme, since $\det(\underline{\omega})$ is an ample line bundle on $\mathbf{Sh}^*_{K,\kappa}$. This is a very convenient feature because it implies the existence of a fundamental system of strict affinoid neighborhoods of $]\mathbf{Sh}^{*,\mathrm{ord}}_{K,\kappa}[$.

Theorem 2.6.4. *We recall that overconvergent cuspforms of weight $(k,w) \in \mathbb{Z}[\mathrm{I}_F]^2$ are defined as $S^\dagger_{k,w}(K;E_\wp) = H^0\big(\mathscr{S}^{tor}_{K,rig}, j^\dagger\big(\underline{\omega}^{(k,w)}_G(-D)\big)\big)$. For any cohomological weight $(k,w) \in \mathbb{Z}[\mathrm{I}_F]^2$, $k - 2w = mt_F$, the hypercohomology group $\mathbb{H}^g\big(\mathscr{S}^{tor}_{K,rig}, j^\dagger DR^\bullet_c\big(\mathcal{F}^{(k,w)}_G\big)\big)$ can be computed either as*

$$\frac{S^\dagger_{k,w}(K;E_\wp)}{\sum_{\tau\in\mathrm{I}_F}\Theta_{\tau,k_\tau-1}\big(S^\dagger_{s_\tau\cdot(k,w)}(K;E_\wp)\big)} \qquad \textit{or} \qquad \frac{H^0_{rig}\big(\mathscr{S}^{tor}_{K,rig}, j^\dagger\big(\mathcal{F}^{(k,w)}_G \otimes \Omega^g_{\mathscr{S}^{tor}_{K,rig}}(G)\big)\big)}{\nabla H^0_{rig}\big(\mathscr{S}^{tor}_{K,rig}, j^\dagger\big(\mathcal{F}^{(k,w)}_G \otimes \Omega^{g-1}_{\mathscr{S}^{tor}_{K,rig}}(G)\big)\big)}.$$

Proof. This theorem can be proved as ([TX16], Theorem 3.5). Indeed, Theorem 2.6.3 gives us a quasi-isomorphism of complexes

$$\mathrm{DR}^\bullet_c(\mathcal{F}^{(k,w)}_G) \cong \mathrm{BGG}^\bullet_c(\mathcal{F}^{(k,w)}_G),$$

then the isomorphism of $\mathbb{H}^g\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger\mathrm{DR}^\bullet_c\big(\mathcal{F}^{(k,w)}_G\big)\big)$ with

$$\mathbb{H}^g\big(\mathscr{S}^*_{K,\mathrm{rig}}, j^\dagger \oint_* \mathrm{BGG}_c(\mathcal{F}^{(k,w)}_G)\big) \cong \mathbb{H}^g\big(\mathscr{S}^*_{K,\mathrm{rig}}, j^\dagger \oint_* \mathrm{DR}^\bullet_c\big(\mathcal{F}^{(k,w)}_G\big)\big)$$

follows by applying the Leray spectral sequence for the composition

$$\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}} \to \mathscr{S}^*_{K,\mathrm{rig}} \to \mathrm{Spa}\mathbb{Q}_p$$

together with the vanishing of the higher derived images of subcanonical automorphic bundles ([Lan17], Theorem 8.2.1.2). We conclude that

$$\mathbb{H}^g\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger\mathrm{DR}^\bullet_c\big(\mathcal{F}^{(k,w)}_G\big)\big) \cong \frac{H^0_{\mathrm{rig}}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger\big(\mathcal{F}^{(k,w)}_G \otimes \Omega^g_{\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}}(G)\big)\big)}{\nabla H^0_{\mathrm{rig}}\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger\big(\mathcal{F}^{(k,w)}_G \otimes \Omega^{g-1}_{\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}}(G)\big)\big)}$$

because there is a fundamental system of affinoid neighborhoods of the ordinary locus on the minimal compactification. $\square$

Remark 2.6.5. *Replacing the group G by G^* in Theorem* 2.6.4, *the conclusion still holds for any* *cohomological weight* $(k,\nu) \in \mathbb{Z}[\mathrm{I}_F] \times \mathbb{Z}$ *and the group* $\mathbb{H}^g(\mathscr{S}^{tor}_{K,rig}, j^\dagger DR^\bullet_c(\mathcal{F}^{(k,\nu)}_{G^*}))$, *if we define overconvergent cuspforms for* G^* *as* $S^{*,\dagger}_{k,\nu}(K;E_\wp) = H^0(\mathscr{S}^{tor}_{K,rig}, j^\dagger(\underline{\omega}^{(k,\nu)}_{G^*}(-D)))$.

Lemma 2.6.6. *Let* $\mathfrak{p} \mid p$ *be a prime* $\mathcal{O}_F$*-ideal. The partial Frobenius* $Fr_\mathfrak{p}$ *([TX16], Section 3.12) acts on the image of* $S^\dagger_{k,w}(K,E_\wp)$ *in the cohomology group* $\mathbb{H}^g(\mathscr{S}^{tor}_{K,rig}, j^\dagger DR^\bullet_c(\mathcal{F}^{(k,w)}_G))$ *as* $Fr_\mathfrak{p} = N_{F/\mathbb{Q}}(\mathfrak{p})V(\mathfrak{p})$.

Proof. Taking into account the action of the partial Frobenius on $j^\dagger\Omega^g_{\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G)}$, the same computation as in ([Col96], Remark p.339) shows that $\mathrm{Fr}_\mathfrak{p}$ acts on the image of $S^\dagger_{k,w}(K,E_\wp)$ in $\mathbb{H}^g(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger\mathrm{DR}^\bullet_c(\mathcal{F}^{(k,w)}_G))$ as $\varpi_\mathfrak{p}^{k-t_F+\frac{(2-m)t_F-k}{2}}[\mathfrak{p}]$, since $[\mathfrak{p}]$ is the operator that acts on q-expansion by $\mathsf{a}(y,\mathsf{f}_{|[\mathfrak{p}]}) = \mathsf{a}(y\varpi_\mathfrak{p}^{-1},\mathsf{f})$. We conclude noting that $[\mathfrak{p}] = \varpi_\mathfrak{p}^{t_F-w}V(\mathfrak{p})$ as operators on $S^\dagger_{k,w}(K,E_\wp)$. □

If we denote by $U_\mathfrak{p}$ the operator defined in ([TX16], Section 3.18), the equality $U_\mathfrak{p}\mathrm{Fr}_\mathfrak{p} = \langle\mathfrak{p}^{-1}\rangle\varpi_\mathfrak{p}^{t_F}$ of ([TX16], Lemma 3.20) implies that $U(\mathfrak{p}) = U_\mathfrak{p}\langle\mathfrak{p}\rangle$ as operators on $S^\dagger_{k,w}(K;E_\wp)$. In particular, we can restate ([TX16], Corollary 3.24) by saying that if $\mathsf{f} \in S^\dagger_{s_J\cdot(k,w)}(K;E_\wp)$ is a generalized eigenform for $U_0(\mathfrak{p})$ with non-zero eigenvalue $\lambda_\mathfrak{p}$, then

$$\mathrm{val}_p(\lambda_\mathfrak{p}) \geq \sum_{\tau\in\mathrm{I}_{F,\mathfrak{p}}\setminus J}(k_\tau - 1) \tag{2.22}$$

where $\mathrm{I}_{F,\mathfrak{p}}$ is the subset of those embeddings $F \hookrightarrow \overline{\mathbb{Q}}$ that induce the prime $\mathfrak{p}$ when composed with the fixed p-adic embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Corollary 2.6.7. *Let* $F/\mathbb{Q}$ *be a real quadratic field in which* $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$ *splits. Let* $\mathsf{f} \in S_{k,w}(K;\overline{\mathbb{Q}})$ *an eigenform of prime to p level. Then the p-adic cuspforms* $d_1^{1-k_1}(\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]})$, $d_2^{1-k_2}(\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]})$ *are overconvergent.*

Proof. We prove the corollary building on an idea of Loeffler, Skinner and Zerbes ([LSZ16], Proposition 4.5.3). Let

$$1 - \mathsf{a}(\varpi_{\mathfrak{p}_2},\mathsf{f})X + \epsilon_\mathsf{f}(\mathfrak{p}_2)\varpi_{\mathfrak{p}_2}^{k-t_F}X^2 = (1-\alpha_{0,2}X)(1-\beta_{0,2}X)$$

be the Hecke polynomial of f for $T_0(\mathfrak{p}_2)$. We denote by $\mathsf{f}_{\alpha_2},\mathsf{f}_{\beta_2}$ the two $\mathfrak{p}_2$-stabilizations of f and without loss of generality suppose $\mathrm{val}_p(\alpha_{0,2}) \leq \mathrm{val}_p(\beta_{0,2})$. If we write $\Theta_i = \Theta_{\tau_i,k_{\tau_i}-1}$ for $i = 1,2$, then the classes of $\mathsf{f}^{[\mathfrak{p}_1]}_{\alpha_2},\mathsf{f}^{[\mathfrak{p}_1]}_{\beta_2}$ are trivial in the quotient

$$\frac{S^\dagger_{k,w}(K;E_\wp)}{\mathrm{Im}(\Theta_1)+\mathrm{Im}(\Theta_2)}$$

because they are annihilated by the invertible operator $U_0(\mathfrak{p}_1)$. For $i = 1,2$ consider the Hecke-equivariant projections

$$\mathrm{pr}_i : \mathrm{Im}(\Theta_1)+\mathrm{Im}(\Theta_2) \to \frac{\mathrm{Im}(\Theta_i)}{\mathrm{Im}(\Theta_1)\cap\mathrm{Im}(\Theta_2)}.$$

We immediately see that $\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}_{\alpha_2}) = 0$ because of the lower bound (2.22) on the slopes of $U_0(\mathfrak{p}_2)$, therefore $\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) = \frac{\beta_{2,0}}{\beta_{2,0}-\alpha_{2,0}}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}_{\beta_2})$ which implies

$$U_0(\mathfrak{p}_2)\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) = \beta_{0,2}\cdot\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}).$$

We claim that $[\mathfrak{p}_2]\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) = \frac{1}{\beta_{0,2}}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]})$. Indeed, the equality of Hecke operators

$$T_0(\mathfrak{p}_2) = U_0(\mathfrak{p}_2) + \alpha_{0,2}\beta_{0,2}[\mathfrak{p}_2]$$

allows us to compute that

$$\begin{aligned}[\mathfrak{p}_2]\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) &= \frac{1}{\alpha_{0,2}\beta_{0,2}}\left[T_0(\mathfrak{p}_2)\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) - U_0(\mathfrak{p}_2)\frac{\beta_{2,0}}{\beta_{2,0}-\alpha_{2,0}}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}_{\beta_2})\right] \\ &= \frac{1}{\alpha_{0,2}\beta_{0,2}}\left[\mathsf{a}(\mathfrak{p}_2,\mathsf{f})\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}) - \beta_{0,2}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]})\right] = \frac{1}{\beta_{0,2}}\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1]}).\end{aligned}$$

Thus, $\mathrm{pr}_2(\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]}) = 0$. By exchanging the roles of the two primes $\mathfrak{p}_1, \mathfrak{p}_2$ we also have that $\mathrm{pr}_1(\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]}) = 0$, which proves $\mathsf{f}^{[\mathfrak{p}_1,\mathfrak{p}_2]} \in \mathrm{Im}(\Theta_1) \cap \mathrm{Im}(\Theta_2)$. □

2.7 A p-adic Gross-Zagier Formula

2.7.1 De Rham realization of modular forms

Let E be a number field, following Voevodsky [Voe00] we consider two categories of motives over E: the category of effective Chow motives denoted $\mathrm{CHM}^{\mathrm{eff}}$ with a natural functor $\mathrm{h} : \mathrm{SmProj}_{/E} \to \mathrm{CHM}^{\mathrm{eff}}$ from the category $\mathrm{SmProj}_{/E}$ of smooth and projective schemes over E, and the triangulated category $\mathrm{DM}^{\mathrm{eff}}$ of effective geometric motives with the natural functor $\mathrm{M_{gm}} : \mathrm{Sm}_{/E} \to \mathrm{DM}^{\mathrm{eff}}$ from the category $\mathrm{Sm}_{/E}$ of smooth schemes over E. Since number fields have characteristic zero, these two categories are related by a full embedding $\mathrm{CHM}^{\mathrm{eff}} \to \mathrm{DM}^{\mathrm{eff}}$ that makes the diagram

$$\begin{tikzcd} \mathrm{SmProj}_{/\mathbb{Q}} \arrow[r] \arrow[d, "\mathrm{h}"'] & \mathrm{Sm}_{/\mathbb{Q}} \arrow[d, "\mathrm{M_{gm}}"] \\ \mathrm{CHM}^{\mathrm{eff}} \arrow[r] & \mathrm{DM}^{\mathrm{eff}} \end{tikzcd}$$

commute ([Voe00], Proposition 2.1.4 and Remark).

Let F be a totally real number field of degree g over $\mathbb{Q}$ and let E be any field containing F^{Gal}. The Shimura variety $\mathrm{Sh}_K(G^*)_{\mathbb{Q}}$ has a universal Hilbert-Blumenthal abelian scheme $\mathcal{A} \to \mathrm{Sh}_K(G^*)$, the $\mathcal{O}_F$-action induces a ring homomorphism $F \hookrightarrow \mathrm{End}_{\mathrm{Sh}_K(G^*)}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote by $\mathrm{CMH}(\mathrm{Sh}_K(G^*))$ the category of Chow motives over $\mathrm{Sh}_K(G^*)$ [DM91]. Since the decomposition of the Chow motive $\mathrm{h}(\mathcal{A}/\mathrm{Sh}_K(G^*)) = \bigoplus_i \mathrm{h}_i(\mathcal{A}/\mathrm{Sh}_K(G^*))$ of $\mathcal{A}$ over $\mathrm{Sh}_K(G^*)$ is functorial ([DM91], Theorem 3.1), there is an isomorphism of $\mathbb{Q}$-vector spaces ([Kin98], Proposition 2.2.1)

$$\mathrm{End}_{\mathrm{Sh}_K(G^*)}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{End}_{\mathrm{CHM}(\mathrm{Sh}_K(G^*))}(\mathrm{h}_1(\mathcal{A}/\mathrm{Sh}_K(G^*))) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

One denotes by $e_\tau \in \mathrm{End}_{\mathrm{CHM}(\mathrm{Sh}_K(G^*))}(\mathrm{h}_1(\mathcal{A}/\mathrm{Sh}_K(G^*))) \otimes_{\mathbb{Z}} E$, $\tau \in \mathrm{I}_F$, the idempotents coming from $\prod_\tau F = F \otimes E \hookrightarrow \mathrm{End}_{\mathrm{CHM}(\mathrm{Sh}_K(G^*))}(\mathrm{h}_1(\mathcal{A}/\mathrm{Sh}_K(G^*))) \otimes_{\mathbb{Z}} E$.

Definition 2.7.1. *Let $k \in \mathbb{N}[\mathrm{I}_F]$, $k \geq 2t_F$. The relative motive $\mathcal{V}^k \in CHM(Sh_K(G^*))_E$ is defined as*

$$\mathcal{V}^k = \bigotimes_{\tau \in \mathrm{I}_F} Sym^{k_\tau - 2} h_1(\mathcal{A}/Sh_K(G^*))^{e_\tau}$$

following the conventions of ([Kin98], p.72) for the symmetric products. The motive $\mathcal{V}^k$ is a direct factor of $h(\mathcal{A}^{|k-2t_F|}/Sh_K(G^*))$, where $\mathcal{A}^{|k-2t_F|}$ denotes the $(|k|-2g)$-fold fiber product of $\mathcal{A}$ over $Sh_K(G^*)$, thus it corresponds to an idempotent

$$e_k \in CH^{g(|k|-2g)}\left(\mathcal{A}^{|k-2t_F|} \times_{Sh_K(G^*)} \mathcal{A}^{|k-2t_F|}\right) \otimes_{\mathbb{Z}} E$$

such that $M_{gm}(\mathcal{A}^{|k-2t_F|})^{e_k} = \mathcal{V}^k$.

Proposition 2.7.2. *([Wil12], Corollary 3.9) Suppose $k > 2t_F$ and let U_{k-2g} be any smooth compactification of $\mathcal{A}^{|k-2t_F|}$, then the graded part of weight zero with respect to the motivic weight structure on CHM_E^{eff}, $Gr_0 M_{gm}\left(\mathcal{A}^{|k-2t_F|}\right)^{e_k}$, is canonically a direct factor of the Chow motive $M_{gm}(U_{k-2g})$. Hence, it corresponds to an idempotent $\theta_k \in CH^{g(|k|-2g+1)}(U_{k-2g} \times_{\mathbb{Q}} U_{k-2g}) \otimes_{\mathbb{Z}} E$.*

Proposition 2.7.3. *Suppose $F = \mathbb{Q}$ and let $k > 2$ be an integer. For any smooth compactification W_{k-2} of the $(k-2)$th-fold product of the universal elliptic curve $\mathcal{E}$ over the modular curve $Sh_{K'}(GL_{2,\mathbb{Q}})$, there exists an idempotent $\theta_k \in CH^{k-1}(W_{k-2} \times_{\mathbb{Q}} W_{k-2}) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that*

$$\theta_k^* H^*_{dR}(W_{k-2}/\mathbb{Q}) = \theta_k^* H^{k-1}_{dR}(W_{k-2}/\mathbb{Q})$$

is functorially isomorphic to parabolic cohomology $H^1_{par}(Sh^{tor}_{K'}, (\mathcal{F}^{(k,k-1)}_{GL_{2,\mathbb{Q}}}, \nabla))$ with its Hodge filtration ([BDP13], Section 2.1).

Proof. Proposition 2.7.2 provides an idempotent θ_k such that

$$\theta_k^* \mathrm{M_{gm}}(W_{k-2}) = \mathrm{Gr_0 M_{gm}}(\mathcal{E}^{k-2})^{e_k}.$$

We claim that the proof of ([BDP13], Lemma 2.2) applies to our situation. Indeed, the main ingredient of that proof is a result of Scholl ([Sch90], Theorem 3.1.0), which can be applied to any smooth compactification W_{k-2} since the motive considered by Scholl is isomorphic to $\mathrm{Gr_0 M_{gm}}(\mathcal{E}^{k-2})^{e_k}$ by ([Wil09] Corollary 3.4(b)). Note that the idempotent e in ([Wil09], Definition 3.1) acts as the idempotent e_k on $\mathrm{M_{gm}}(\mathcal{E}^{k-2})$ because the action of the torsion appearing in e is trivial since $\mathcal{E}^{k-2} \to \mathrm{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})$ is an abelian scheme. □

Proposition 2.7.4. *Let $L/\mathbb{Q}$ be a real quadratic extension and $\ell \in \mathbb{N}[I_L]$, $\ell > 2t_F$ a non-parallel weight. For any smooth compactification $U_{\ell-4}$ of the $(|\ell|-4)$th-fold product of the universal abelian surface over $Sh_K(G_L^*)$, there exists an idempotent*

$$\theta_\ell \in CH^{2(|\ell|-3)}(U_{\ell-4} \times_{\mathbb{Q}} U_{\ell-4}) \otimes_{\mathbb{Z}} L$$

such that $\theta_\ell^* H^{i+|\ell|-4}_{dR}(U_{\ell-4}/\mathbb{Q})$ is functorially isomorphic to $\mathbb{H}^i(Sh_K, DR^\bullet(\mathcal{F}^{(\ell,|\ell-t_L|)}_{G_L^*}))$ with its Hodge filtration.

Proof. Since the weight ℓ is not parallel, Proposition 2.7.2 and ([Wil12], Theorem 3.6) provide an idempotent θ_ℓ such that $\theta_\ell^* \mathrm{M_{gm}}(U_{\ell-4}) = \mathcal{V}^\ell$. Then Kings proved in ([Kin98], Corollary 2.3.4) that the $(i+|\ell|-4)$-th cohomology of the de Rham realization of $\mathcal{V}^\ell$ is isomorphic to $\mathbb{H}^i(\mathrm{Sh}_K, \mathrm{DR}^\bullet(\mathcal{F}^{(\ell,|\ell-t_L|)}_{G_L^*}))$. □

2.7.2 Generalized Hirzebruch-Zagier cycles

Let $L/\mathbb{Q}$ be a real quadratic extension, $K \subset V_{11}(\mathfrak{A}\mathcal{O}_L)$ a small enough (Definition 2.5.2) congruence subgroup, $K' = K \cap \mathrm{GL}_2(\mathbb{A}^\infty)$, and let $\zeta : \mathrm{Sh}_K(G_L^*) \to \mathrm{Sh}_K(G_L)$ be the map of Shimura varieties derived from the inclusion $G_L^* \hookrightarrow G_L$.

Let $\breve{\mathrm{g}} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; \overline{\mathbb{Q}})$ be a eigenform of either parallel weight $\ell = 2t_L$ or non-parallel weight $\ell > 2t_L$ such that $\ell - 2x = nt_L$. Let $\mathrm{f} \in S_{k,w}(V_1(\mathfrak{N}); \overline{\mathbb{Q}})$ be an elliptic newform such that $k - 2w = m$. We suppose that the weights of $\breve{\mathrm{g}}$ and f are balanced. We consider $E/\mathbb{Q}$ a finite Galois extension containing the Fourier coefficients of $\breve{\mathrm{g}}$ and f. We want to realize these modular forms in the de Rham cohomology of some proper and smooth variety. The pullback $\zeta^*\breve{\mathrm{g}}$ lives in $S^*_{\ell,x}(K; L; E)$, which by (2.2) is isomorphic to $S^*_{\ell,\ell-t_L}(K; L; E)$. Thanks to Theorem 2.6.3 we can realize the latter space as a subgroup of the hypercohomology group $\mathbb{H}^2(\mathrm{Sh}^{\mathrm{tor}}_{K,E}, \mathrm{DR}^\bullet(\mathcal{F}^{\ell,|\ell-t_L|}_{G_L^*}))$, which is simply the de Rham cohomology group $H^2_{\mathrm{dR}}(\mathrm{Sh}_K(G_L^*)/E)$ when $\ell = 2t_L$. Instead, when $\ell > 2t_L$ is not parallel, let $U_{\ell-4}$ be any smooth compactification of $\mathcal{A}^{|\ell|-4}$; then, we can invoke Proposition 2.7.4 to establish that the differential attached to $\Psi_{x,|\ell-t_L|}(\zeta^*\breve{\mathrm{g}})$, where $\Psi_{x,|\ell-t_L|}$ is defined in (2.2), lives in $\mathrm{F}^{|\ell|-2}H^{|\ell|-2}_{\mathrm{dR}}(U_{\ell-4}/E)$. Similarly, if $k=2$, $\Psi_{w,1}(\mathrm{f}) \in S_{2,1}(K'; E) \subset \mathrm{F}^1 H^1_{\mathrm{dR}}(\mathrm{Sh}^{\mathrm{tor}}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})/E)$, while when $k > 2$ we can consider any smooth compactification W_{k-2} of $\mathcal{E}^{k-2}$ to see that the class of the differential $\omega_{\Psi_{w,k-1}(\mathrm{f})}$ lives in $H^{k-1}_{\mathrm{dR}}(W_{k-2}/E)$, by Proposition 2.7.3.

Definition 2.7.5. *Choose a prime p coprime to M. Let $E_\wp$ be the closure of $\iota_p(E)$ in $\overline{\mathbb{Q}}_p$ and suppose that $\breve{\mathrm{g}}, \mathrm{f}$ are p-nearly ordinary. We write ω for the differential $\omega_{\Psi_{x,|\ell-t_L|}(\zeta^*\breve{\mathrm{g}})}$ and we take η to be the class in the $\Psi_{w,k-1}(\mathrm{f})$-isotypic part of $H^1_{par}\big(Sh^{tor}_{K'}, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big)^{u.r.}$ whose image in the 0-th graded piece, $H^1(Sh^{tor}_{K',E_\wp}, \underline{\omega}^{2-k}_{\mathrm{GL}_{2,\mathbb{Q}}})$, is equal to the image of $\frac{\mathrm{vol}(K')}{\langle \mathrm{f}^*,\mathrm{f}^*\rangle} \cdot \overline{\omega_{\Psi_{w,k-1}(\mathrm{f}^*)}}$.*

The class $\eta \in H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K'}, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big)^{\mathrm{u.r.}}$ satisfies

$$\mathrm{Fr}_p(\eta) = \alpha_{\mathrm{f}^*} p^{w-1}\eta, \tag{2.23}$$

where the eigenvalue is a p-adic unit since f^* is p-nearly ordinary. Indeed, by definition $\eta = [c \cdot \Psi_{w,k-1}(\mathrm{f}_\beta)]$ for some non-zero constant c, and applying Lemmas 2.1.7 and 2.6.6 we can compute

$$\begin{aligned}\mathrm{Fr}_p(\eta) &= pV(p)[c \cdot \Psi_{w,k-1}(\mathrm{f}_\beta)] = p \cdot p^{k-1-w}[c \cdot \Psi_{w,k-1}(V(p)\mathrm{f}_\beta)] \\ &= p^{k-w}[c \cdot \Psi_{w,k-1}(U(p)^{-1}\mathrm{f}_\beta)] = p^{k-w}\beta_{\mathrm{f}}^{-1}\eta = \alpha_{\mathrm{f}^*}p^{w-1}\eta,\end{aligned}$$

since $\beta_{\mathrm{f}}^{-1} = \alpha_{\mathrm{f}}\psi_{\mathrm{f}}(p)^{-1}p^{-1} = \alpha_{\mathrm{f}^*}p^{-m-1}$.

For all $s \geq 0$ we want to consider the cohomology class

$$\pi_1^*\omega \cup \pi_2^*\eta \in \mathrm{F}^{|\ell|-2-s}H^{|\ell|+k-3}_{\mathrm{dR}}\big(U_{\ell-4} \times_{E_\wp} W_{k-2}\big).$$

Our goal is to define a null-homologous cycle on $U_{\ell-4} \times_E W_{k-2}$ whose syntomic Abel-Jacobi map can be evaluated at $\pi_1^*\omega \cup \pi_2^*\eta$. Let $\mathscr{Z}_{\ell,k}$ be a proper smooth model of $U_{\ell-4} \times_{E_\wp} W_{k-2}$ over $\mathcal{O}_{E_\wp}$ of relative dimension d, and denote by $Z_{\ell,k}$ its generic fiber. For all integers $i \geq 0$, the syntomic cohomology groups of $\mathscr{Z}_{\ell,k}$ sit in a short exact sequence of the form

$$0 \longrightarrow H^{2i-1}_{\mathrm{dR}}(Z_{\ell,k})/F^i \xrightarrow{\ \iota\ } H^{2i}_{\mathrm{syn}}(\mathscr{Z}_{\ell,k}, i) \xrightarrow{\ \pi\ } \mathrm{F}^i H^{2i}_{\mathrm{dR}}(Z_{\ell,k}).$$

The syntomic cycle class map ([Bes00], Proposition 5.4) is compatible with the de Rham cycle class map producing a commuting diagram

$$\begin{tikzcd} \mathrm{CH}^i(\mathscr{Z}_{\ell,k}) \arrow[r, "\mathrm{cl}_{\mathrm{syn}}"] \arrow[d, "\mathrm{Res}"] & H^{2i}_{\mathrm{syn}}(\mathscr{Z}_{\ell,k}, i) \arrow[d, "\pi"] \\ \mathrm{CH}^i(Z_{\ell,k}) \arrow[r, "\mathrm{cl}_{\mathrm{dR}}"] & F^i H^{2i}_{\mathrm{dR}}(Z_{\ell,k}), \end{tikzcd}$$

where on the left hand side are the Chow groups of algebraic cycles modulo rational equivalence. The restriction of the syntomic cycle class map $\mathrm{cl}_{\mathrm{syn}}$ to the subgroup of de Rham null-homologous cycles $\mathrm{CH}^i(\mathscr{Z}_{\ell,k})_0$, i.e., the kernel of the composition $\mathrm{cl}_{\mathrm{dR}} \circ \mathrm{Res}$, has image landing in $H^{2i-1}_{\mathrm{dR}}(Z_{\ell,k})/F^i$. The syntomic Abel-Jacobi map

$$\mathrm{AJ}_p : \mathrm{CH}^i(\mathscr{Z}_{\ell,k})_0 \longrightarrow \left(F^{d-i+1} H^{2(d-i)+1}_{dR}(Z_{\ell,k})\right)^\vee \tag{2.24}$$

is obtained by identifying the target using Poincaré duality.

We determine the positive integer s and make sure the numerology works. The dimension of the variety $U_{\ell-4} \times_E W_{k-2}$ is $d = 2|\ell| + k - 7$, therefore the cycle we want has to be of dimension $d - i$ such that $2(d-i) + 1 = |\ell| + k - 3$, and $s \geq 0$ has to satisfy $|\ell| - 2 - s = (d-i) + 1$. Hence

$$(d-i) = \frac{|\ell| + k - 4}{2}, \qquad s = \frac{|\ell| - k - 2}{2} \tag{2.25}$$

with $s \geq 0$ since the weights are balanced.

Definition of the cycles

We treat separately the case $(\ell, k) = (2t_L, 2)$ and the general case $(\ell, k) > (2t_L, 2)$ with ℓ not parallel. Set $\gamma + 1 = \frac{|\ell|+k-4}{2}$ and consider the finite map

$$\begin{aligned} \varphi : \mathcal{E}^\gamma &\longrightarrow \mathcal{A}^{|\ell|-4} \times_E \mathcal{E}^{k-2} \\ (x, P_1, \ldots, P_\gamma) &\mapsto (\zeta(x), P'_1 \otimes 1, \ldots, P'_{|\ell|-4} \otimes 1; x, P'_{|\ell|-3}, \ldots, P'_{2\gamma}) \end{aligned}$$

where $(P'_1, \ldots, P'_{2\gamma}) = (P_1, \ldots, P_\gamma, P_1, \ldots, P_\gamma)$ and $P'_i \otimes 1$ is the point $P'_i \otimes 1 \to \mathcal{E} \otimes_{\mathbb{Z}} \mathcal{O}_F \to \mathcal{A}$. The definition makes sense because $2\gamma = |\ell| - 4 + k - 2$. The variety $\mathcal{E}^\gamma$ has dimension equal to $\gamma + 1$ and we will define the null-homologous cycle by first compactifying and then by applying an appropriate correspondence. Let W_0 be the smooth and projective compactification of the modular curve $\mathrm{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})$. We consider $W_\gamma, U_{\ell-4}, W_{k-2}$ smooth and projective compactifications of $\mathcal{E}^\gamma, \mathcal{A}^{|\ell|-4}, \mathcal{E}^{k-2}$ respectively, such that W_γ has a map $W_\gamma \to W_0$ extending $\mathcal{E}^\gamma \to \mathrm{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})$; then the map φ defines a rational morphism $\varphi : W_\gamma \dashrightarrow U_{\ell-4} \times_E W_{k-2}$. Using Hironaka's work on resolution of singularities ([Hir64], Chapter 0.5, Question (E)), we can assume the rational map φ has a representative $\varphi : W_\gamma \longrightarrow U_{\ell-4} \times_E W_{k-2}$ defined everywhere, up to replacing the smooth and projective compactification of $\mathcal{E}^\gamma$. Furthermore, by desingularizing the fibers over the cusps, we can assume that $W_\gamma \to W_0$ is smooth. By spreading out, there is an open of $\mathrm{Spec}(\mathcal{O}_E)$ over which all our geometric objects can be defined simultaneously and retain their relevant features: we have smooth and projective models $\mathscr{W}_\gamma, \mathscr{U}_{\ell-4}, \mathscr{W}_{k-2}$ of $W_\gamma, U_{\ell-4}, W_{k-2}$ respectively, the map φ extends to a map $\tilde{\varphi} : \mathscr{W}_\gamma \longrightarrow \mathscr{U}_{\ell-4} \times \mathscr{W}_{k-2}$ and $\mathscr{W}_\gamma \to \mathscr{W}_0$ is smooth.

Remark 2.7.6. *If $X \to Spec(E)$ is a quasi-compact scheme of finite type and $\mathscr{X}_1, \mathscr{X}_2$ are two models defined over some open $U \subset Spec(\mathcal{O}_E)$, then every identification $(\mathscr{X}_1)_E \cong (\mathscr{X}_2)_E$ of their generic fibers can be spread out to an isomorphism $(\mathscr{X}_1)_V \cong (\mathscr{X}_2)_V$ over some open $V \subset U$.*

In particular, up to shrinking further the open subset of $\mathrm{Spec}(\mathcal{O}_E)$, we can also assume that the fibers of $\mathscr{W}_\gamma$ and $\mathscr{W}_{k-2}$ over the integral model of the open modular curve, and the fiber of $\mathscr{U}_{\ell-4}$ over the integral model of the open Hilbert modular surface come equipped with isomorphisms with the canonical models $\mathscr{E}^\gamma, \mathscr{E}^{k-2}$ and $\mathscr{A}^{|\ell|-4}$ – obtained as the solution of the relevant moduli problems – of $\mathcal{E}^\gamma, \mathcal{E}^{k-2}$ and $\mathcal{A}^{|\ell|-4}$ respectively.

When $\ell = 2t_L$ and $k = 2$ we define correspondences on $\mathscr{U}_0 \times \mathscr{W}_0$ as follows. We assume the number field E is large enough such that $U_{0/E}$ (resp. $W_{0/E}$) is the disjoint union of its geometrically connected components $U_{0/E} = \coprod_i U_{0,i}$ (resp. $W_{0/E} = \coprod_j W_{0,j}$) and we pick an E-rational point $a_i \in U_{0,i}$ (resp. $b_j \in W_{0,j}$) for every such component. Consider the following morphisms: for every pair (i,j) indexing a geometrically irreducible component of $Z = U_0 \times_E W_0$, we define $q_{i,j} : Z \to U_{0,i} \times_E W_{0,j} \hookrightarrow Z$ as the map that restricts to the natural inclusion of $U_{0,i} \times_E W_{0,j}$ into Z and maps any other geometrically irreducible component to the point (a_i, b_j). Similarly, we define the morphisms $q_{a_i,j} : Z \to \{a_i\} \times W_{0,j} \hookrightarrow Z$, $q_{i,b_j} : Z \to U_{0,i} \times \{b_j\} \hookrightarrow Z$ and $q_{a_i,b_j} : Z \to \{a_i\} \times \{b_j\} \hookrightarrow Z$. Consider the correspondences

$$P_{i,j} = \mathrm{graph}(q_{i,j}),\ \ P_{a_i,j} = \mathrm{graph}(q_{a_i,j}),\ \ P_{i,b_j} = \mathrm{graph}(q_{i,b_j}),\ \ P_{a_i,b_j} = \mathrm{graph}(q_{a_i,b_j}),$$

in $\mathrm{CH}^6(Z \times_E Z)$. We define

$$P = \sum_{i,j} \left(P_{i,j} - P_{a_i,j} - P_{i,b_j} + P_{a_i,b_j} \right)$$

that acts on $\mathrm{CH}^\bullet(Z)$ by $P_* = \mathrm{pr}_{2,*}(P \cdot \mathrm{pr}_1^*)$; in particular, for any cycle $S \in \mathrm{CH}^\bullet(Z)$, we have

$$P_*(S) = \sum_{i,j} \left[(q_{i,j})_* - (q_{a_i,j})_* - (q_{i,b_j})_* + (q_{a_i,b_j})_* \right](S).$$

For i, j running in the set of indices of the geometrically connected components of U_0 and W_0 the correspondences $(P_{i,j} - P_{a_i,j} - P_{i,b_j} + P_{a_i,b_j})$ are idempotents and orthogonal to each other, hence $P \circ P = P$ in $\mathrm{CH}^6(Z \times_E Z)$, i.e., P is a projector. We denote by $\tilde{P}$ the correspondence on $\mathscr{U}_0 \times \mathscr{W}_0$ defined over some open of $\mathrm{Spec}(\mathcal{O}_E)$ obtained by spreading out P.

When $(\ell, k) > (2t_L, 2)$ with ℓ non-parallel, we obtain a correspondence on $\mathscr{U}_{\ell-4} \times \mathscr{W}_{k-2}$ by spreading out those correspondences considered in Section 2.7.1. Indeed, the idempotents $\theta_\ell \in \mathrm{CH}^{2(|\ell|-3)}(U_{\ell-4} \times_E U_{\ell-4}) \otimes_{\mathbb{Z}} L$ and $\theta_k \in \mathrm{CH}^{|k|-1}(W_{k-2} \times_E W_{k-2}) \otimes_{\mathbb{Z}} \mathbb{Q}$ extend to correspondences on the integral models $\tilde{\theta}_\ell \in \mathrm{CH}^{2(|\ell|-3)}(\mathscr{U}_{\ell-4} \times \mathscr{U}_{\ell-4}) \otimes_{\mathbb{Z}} L$ and $\tilde{\theta}_k \in \mathrm{CH}^{|k|-1}(\mathscr{W}_{k-2} \times \mathscr{W}_{k-2}) \otimes_{\mathbb{Z}} \mathbb{Q}$ respectively.

Definition 2.7.7. *For all but finitely many primes p, we define the Hirzebruch-Zagier cycle of weight $(2t_L, 2)$ to be*

$$\Delta_{2t_L,2} = \tilde{P}_* \tilde{\varphi}_*[\mathscr{W}_0] \in CH^2(\mathscr{U}_0 \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_0).$$

Proposition 2.7.8. *The Hirzebruch-Zagier cycle $\Delta_{2t_L,2} \in CH^2(\mathscr{U}_0 \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_0)$ is de Rham null-homologous.*

Proof. To verify that $\mathrm{cl}_{\mathrm{dR}}(\Delta_{2t_L,2})$ is zero in $H^4_{\mathrm{dR}}(Z/E_\wp)$, it suffices to show that the projection $P_* H^4_{\mathrm{dR}}(Z/E)$ is trivial since the cycle is defined over E. After base-change to $\mathbb{C}$, via the fixed complex embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, Poincaré duality tells us that it is enough to prove the projector annihilates the second singular homology, i.e., $P_* H_2(Z(\mathbb{C})) = 0$. By Kunneth formula and the fact that each connected component of $U_0(\mathbb{C})$ is simply connected, we compute that $P_* H_2(Z(\mathbb{C})) = P_*(H_0(U_0(\mathbb{C})) \otimes H_2(W_0(\mathbb{C})) \oplus H_2(U_0(\mathbb{C})) \otimes H_0(W_0(\mathbb{C})))$, which we can show to be zero by the explicit definition of the projector P. Indeed, let $[x] \otimes [C] \in H_0(U_0(\mathbb{C})) \otimes H_2(W_0(\mathbb{C}))$ be a simple tensor for $x \in U_0(\mathbb{C})$ a point, then for all i, j we find

$$\begin{aligned}\left(P_{i,j} - P_{a_i,j} - P_{i,b_j} + P_{a_i,b_j}\right)([x] \otimes [C]) &= \left((q_{i,j})_* - (q_{a_i,j})_* - (q_{i,b_j})_* + (q_{a_i,b_j})_*\right)([x] \otimes [C]) \\ &= [a_i] \otimes [C_j] - [a_i] \otimes [C_j] = 0,\end{aligned}$$

where $(q_{i,b_j})_*([x] \otimes [C]) = 0 = (q_{a_i,b_j})_*([x] \otimes [C])$ because the dimension of the pushforward drops. Similarly, if $[D] \otimes [y] \in H_2(U_0(\mathbb{C})) \otimes H_0(W_0(\mathbb{C}))$ is a simple tensor for $y \in W_0(\mathbb{C})$ a point, then $\left(P_{i,j} - P_{a_i,j} - P_{i,b_j} + P_{a_i,b_j}\right)([D] \otimes [y]) = 0$ for all i, j. $\square$

Definition 2.7.9. *Let $\ell \in \mathbb{Z}[I_L]$, $\ell > 2t_L$, be a non-parallel weight and $k > 2$ an integer such that (ℓ, k) is a balanced triple. For all but finitely many primes p, the generalized Hirzebruch-Zagier cycle of weight (ℓ, k) is*

$$\Delta_{\ell,k} = (\tilde{\theta}_\ell, \tilde{\theta}_k)_* \tilde{\varphi}_* [\mathscr{W}_\gamma] \in CH^i(\mathscr{U}_{\ell-4} \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_{k-2}) \otimes_{\mathbb{Z}} L.$$

Proposition 2.7.10. *Let $\ell \in \mathbb{Z}[I_L]$, $\ell > 2t_L$, be a non-parallel weight and $k > 2$ an integer such that (ℓ, k) is a balanced triple. The generalized Hirzebruch-Zagier cycle $\Delta_{\ell,k} \in CH^i(\mathscr{U}_{\ell-4} \times_{\mathcal{O}_{E,\wp}} \mathscr{W}_{k-2}) \otimes_{\mathbb{Z}} L$ is de Rham null-homologous.*

Proof. The class $\mathrm{cl}_{\mathrm{dR}}(\Delta_{\ell,k})$ belongs to $(\theta_\ell, \theta_k)_* H^{2i}_{\mathrm{dR}}(U_{\ell-4} \times_{E_\wp} W_{k-2})$ and by Poincaré duality, it is trivial if and only if

$$(\theta_\ell, \theta_k)^* H^{2(d-i)}_{\mathrm{dR}}(U_{\ell-4} \times_{E_\wp} W_{k-2}) = \bigoplus_{\mu+\nu=2(d-i)} (\theta_\ell)^* H^\mu_{\mathrm{dR}}(U_{\ell-4}) \otimes (\theta_k)^* H^\nu_{\mathrm{dR}}(W_{k-2}) \tag{2.26}$$

is trivial. By Propositions 2.7.4 and 2.7.3, we have

$$\theta_\ell^* H^\mu_{\mathrm{dR}}(U_{\ell-4}) = \mathbb{H}^{\mu-|\ell|+4}(\mathrm{Sh}_K, \mathrm{DR}^\bullet(\mathcal{F}^{(\ell,\ell-t)}_{G_L^*}))$$

and $\theta_k^* H^*_{\mathrm{dR}}(W_{k-2}) = \theta_k^* H^{k-1}_{\mathrm{dR}}(W_{k-2}) = H^1_{\mathrm{par}}(\mathrm{Sh}^{\mathrm{tor}}_K, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla))$. Hence, $\nu = k-1$ forces μ to be $\mu = |\ell| - 3$ and the group

$$\theta_\ell^* H^{|\ell|-3}_{\mathrm{dR}}(U_{\ell-4}) = \mathbb{H}^1(\mathrm{Sh}_K, \mathrm{DR}^\bullet(\mathcal{F}^{(\ell,\ell-t)}_{G_L^*})) \tag{2.27}$$

is trivial. Indeed, by ([Nek18], proof of A6.17 and A6.20), the cohomology group

$$\mathbb{H}^1(\mathrm{Sh}_K, \mathrm{DR}^\bullet(\mathcal{F}^{(\ell,\ell-t)}_{G_L^*}))$$

is identified with the intersection cohomology of the BB-compactification of $\mathrm{Sh}_K(G_L^*)$, that in turn is trivial in degree 1 by computations using Lie algebra cohomology ([Nek18], Sections 5.11, 6.3, 6.4). $\square$

2.7.3 Evaluation of syntomic Abel-Jacobi

We are interested in computing $\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega \cup \pi_2^*\eta)$ and to relate it to some value of the twisted triple product p-adic L-function outside the range of interpolation. Let $\tilde{\omega}$ (resp. $\tilde{\eta}$) be a lift of ω (resp. η) to fp-cohomology; since the Hirzebruch-Zagier cycle is null-homologous the computation is independent of the choice of lifts. We start by treating the case $(\ell, k) = (2t_L, 2)$:

$$\begin{aligned}
\mathrm{AJ}_p(\Delta_{2t_L,2})(\pi_1^*\omega \cup \pi_2^*\eta) &= \langle \mathrm{cl}_{\mathrm{syn}}(\Delta_{2t_L,2}), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \langle \tilde{P}_*\mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_0]), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_0]), \textstyle\sum_{i,j}(\tilde{P}_{i,j} - \tilde{P}_{a_i,j} - \tilde{P}_{i,b_j} + \tilde{P}_{a_i,b_j})^*(\pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta})\rangle_{\mathrm{fp}} \\
&= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_0]), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \mathrm{tr}_{\mathscr{W}_0}(\tilde{\varphi}^*(\pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta})) = \mathrm{tr}_{\mathscr{W}_0}(\tilde{\zeta}^*\tilde{\omega} \cup \tilde{\eta}).
\end{aligned}$$

The fourth equality is justified by the vanishing of the groups

$$H^1_{\mathrm{fp}}(\mathrm{Spec}(\mathcal{O}_{E,\wp}), 0) \quad \text{and} \quad H^2_{\mathrm{fp}}(\mathrm{Spec}(\mathcal{O}_{E,\wp}), 2),$$

which implies that $\sum_{i,j}\tilde{P}_{i,j}^* = (\mathrm{id}_{\mathscr{U}_0\times\mathscr{W}_0})^*$ and that all the other pullbacks are zero.

To deal with the general case, we first need to analyze the action of the correspondences $\tilde{\theta}_k, \tilde{\theta}_\ell$ on fp-cohomology. The exact sequence in ([Bes00] (8)) induces a functorial isomorphism $H^{k-1}_{\mathrm{fp}}(\mathscr{W}_{k-2}, 0) \cong H^{k-1}_{\mathrm{dR}}(W_{k-2})$, we denote by $\tilde{\eta}$ the preimage of $\eta \in \theta_k^* H^{k-1}_{\mathrm{dR}}(W_{k-2})$ that satisfies $\tilde{\theta}_k^*\tilde{\eta} = \tilde{\eta}$ since $\theta_k^*\eta = \eta$. By functoriality of the short exact sequence ([Bes00] (8)), there is a commuting diagram

$$\begin{tikzcd}
H^{|\ell|-3}_{\mathrm{dR}}(U_{\ell-4})/F^{|\ell|-2-s} \arrow[r, hook, "\iota"] \arrow[d, "\theta_\ell^*=0"] & H^{|\ell|-2}_{\mathrm{fp}}(\mathscr{U}_{\ell-4}, |\ell|-2-s) \arrow[r, two heads, "\pi"] \arrow[d, "\tilde{\theta}_\ell^*"] & \mathrm{F}^{|\ell|-2-s}H^{|\ell|-2}_{\mathrm{dR}}(U_{\ell-4}) \arrow[d, "\theta_\ell^*"] \arrow[dl, dotted] \\
H^{|\ell|-3}_{\mathrm{dR}}(U_{\ell-4})/F^{|\ell|-2-s} \arrow[r, hook, "\iota"] & H^{|\ell|-2}_{\mathrm{fp}}(\mathscr{U}_{\ell-4}, |\ell|-2-s) \arrow[r, two heads, "\pi"] & \mathrm{F}^{|\ell|-2-s}H^{|\ell|-2}_{\mathrm{dR}}(U_{\ell-4}),
\end{tikzcd}$$

where the leftmost vertical arrow is zero because of the vanishing (2.27). Therefore, there is a canonical lift $\tilde{\omega} = \tilde{\theta}_\ell^*\omega$ to $H^{|\ell|-2}_{\mathrm{fp}}(\mathscr{U}_{\ell-4}, |\ell|-2-s)$ of any class $\omega \in \theta_\ell^*\mathrm{F}^{|\ell|-2-s}H^{|\ell|-2}_{\mathrm{dR}}(U_{\ell-4})$, with the property $\tilde{\theta}_\ell^*\tilde{\omega} = \tilde{\omega}$. At this point we can compute

$$\begin{aligned}
\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega \cup \pi_2^*\eta) &= \langle \mathrm{cl}_{\mathrm{syn}}(\Delta_{\ell,k}), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \langle (\tilde{\theta}_\ell, \tilde{\theta}_k)_*\mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_\gamma]), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_\gamma]), \pi_1^*\tilde{\theta}_\ell^*\tilde{\omega} \cup \pi_2^*\tilde{\theta}_k^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{W}_\gamma]), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\
&= \mathrm{tr}_{\mathscr{W}_\gamma}(\tilde{\varphi}^*(\pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta})) = \mathrm{tr}_{\mathscr{W}_\gamma}(\tilde{\varphi}_1^*\tilde{\omega} \cup \tilde{\varphi}_2^*\tilde{\eta}),
\end{aligned}$$

where $\tilde{\varphi}_i = (\pi_i \circ \tilde{\varphi})$. The fundamental exact sequence of fp-cohomology induces an isomorphism $\iota : H^{|\ell|-3}_{\mathrm{dR}}(W_\gamma) \xrightarrow{\sim} H^{|\ell|-2}_{\mathrm{fp}}(\mathscr{W}_\gamma, |\ell|-2-s)$ since the filtered piece $\mathrm{F}^n H^j_{\mathrm{dR}}(W_\gamma)$ is trivial for $n > \dim_{E_\wp} W_\gamma$ and indeed $|\ell|-2-s > \dim_{E_\wp} W_\gamma = \gamma+1$. Therefore, if we write $\tilde{\varphi}_1^*\tilde{\omega} = \iota\Upsilon(\omega)$, we can rewrite the quantity we want to evaluate as

$$\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega \cup \pi_2^*\eta) = \mathrm{tr}_{W_\gamma}(\Upsilon(\omega) \cup_{\mathrm{dR}} \varphi_2^*\eta) = \langle \Upsilon(\omega), \varphi_2^*\eta\rangle_{\mathrm{dR}}, \tag{2.28}$$

for the Poincaré pairing $\langle\ ,\ \rangle_{\mathrm{dR}} : H^{|\ell|-3}_{\mathrm{dR}}(W_\gamma) \times H^{k-1}_{\mathrm{dR}}(W_\gamma) \xrightarrow{\cup} H^{|\ell|+k-4}_{\mathrm{dR}}(W_\gamma) \xrightarrow{\mathrm{tr}_{\mathrm{dR}}} E_\wp$.

2.7.4 Description of $\mathbf{AJ}_p(\Delta_{\ell,k})$ in terms of p-adic modular forms

Let $\mathscr{Y}_{K'} \hookrightarrow \mathbf{Sh}_{K'}(\mathrm{GL}_{2,\mathbb{Q}})_{\mathcal{O}_{E_\wp}}$ be the $\mathcal{O}_{E_\wp}$-scheme defined as the complement of the supersingular points and let $\mathscr{E} \to \mathscr{Y}_{K'}$ be the universal elliptic curve over it.

Proposition 2.7.11. *There are natural inclusions of parabolic cohomology in the de-Rham cohomology of proper and smooth compactifications of Kuga-Sato varieties*

$$H^1_{par}\big(Sh^{tor}_{K',E_\wp}, (\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big) \hookrightarrow H^{|\ell|-3}_{dR}(W_\gamma),$$

$$H^1_{par}\big(Sh^{tor}_{K',E_\wp}, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big) \hookrightarrow H^{k-1}_{dR}(W_\gamma)$$

compatible with Poincaré duality.

Proof. Let $\mathscr{D}_{\gamma,\kappa}$ be the inverse image of cusps and supersingular points under $\mathscr{W}_{\gamma,\kappa} \to \mathscr{W}_{0,\kappa}$; then $\mathscr{D}_{\gamma,\kappa} = \mathscr{W}_{\gamma,\kappa} \setminus \mathscr{E}^\gamma_\kappa$ and it is a smooth and projective subscheme of codimension 1 in $\mathscr{W}_{\gamma,\kappa}$. Consider the diagram

$$\begin{tikzcd}
H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{W}_{\gamma,\kappa}) \arrow[r] & H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{E}^\gamma_\kappa) \arrow[r] & H^{|\ell|-4}_{\mathrm{rig}}(\mathscr{D}_{\gamma,\kappa})(-1) \\
H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp}, (\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big) \arrow[u, hook, dotted] \arrow[r, hook] & \mathbb{H}^1\big(\mathscr{S}^{\mathrm{tor}}_{K',\mathrm{rig}}, j^\dagger \mathrm{DR}^\bullet(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}})\big) \arrow[u, hook] &
\end{tikzcd}$$

where the top horizontal arrow is exact and comes from excision. The composition

$$H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp}, (\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big) \longrightarrow H^{|\ell|-4}_{\mathrm{rig}}(\mathscr{D}_{r,k})(-1)$$

is identically zero because the two cohomology groups are pure of different weights. Thus, $H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp}, (\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big) \hookrightarrow H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{W}_{\gamma,\kappa}) \cong H^{|\ell|-3}_{\mathrm{dR}}(W_\gamma)$. A similar argument provides the other inclusion $H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp}, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big) \hookrightarrow H^{k-1}_{\mathrm{rig}}(\mathscr{W}_{\gamma,\kappa}) \cong H^{k-1}_{\mathrm{dR}}(W_\gamma)$. ☐

It is clear that $\varphi_2^*\eta \in H^{k-1}_{\mathrm{dR}}(W_\gamma)$ is equal to $\eta \in H^1_{\mathrm{par}}\big(\mathrm{Sh}^{\mathrm{tor}}_{K',E_\wp}, (\mathcal{F}^{(k,k-1)}_{\mathrm{GL}_{2,\mathbb{Q}}}, \nabla)\big) \hookrightarrow H^{k-1}_{\mathrm{dR}}(W_\gamma)$, so our task is to describe $\Upsilon(\omega) \in H^{|\ell|-3}_{\mathrm{dR}}(W_\gamma)$ using p-adic modular forms.

Let $\mathscr{X}_K \hookrightarrow \mathbf{Sh}_K(G^*_L)_{\mathcal{O}_{E_\wp}}$ be the $\mathcal{O}_{E_\wp}$-scheme defined as the complement of the supersingular locus and $\zeta : \mathscr{Y}_{K'} \longrightarrow \mathscr{X}_K$ the diagonal morphism. Let $\mathscr{A} \to \mathscr{X}_K$ be the universal abelian surface, then we have a commuting diagram

$$\begin{tikzcd}
\mathscr{E}^\gamma \arrow[r, "\tilde{\varphi}_1"] \arrow[d, "v"] & \mathscr{A}^{|\ell|-4} \arrow[d, "v"] \\
\mathscr{W}_\gamma \arrow[r, "\tilde{\varphi}_1"] & \mathscr{U}_{\ell-4}
\end{tikzcd}$$

that induces

$$\begin{tikzcd} \tilde{\theta}_\ell^* \tilde{H}_{\mathrm{f},Q}^{|\ell|-2}(\mathscr{U}_{\ell-4}, |\ell|-2-s) \arrow[r, "v^*"] \arrow[d, "\tilde{\varphi}_1^*"] & \tilde{\theta}_\ell^* \tilde{H}_{\mathrm{f},Q}^{|\ell|-2}(\mathscr{A}^{|\ell|-4}, |\ell|-2-s) \arrow[d, "\tilde{\varphi}_1^*"] \\ \tilde{H}_{\mathrm{f},Q}^{|\ell|-2}(\mathscr{W}_\gamma, |\ell|-2-s) \arrow[r, "v^*"] & \tilde{H}_{\mathrm{f},Q}^{|\ell|-2}(\mathscr{E}^\gamma, |\ell|-2-s), \end{tikzcd}$$

where we consider the Gros-style version of fp-cohomology ([Bes00], Section 9) for a suitable choice of polynomial Q. We choose to work with the Gros-style version because for schemes that can be embedded in a smooth and proper scheme it is defined using rigid complexes in place of de Rham ones; in particular, the two versions coincide for proper and smooth schemes.

The pull back $v^*\tilde{\omega} \in \tilde{\theta}_\ell^* \tilde{H}_{\mathrm{f},Q}^{|\ell|-2}(\mathscr{A}^{|\ell|-4}, |\ell|-2-s)$ can be directly described in terms of p-adic modular forms. Indeed, we can write $v^*\tilde{\omega} = [\omega, f]$ for

$$\omega \in H^0\big(\mathscr{S}_{K,\mathrm{rig}}^{\mathrm{tor}}, j^\dagger(\mathcal{F}_{G_L^*}^{(\ell,\ell-t_L)} \otimes \Omega^2(\log \mathrm{D}))\big)$$

and

$$f \in H^0\big(\mathscr{S}_{K,\mathrm{rig}}^{\mathrm{tor}}, j^\dagger(\mathcal{F}_{G_L^*}^{(\ell,\ell-t_L)} \otimes \Omega^1(\log \mathrm{D}))\big)$$

satisfying $Q(\mathrm{Fr}_p)\omega = \nabla f$, as the group $\tilde{\theta}_\ell^* H_{\mathrm{rig}}^{\ell-i}(\mathscr{A}_k^{|\ell|-4}/E_\wp)$ is the same as the cohomology of the rigid realization of the motive $\mathcal{V}^\ell$ over $]\mathbf{Sh}_K(G_L^*)_k^{\mathrm{ord}}[$, that is, the rigid cohomology $\mathbb{H}^i\big(\mathscr{S}_{K,\mathrm{rig}}^{\mathrm{tor}}; j^\dagger \mathrm{DR}^\bullet(\mathcal{F}_{G_L^*}^{(\ell,\ell-t_L)})\big)$, for $i = 1, 2$.

To express the class $v^*\tilde{\omega}$ explicitly we need to make a judicious choice of a polynomial. From now on we assume that p splits in $L/\mathbb{Q}$, $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$. By observing the form of the Euler factors appearing in Theorem 2.4.10 and the formulas in Corollary 2.6.7 we are led to consider the polynomial $P(T) = \prod_{\bullet,\star\in\{\alpha,\beta\}}(1 - \bullet_1 \star_2 T)$. Following ([LSZ16], Proposition 4.5.5), if we set $T = T_1T_2$, we can write $P(T_1, T_2) = a_2(T_1, T_2)P_1(T_1) + b_1(T_1, T_2)P_2(T_2)$ for $P_i(T_i) = (1 - \alpha_i T_i)(1 - \beta_i T_i)$ and

$$a_2(T_1, T_2) = \alpha_1\beta_1\alpha_2\beta_2(\alpha_2 + \beta_2)T_1^2T_2^3 - \alpha_1\beta_1\alpha_2\beta_2T_1^2T_2^2 - \alpha_2\beta_2(\alpha_1 + \beta_1)T_1T_2^2 + 1,$$

$$b_1(T_1, T_2) = \alpha_1^2\beta_1^2\alpha_2\beta_2T_1^4T_2^2 - \alpha_1\beta_1(\alpha_2 + \beta_2)T_1^2T_2 - \alpha_1\beta_1T_1^2 + (\alpha_1 + \beta_1)T_1.$$

The index 2 in a_2 (resp. the index 1 in b_1) is there to remind us that the monomials composing the polynomial are of the form $T_1^{a_1}T_2^{a_2}$ with $a_1 \le a_2$ (resp. $T_1^{b_1}T_2^{b_2}$ with $b_1 > b_2$). The polynomial $P(T_1, T_2)$ is symmetric in the indices $1, 2$, hence we can also write

$$P(T_1, T_2) = a_1(T_1, T_2)P_2(T_1) + b_2(T_1, T_2)P_1(T_2)$$

where $a_1(T_1, T_2)$ (resp. $b_2(T_1, T_2)$) is obtained from $a_2(T_1, T_2)$ (resp. $b_1(T_1, T_2)$) by swapping all the indices. Therefore,

$$\begin{aligned} P(T_1, T_2)^2 &= a_1a_2P_1P_2 + a_2P_1b_2P_1 + a_1P_2b_1P_2 + b_1b_2P_1P_2 \\ &= a_1a_2P_1P_2 + (P - b_1P_2)b_2P_1 + (P - b_2P_1)b_1P_2 + b_1b_2P_1P_2 \\ &= (a_1a_2 - b_1b_2)P_1P_2 + P(b_2P_1 + b_1P_2) \\ &= P(1 - \alpha_1\beta_1\alpha_2\beta_2T^2)P_1P_2 + P(b_2P_1 + b_1P_2). \end{aligned}$$

We are going to use the handy identity

$$P(T_1, T_2) = (1 - \alpha_1\beta_1\alpha_2\beta_2 T^2)P_1(T_1)P_2(T_2) + (b_2(T_1, T_2)P_1(T_1) + b_1(T_1, T_2)P_2(T_2)).$$

The class of $\omega_{\breve{g}^{[\mathfrak{p}_i]}}$ is zero in $\mathbb{H}^2(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}, j^\dagger \mathrm{DR}^\bullet_c(\mathcal{F}^{(\ell,x)}))$, hence there are overconvergent cuspforms $\mathsf{g}^{(i)}_j \in S^\dagger_{s_{\tau_j}\cdot(\ell,x)}(K; E_\wp)$ such that $\breve{\mathsf{g}}^{[\mathfrak{p}_i]} = d_1^{\ell_1-1}(\mathsf{g}^{(i)}_1) + d_2^{\ell_2-1}(\mathsf{g}^{(i)}_2)$. Furthermore, the p-adic modular form $d_1^{1-\ell_1}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}$ overconverges by Corollary 2.6.7. It follows we can write $P(V(p))\breve{\mathsf{g}}$ as

$$\begin{aligned} P(V(p))\breve{\mathsf{g}} &= (1 - \alpha_1\beta_1\alpha_2\beta_2 V(p)^2)\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]} + b_2(V(\mathfrak{p}_1), V(\mathfrak{p}_2))\breve{\mathsf{g}}^{[\mathfrak{p}_1]} + b_1(V(\mathfrak{p}_1), V(\mathfrak{p}_2))\breve{\mathsf{g}}^{[\mathfrak{p}_2]} \\ &= d_1^{\ell_1-1}(\mathsf{h}) + d_1^{\ell_1-1}(\mathsf{h}_1) + d_2^{\ell_2-1}(\mathsf{h}_2), \end{aligned}$$

where $\mathsf{h} = (1 - \alpha_1\beta_1\alpha_2\beta_2 V(p)^2)d_1^{1-\ell_1}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}$, $\mathsf{h}_1 = b_2\mathsf{g}^{(1)}_1 + b_1\mathsf{g}^{(2)}_1$ and $\mathsf{h}_2 = b_2\mathsf{g}^{(1)}_2 + b_1\mathsf{g}^{(2)}_2$.

Proposition 2.7.12. *Let $L/\mathbb{Q}$ be a real quadratic extension and $g \in S^\dagger_{\ell,x}(K, L; E_\wp)$ an overconvergent cuspform whose class ω_g in $\mathbb{H}^2(\mathscr{S}^{tor}_{K,rig}(G_L), j^\dagger DR^\bullet_c(\mathcal{F}^{(\ell,x)}))$ is trivial. By Theorem 2.6.4 there are p-adic modular forms $g_j \in S^\dagger_{s_j\cdot(\ell,x)}(K; E_\wp)$ for $j = 1, 2$, such that $g = d_1^{\ell_1-1}(g_1) + d_2^{\ell_2-1}(g_2)$. We can use them to explicitly construct sections*

$$G_j \in H^0(\mathscr{S}^{tor}_{K,rig}(G_L), j^\dagger(\mathcal{F}^{(\ell,x)} \otimes \Omega^1_{\tau_j})), \quad j = 1, 2,$$

that satisfy

$$\omega_g = \nabla(G_1 + G_2)$$

in $H^0(\mathscr{S}^{tor}_{K,rig}(G_L), j^\dagger(\mathcal{F}^{(\ell,x)} \otimes \Omega^2))$.

Proof. For $j = 1, 2$, let ω_j, η_j be a local basis of the τ_j-part of the first de Rham cohomology of the universal abelian surface. Set $v_j^{(a,b)} = \omega_j^a\eta_j^b$, $w_j = \omega_j \wedge \eta_j$ and consider the sections

$$G_1 = \sum_{i=0}^{\ell_1-2}(-1)^i\frac{(\ell_1-2)!}{(\ell_1-2-i)!}d_1^{\ell_1-2-i}(g_1)\big(w_2^{\frac{2-n-\ell_2}{2}} \otimes v_2^{(\ell_2-2,0)} \otimes w_1^{\frac{2-n-\ell_1}{2}} \otimes v_1^{(\ell_1-2-i,i)}\big) \otimes \frac{dq_2}{q_2}.$$

$$G_2 = -\sum_{i=0}^{\ell_2-2}(-1)^i\frac{(\ell_2-2)!}{(\ell_2-2-i)!}d_1^{\ell_2-2-i}(g_2)\big(w_1^{\frac{2-n-\ell_1}{2}} \otimes v_1^{(\ell_1-2,0)} \otimes w_2^{\frac{2-n-\ell_2}{2}} \otimes v_2^{(\ell_2-2-i,i)}\big) \otimes \frac{dq_1}{q_1}.$$

of $H^0(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}(G_L), j^\dagger(\mathcal{F}^{(\ell,x)} \otimes \Omega^1))$. Differentiating them we obtain telescopic sums which collapse to

$$\nabla(G_j) = d_j^{\ell_j-1}(g_j)\bigotimes_{c=1}^{2}\big(w_c^{\frac{2-n-\ell_c}{2}} \otimes v_c^{(\ell_c-2,0)}\big) \otimes (\frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2}).$$

Therefore, $\omega_g = \nabla(G_1) + \nabla(G_2)$ as claimed. □

It follows that there are sections $G_{\mathsf{h}}, G_{\mathsf{h}_1}, G_{\mathsf{h}_2}$ associated with h, h_1, h_2 respectively, that satisfy $P(p^{-t_L}\mathrm{Fr}_p)\omega_{\mathsf{g}} = \nabla(G_{\mathsf{h}} + G_{\mathsf{h}_1} + G_{\mathsf{h}_2})$ since $\mathrm{Fr}_p = p^{t_L}V(p)$ in cohomology (Lemma 2.6.6). The pullback by the morphism $\xi : \mathrm{Sh}_K(G^*_L) \to \mathrm{Sh}_K(G_L)$ gives $P(p^{-t_L}\mathrm{Fr}_p)\omega_{\xi^*\mathsf{g}} = \nabla(G_{\xi^*\mathsf{h}} + G_{\xi^*\mathsf{h}_1} + G_{\xi^*\mathsf{h}_2})$ and to land in the right cohomology group we need to change the central character using the isomophism $\Psi = \Psi_{x,|\ell-t_L|}$. Lemma 2.1.7 implies

$$P(p^{x-\ell}\mathrm{Fr}_p)\omega_{\Psi\xi^*\mathsf{g}} = \nabla(G_{\Psi\xi^*\mathsf{h}} + G_{\Psi\xi^*\mathsf{h}_1} + G_{\Psi\xi^*\mathsf{h}_2}).$$

We set $G = G_{\Psi\zeta^*\mathsf{h}} + G_{\Psi\zeta^*\mathsf{h}_1} + G_{\Psi\zeta^*\mathsf{h}_2}$ and we let $\varepsilon_\ell : \bigotimes_\tau(\mathbb{H}^1_\tau)^{\ell_\tau-2} \to \bigotimes_\tau \mathrm{Sym}^{\ell_\tau-2}\mathbb{H}^1_\tau$ be the symmetrization projector which identifies the target sheaf with a subsheaf of the first. Finally, if we set $Q(T) = P(p^{x-\ell}T)$, then the cohomology class $v^*\tilde{\omega}$ is represented by $[\omega, \varepsilon_\ell G]$ in $\tilde{H}^{|\ell|-2}_{\mathrm{f},Q}\big(\mathscr{A}^{|\ell|-4}, |\ell|-2-s\big)$.

Proposition 2.7.13. *The class $\nu^*(\tilde{\varphi}_1^*\tilde{\omega})$ is represented by $[0, \tilde{\varphi}_1^*\varepsilon_\ell G]$ in $\tilde{H}^{|\ell|-2}_{f,Q}\big(\mathscr{E}^\gamma, |\ell|-2-s\big)$ and the image of $\tilde{\varphi}_1^*\varepsilon_\ell G$ under the unit-root splitting is equal to the p-adic modular form*

$$Spl_{ur}(\tilde{\varphi}_1^*\varepsilon_\ell G) = (-1)^s s!\Psi_{w,k-1+s}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h}) + d_1^{\ell_1-2-s}(\mathsf{h}_1) + d_2^{\ell_2-2-s}(\mathsf{h}_2)\big)$$

in $S^{p\text{-adic}}_{k,k-1+s}(K', E_\wp)$.

Proof. The class $\nu^*(\tilde{\varphi}_1^*\tilde{\omega}) = \tilde{\varphi}_1^* v^*(\tilde{\omega}) = [0, \tilde{\varphi}_1^*\varepsilon_\ell G]$ because $\varphi_1^*\omega = 0$ as a section of

$$\varphi_1^*\big(\mathcal{F}^{(\ell,\ell-t_L)}_{G_L^*} \otimes \Omega^2\big) = 0.$$

The diagonal morphism $\tilde{\varphi}_1 : \mathscr{E}^\gamma \to \mathscr{A}^{|\ell|-4}$ is a map of $\mathscr{X}_K$-schemes, so the pull-back $\tilde{\varphi}_1^* : H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{A}^{|\ell|-4}_\kappa) \to H^{|\ell|-3}_{\mathrm{rig}}(\mathscr{E}^\gamma_\kappa)$ is compatible with the pull-backs between the terms of the Leray spectral sequences for $\mathscr{A}^{|\ell|-4} \to \mathscr{X}_K \to \mathrm{Spec}\mathcal{O}_{E_\wp}$ and $\mathscr{E}^\gamma \to \mathscr{X}_K \to \mathrm{Spec}\mathcal{O}_{E_\wp}$. Since $\zeta : \mathscr{Y}_{K'} \to \mathscr{X}_K$ is a finite morphism, we have an induced map

$$\tilde{\varphi}_1^* : \mathbb{H}^1\big(\mathscr{S}^{\mathrm{tor}}_{K,\mathrm{rig}}; j^\dagger\mathrm{DR}^\bullet(\mathcal{F}^{(\ell,\ell-t_L)}_{G_L^*})\big) \longrightarrow \mathbb{H}^1\big(\mathscr{S}^{\mathrm{tor}}_{K',\mathrm{rig}}; j^\dagger\mathrm{DR}^\bullet(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}})\big).$$

It is possible to describe explicitly the pullback $\tilde{\varphi}_1^*\varepsilon_\ell G$ as in ([DR14], Proposition 2.9) and a direct calculation reveals that

$$\begin{aligned}\mathrm{Spl}_{\mathrm{ur}}\zeta^*(\varepsilon_{\underline{\ell}} G_j) &= (-1)^s s!\zeta^*\Psi_{x,|\ell-t_L|}\big(d_1^{\ell_1-2-s}(\zeta^*\mathsf{h}) + d_1^{\ell_1-2-s}(\zeta^*\mathsf{h}_1) + d_2^{\ell_2-2-s}(\zeta^*\mathsf{h}_2)\big)\\ &= (-1)^s s!\Psi_{x-s-1,k-1+s}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h}) + d_1^{\ell_1-2-s}(\mathsf{h}_1) + d_2^{\ell_2-2-s}(\mathsf{h}_2)\big)\\ &= (-1)^s s!\Psi_{w,k-1+s}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h}) + d_1^{\ell_1-2-s}(\mathsf{h}_1) + d_2^{\ell_2-2-s}(\mathsf{h}_2)\big),\end{aligned}$$

as p-adic modular forms. □

Remark 2.7.14. *We proved that the image of $\Upsilon(\omega)$ under $H^{|\ell|-3}_{rig}(\mathscr{W}_{\gamma,\kappa}) \to H^{|\ell|-3}_{rig}(\mathscr{E}^\gamma_\kappa)$ is given by $[\tilde{\varphi}_1^*\varepsilon_\ell G] \in H^1_{par}\big(Sh^{tor}_{K',E_\wp}, (\mathcal{F}^{(k,k-1+s)}_{GL_{2,\mathbb{Q}}}, \nabla)\big) \subset \mathbb{H}^1\big(\mathscr{S}^{tor}_{K',rig}, j^\dagger DR^\bullet_c(\mathcal{F}^{(k,k-1+s)}_{GL_{2,\mathbb{Q}}})\big)$.*

Lemma 2.7.15. *Let $(\omega, \eta) \in H^1_{par}\big(Sh^{tor}_{K',E_\wp}, (\mathcal{F}^{(k,k-1+s)}_{GL_{2,\mathbb{Q}}}, \nabla)\big) \times H^1_{par}\big(Sh^{tor}_{K',E_\wp}, (\mathcal{F}^{(k,k-1)}_{GL_{2,\mathbb{Q}}}, \nabla)\big)$ be a pair such that $Fr_p\eta = \alpha\eta$ for α a p-adic unit, then $\langle\omega, \eta\rangle = \langle e_{n.o.}\omega, \eta\rangle$.*

Proof. We have the equalities of operators $\mathrm{Fr}_p = pV(p)$ and $U_0(p) = p^r U(p)$, therefore the computation

$$\begin{aligned}\langle\omega, \eta\rangle &= \alpha^{-1}\langle\omega, \mathrm{Fr}_p\eta\rangle = \alpha^{-1}\mathrm{Fr}_p\langle\mathrm{Fr}_p^{-1}\omega, \eta\rangle\\ &= \alpha^{-1}p^{r+1}\langle\mathrm{Fr}_p^{-1}\omega, \eta\rangle = \alpha^{-1}p^{r+1}\langle p^{-1}U(p)\omega, \eta\rangle = \alpha^{-1}\langle U_0(p)\omega, \eta\rangle,\end{aligned}$$

implies that $\langle\omega, \eta\rangle = \lim_{n\to\infty}\alpha^{-n!}\langle U_0(p)^{n!}\omega, \eta\rangle = \langle e_{\mathrm{n.o.}}\omega, \eta\rangle$. □

Theorem 2.7.16. *Let $L/\mathbb{Q}$ be a real quadratic extension. Consider $\breve{\mathsf{g}} \in S_{\ell,x}(V_{11}(\mathfrak{A}\mathcal{O}_L); L; \overline{\mathbb{Q}})$ a cuspform of either parallel weight $\ell = 2t_L$ or non-parallel weight $\ell > 2t_L$ over L and $\mathsf{f} \in S_{k,w}(V_1(\mathfrak{N}); \overline{\mathbb{Q}})$ an elliptic newform. Suppose their weights are balanced and choose a prime p splitting in F, $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$, coprime to $\mathfrak{A}$, such that both cuspforms are p-nearly ordinary and the cycle*

$\Delta_{\ell,k}$ *is defined. Then*

$$AJ_p(\Delta_{\ell,k})(\pi_1^*\omega \cup \pi_2^*\eta) = s!(-1)^s \frac{1-\alpha_1\beta_1\alpha_2\beta_2(\alpha_{\mathsf{f}^*}^{-1}p^{-1})^2}{\prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\,\alpha_{\mathsf{f}^*}^{-1}p^{-1})} \frac{\langle e_{n.o.}\zeta^*(d_1^{-1-s}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}),\mathsf{f}^*\rangle}{\langle \mathsf{f}^*,\mathsf{f}^*\rangle},$$

where ω *and* η *are the classes in Definition* 2.7.5 *and* $s=\frac{|\ell|-k-2}{2}$.

Proof. Recall that (2.28) states that $\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega\cup\pi_2^*\eta)=\langle \Upsilon(\omega),\eta\rangle_{\mathrm{dR}}$, where the Poincaré pairing takes values in $E_\wp(-(\gamma+1))$, a one dimensional space on which Fr_p acts as multiplication by $p^{\gamma+1}$. The isomorphism $\iota: H^{|\ell|-3}_{\mathrm{dR}}(W_\gamma)\overset{\sim}{\to} H^{|\ell|-2}_{\mathrm{f},\mathbb{Q}}(\mathscr{W}_\gamma,|\ell|-2-s)$ is given by $\iota(-)=[0,Q(\mathrm{Fr}_p)(-)]$, therefore $Q(\mathrm{Fr}_p)\Upsilon(\omega)=\tilde{\varphi}_1^*(\epsilon_\ell G)$. On the one hand,

$$\langle Q(\mathrm{Fr}_p)\Upsilon(\omega),\eta\rangle_{\mathrm{dR}} = Q(p^{\gamma+1}\alpha_{\mathsf{f}^*}^{-1}p^{1-w})\langle\Upsilon(\omega),\eta\rangle_{\mathrm{dR}},$$

because we computed in (2.23) that $\mathrm{Fr}_p\big(\nu^*\varphi_2^*\eta\big)=\alpha_{\mathsf{f}^*}p^{w-1}\big(\nu^*\varphi_2^*\eta\big)$. On the other hand,

$$\begin{aligned}\langle Q(\mathrm{Fr}_p)\Upsilon(\omega),\eta\rangle_{\mathrm{dR}} &= \langle\tilde{\varphi}_1^*(\epsilon_\ell G),\eta\rangle_{\mathrm{dR}}\\ &= s!(-1)^s\frac{\langle\Psi_{w,k-1}e_{\mathrm{n.o.}}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h})+d_1^{\ell_1-2-s}(\mathsf{h}_1)+d_2^{\ell_2-2-s}(\mathsf{h}_2)\big),\Psi_{w,k-1}(\mathsf{f}^*)\rangle}{\langle\mathsf{f}^*,\mathsf{f}^*\rangle}\\ &= s!(-1)^s\frac{\langle e_{\mathrm{n.o.}}\zeta^*\big(d_1^{\ell_1-2-s}(\mathsf{h})+d_1^{\ell_1-2-s}(\mathsf{h}_1)+d_2^{\ell_2-2-s}(\mathsf{h}_2)\big),\mathsf{f}^*\rangle}{\langle\mathsf{f}^*,\mathsf{f}^*\rangle}.\end{aligned}$$

Indeed, the class of $\tilde{\varphi}_1^*(\epsilon_\ell G)$ in $\mathbb{H}^1\big(\mathscr{S}^{\mathrm{tor}}_{K',\mathrm{rig}},j^\dagger\mathrm{DR}_c^\bullet(\mathcal{F}^{(k,k-1+s)}_{\mathrm{GL}_{2,\mathbb{Q}}})\big)$ is represented by an overconvergent cuspform whose nearly ordinary projection is equal to $e_{\mathrm{n.o.}}\mathrm{Spl}_{\mathrm{ur}}\tilde{\varphi}_1^*(\epsilon_\ell G)$ (see [DR14], Lemma 2.7), then Lemma 2.7.15 justifies the computation.

For $j=1,2$ the nearly ordinary projection

$$e_{\mathrm{n.o.}}\zeta^*d_j^{\ell_j-2-s}(\mathsf{h}_j)=e_{\mathrm{n.o.}}\zeta^*[d_j^{\ell_j-2-s}(b_2\mathsf{g}_j^{(1)}+b_1\mathsf{g}_j^{(2)})]=0$$

thanks to Lemma 2.4.7 because the cuspform $\mathsf{g}_j^{(i)}$ is $\mathfrak{p}_i$-depleted, $i=1,2$ and b_ι, $\iota=1,2$, can be written as a polynomial only in the variables $V(p),V(\mathfrak{p}_\iota)$ divisible by $V(\mathfrak{p}_\iota)$. Moreover,

$$\begin{aligned}e_{\mathsf{f}^*,\mathrm{n.o.}}\zeta^*(d_1^{\ell_1-2-s}(\mathsf{h})) &= e_{\mathsf{f}^*,\mathrm{n.o.}}\zeta^*(1-\alpha_1\beta_2\alpha_2\beta_2V(p)^2)(d_1^{-1-s}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]})\\ &= (1-\alpha_1\beta_1\alpha_2\beta_2(\alpha_{\mathsf{f}^*}^{-1}p^{-1})^2)e_{\mathsf{f}^*,\mathrm{n.o.}}\zeta^*(d_1^{-1-s}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}).\end{aligned}$$

Finally, the last bit we need to unravel is the polynomial $Q(p^{\gamma+1}\alpha_{\mathsf{f}^*}^{-1}p^{1-w})$; we compute

$$\begin{aligned}Q(p^{\gamma+1}\alpha_{\mathsf{f}^*}^{-1}p^{1-w}) &= \prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\,p^{x-\ell}p^{\gamma+1}\alpha_{\mathsf{f}^*}^{-1}p^{1-w})\\ &= \prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\,\alpha_{\mathsf{f}^*}^{-1}p^{-n+\frac{m}{2}-1}) = \prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\,\alpha_{\mathsf{f}^*}^{-1}p^{-1})\end{aligned}$$

since under our assumptions $2n=m$. Hence, putting all together

$$\mathrm{AJ}_p(\Delta_{\ell,k})(\pi_1^*\omega \cup \pi_2^*\eta) = s!(-1)^s \frac{1-\alpha_1\beta_1\alpha_2\beta_2(\alpha_{\mathsf{f}^*}^{-1}p^{-1})^2}{\prod_{\bullet,\star\in\{\alpha,\beta\}}(1-\bullet_1\star_2\,\alpha_{\mathsf{f}^*}^{-1}p^{-1})} \frac{\langle e_{\mathrm{n.o.}}\zeta^*(d_1^{-1-s}\breve{\mathsf{g}}^{[\mathfrak{p}_1,\mathfrak{p}_2]}),\mathsf{f}^*\rangle}{\langle \mathsf{f}^*,\mathsf{f}^*\rangle}.$$

□

Remark 2.7.17. *The right-hand side of the equality in Theorem* 2.7.16 *is independent of the particular* *choice of small enough levels K, K' because of the normalization of the cohomology class η (Definition* 2.7.5*).*

Corollary 2.7.18. *Let $L/\mathbb{Q}$ be a real quadratic field and (ℓ, k) a balanced triple. Let p be a prime splitting in L for which the generalized Hirzebruch-Zagier cycle $\Delta_{\ell,k}$ is defined. Then for all $(P, Q) \in \mathcal{C}^{\theta,\bar{r}}_{\mathrm{bal}}(\ell, k)$ we have*

$$_{\bar{r}}\mathscr{L}^{\theta}_{p}(\breve{\mathscr{G}}, \mathscr{F})(P, Q) = \frac{\pm 1}{s!\mathsf{E}(\mathsf{f}^*_Q)} \frac{\mathcal{E}^{\mathrm{sp}}_p(\mathsf{g}_P, \mathsf{f}^*_Q)}{\mathcal{E}_{1,p}(\mathsf{f}^*_Q)} AJ_p(\Delta_{\ell,k})(\pi_1^*\omega_P \cup \pi_2^*\eta_Q).$$

Proof. It follows from the formula (2.16), Proposition 2.4.3 and Theorem 2.7.16. □

2.8 An application to Bloch-Kato Selmer groups

Let A be an elliptic curve over L of conductor $\mathfrak{Q}$ and B a rational elliptic curve of conductor $\mathfrak{N}$, both without complex multiplication over $\overline{\mathbb{Q}}$. We denote by $(\mathrm{M}_{A,B})_p$ the Galois representation $\mathrm{AsV}_p(A)(-1) \otimes_{\mathbb{Q}_p} \mathrm{V}_p(B)$ of the absolute Galois group of $\mathbb{Q}$. We can use Corollary 2.7.18 to give a criterion for the Bloch-Kato Selmer group $H^1_f(\mathbb{Q}, (\mathrm{M}_{A,B})_p)$ to be of dimension one in terms of the non-vanishing of a value of one of our twisted triple product p-adic L-functions. This builds on the recent work of Liu [Liu16], where he computes the dimension of $H^1_f(\mathbb{Q}, (\mathrm{M}_{A,B})_p)$ assuming the non-vanishing of the étale Abel-Jacobi map of certain cycle $\Delta_{A,B}$.

Let $\mathsf{g}_A \in S_{2t_L,t_L}(V_1(\mathfrak{Q}); L; \mathbb{Q})$, $\mathsf{f}_B \in S_{2,1}(V_1(\mathfrak{N}); \mathbb{Q})$ be the newforms attached to A and B by modularity, π_A, σ_B the automorphic representations they respectively generate. Let p a rational prime coprime to $\mathfrak{N} \cdot \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q}) \cdot d_{L/F}$, if $\mathsf{g}_A, \mathsf{f}_B$ are p-nearly ordinary we denote by $\mathscr{G}, \mathscr{F}$ the Hida families passing through the p-nearly ordinary stabilizations $\mathscr{G}_{\mathrm{P}_A} = \mathsf{g}_A^{(p)}$ and $\mathscr{F}_{\mathrm{Q}_B} = \mathsf{f}_B^{(p)}$. We recall some of the definitions in [Liu16]. Let X be the minimal resolution of the Baily-Borel compactification of the Hilbert modular surface over L of Γ_0-level $\mathfrak{N} \cdot \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})$, Y the compactified modular curve of Γ_0-level $\mathfrak{N} \cdot \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})$ and $\zeta : Y \to X$ the diagonal morphism. According to Liu, there are idempotents $\mathscr{P}_A \in \mathrm{Corr}(X, X)$, $\mathscr{P}_B \in \mathrm{Corr}(Y, Y)$ acting as projectors

$$\mathscr{P}_{A,*} : H^*_{\mathrm{dR}}(X) \to H^2_{\mathrm{dR}}(X)[\pi_A], \qquad \mathscr{P}_{B,*} : H^*_{\mathrm{dR}}(Y) \to H^1_{\mathrm{dR}}(Y)[\sigma_B].$$

The null-homologous cycle $\Delta_{A,B} \in \mathrm{CH}^2(X \times Y) \otimes \mathbb{Q}$ is defined as $\Delta_{A,B} = (\mathscr{P}_A \times \mathscr{P}_B)_*\Delta$ for $\Delta = \mathrm{graph}(\zeta)$. By spreading out we can consider smooth models $\mathscr{X}, \mathscr{Y}$ over $\mathbb{Z}_p$ for almost all p, and $\tilde{\mathscr{P}}_A \times \tilde{\mathscr{P}}_B \in \mathrm{Corr}(\mathscr{X} \times \mathscr{Y}, \mathscr{X} \times \mathscr{Y})$.

Corollary 2.8.1. *Suppose that $\mathfrak{N}$ and $N_{L/\mathbb{Q}}(\mathfrak{Q}) \cdot d_{L/\mathbb{Q}}$ are coprime ideals and that all the primes dividing $\mathfrak{N}$ split in L. For all but finitely many primes p that are split in L and such that $\mathsf{g}_A, \mathsf{f}_B$ are p-nearly ordinary we have*

$$_{\bar{r}}\mathscr{L}^{\theta}_{p}(\breve{\mathscr{G}}, \mathscr{F})(P_A, Q_B) \neq 0 \quad \Longrightarrow \quad \dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}, (M_{A,B})_p) = 1,$$

where $\theta = -\mu + \mu' \in \mathbb{Z}[\mathrm{I}_L]$, $\bar{r} = -\mu$.

Proof. Let $\tilde{\varphi} : \mathscr{Y} \to \mathscr{X} \times \mathscr{Y}$ be the map $(\tilde{\zeta}, \mathrm{id}_{\mathscr{Y}})$, and set $\tilde{\Delta}_{A,B} = (\tilde{\mathscr{P}}_A \times \tilde{\mathscr{P}}_B)_* \tilde{\varphi}_*[\mathscr{Y}]$. For any $\omega \in H^2_{\mathrm{dR}}(X)[\pi_A], \eta \in H^1_{\mathrm{dR}}(Y)[\sigma_B]$ and lifts $\tilde{\omega}, \tilde{\eta}$ to fp-cohomology we can compute

$$\begin{aligned}\mathrm{AJ}_p(\Delta_{A,B})(\pi_1^*\omega \cup \pi_2^*\eta) &= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\Delta}_{A,B}), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} \\ &= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{Y}]), (\mathscr{P}_A \times \mathscr{P}_B)^*(\pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta})\rangle_{\mathrm{fp}} \\ &= \langle \mathrm{cl}_{\mathrm{syn}}(\tilde{\varphi}_*[\mathscr{Y}]), \pi_1^*\tilde{\omega} \cup \pi_2^*\tilde{\eta}\rangle_{\mathrm{fp}} = \mathrm{tr}_{\mathscr{Y}}(\tilde{\zeta}^*\tilde{\omega} \cup \tilde{\eta})\end{aligned}$$

as in Section 2.7.3. If $\alpha_1 : \mathscr{U}_0 \to \mathscr{X}$, $\alpha_2 : \mathscr{W}_0 \to \mathscr{Y}$ are the natural finite degeneracy maps, we know that $\mathrm{AJ}_p(\Delta_{2t_L,2})(\pi_1^*(\alpha_1^*\omega) \cup \pi_2^*(\alpha_2^*\eta)) = \mathrm{tr}_{\mathscr{W}_0}(\tilde{\zeta}^*(\alpha_1^*\tilde{\omega}) \cup (\alpha_2^*\tilde{\eta}))$. Therefore,

$$\mathrm{AJ}_p(\Delta_{2t_L,2})(\pi_1^*(\alpha_1^*\omega) \cup \pi_2^*(\alpha_2^*\eta)) = \deg(\alpha_1)\deg(\alpha_2) \cdot \mathrm{AJ}_p(\Delta_{A,B})(\pi_1^*\omega \cup \pi_2^*\eta)$$

and the LHS vanishes if and only if the RHS vanishes. It follows that the non-vanishing of the p-adic L-function implies the non-vanishing of the syntomic Abel-Jacobi image of both $\Delta_{2t_L,2}$ and $\Delta_{A,E}$ by Corollary 2.7.18, which in turn forces the non-vanishing of the p-adic étale Abel-Jacobi image of the cycle $\Delta_{A,E}$ ([BDP13], Section 3.4). Then Liu's theorem ([Liu16], Theorem 1.5) gives the dimension of the Bloch-Kato Selmer group. □

Chapter 3

Future plans

After the results attained in this thesis, a natural next step is to extend Darmon and Rotger's ideas [DR17a] to the twisted triple product setting. Currently, Zhaorong Jin and I are pursuing this project, and I will present, as a preview, the construction of big Hirzebruch-Zagier classes I carried out following [DR17b].

Our objective is to provide a geometric construction of twisted triple product p-adic L-functions with applications to the equivariant BSD-conjecture in rank zero. Let L be a real quadratic field, $E_{/\mathbb{Q}}$ a rational elliptic curve of conductor N and $\varrho : \Gamma_L \to \mathrm{GL}_2(\mathbb{C})$ a totally odd two-dimensional Artin representation of conductor $\mathfrak{Q}$ factoring through the Galois group of a finite extension $H/\mathbb{Q}$. For any rational prime p, we consider the p-adic Galois representation of $\Gamma_{\mathbb{Q}}$

$$\mathrm{V}_{\varrho,E} = \otimes\text{-}\mathrm{Ind}_L^{\mathbb{Q}}(\varrho) \otimes \mathrm{V}_p(E)(1).$$

When the tensor induction of $\det(\varrho)$ is the trivial character, the representation $\mathrm{V}_{\varrho,E}$ is self-dual and its L-function $L\big(\mathrm{V}_{\varrho,E}, s\big)$ has meromorphic continuation to $\mathbb{C}$ and a functional equation centered at $s = 0$, at which the L-function is holomorphic. Indeed, by modularity ([Wil95],[TW95], [PS16]) there is an automorphic representation $\Pi_{\varrho,E}$ of $\mathrm{Res}_{L\times\mathbb{Q}/\mathbb{Q}}\mathrm{GL}_2$ such that its twisted triple product L-function $L(s, \Pi_{\varrho,E}, \mathrm{r})$ equals $L\big(\mathrm{V}_{\varrho,E}, s - \frac{1}{2}\big)$. The analytic rank $r_{\mathrm{an}}(E, \varrho)$ is defined as the order of vanishing of $L\big(\mathrm{V}_{\varrho,E}, s\big)$ at the center. Moreover, if we let $E(H)^{\varrho} = \mathrm{Hom}_{\Gamma_{\mathbb{Q}}}(\varrho, E(H) \otimes \mathbb{C})$ denote the ϱ-isotypical component of the Mordell-Weil group $E(H)$, we can define the algebraic rank $r_{\mathrm{alg}}(E, \varrho)$ of E twisted by ϱ as the dimension $\dim_{\mathbb{C}} E(H)^{\varrho}$. The equivariant refinement of the BSD-conjecture predicts that the two ranks are always equal. If $(N, \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{Q})) = 1$ and all the primes dividing N split in L, then the sign of the functional equation of $L(s, \Pi_{\varrho,E})$ is $\epsilon = +1$ and we can hope to prove the following implication:

$$r_{\mathrm{an}}(E, \varrho) = 0 \overset{?}{\Longrightarrow} r_{\mathrm{alg}}(E, \varrho) = 0.$$

By a judicious choice of the Artin representation ϱ as in Corollary 1.3.3, we would then deduce cases of the BSD-conjecture in rank zero over non-solvable quintic fields: when $K/\mathbb{Q}$ is a non-totally real S_5-quintic extension of discriminant $d_K > 0$ and N is odd and split in the real quadratic field $\mathbb{Q}(\sqrt{d_K})$, then we expect to deduce that

$$r_{\mathrm{an}}(E/K) = r_{\mathrm{an}}(E/\mathbb{Q}) \overset{?}{\Longrightarrow} r_{\mathrm{alg}}(E/K) = r_{\mathrm{alg}}(E/\mathbb{Q}).$$

The winding path we plan take to connect the two sides of the implication is well-known by now ([DR17a], [KLZ17]). Even though it is not apparent, p-adic deformation should play a crucial role in our strategy. Our planned route can be divided into two main parts: the first consists in producing enough annihilators of the Bloch-Kato Selmer group, and the second in using the non-vanishing at the center of the automorphic L-function to show that the annihilators are non-trivial. In our setting, it is not known how to produce interesting cohomology classes directly from geometry, so the dévissage is to fit the automorphic representation $\Pi_{\varrho,E}$ into a Hida family. There are certain *balanced* points in the family for which geometric classes exist; those classes can be interpolated into *big cohomology classes*, called big Hirzebruch-Zagier classes, which can then be specialized to the point corresponding to $\Pi_{\varrho,E}$.

The second part bridges the distance between the analytic and the algebraic worlds, and it entails the comparison of two rigid-analytic meromorphic functions: the analytic and the motivic p-adic L-function. On the one hand, the analytic twisted triple product p-adic L-function, built in Chapter 2, has been constructed interpolating the algebraic part of central L-values of automorphic L-functions corresponding to $\mathbb{Q}$-*dominated* points in the Hida family. Hence, it is embedded in its definition the information of whether $L\left(\frac{1}{2},\Pi_{\varrho,E},\mathrm{r}\right)$ vanishes or not. On the other hand, the motivic p-adic L-function is produced out of big Hirzebruch-Zagier classes and by construction it is well-understood over balanced points. The identification of the two functions is obtained by comparing their values on a dense subset of balanced points via a p-adic Gross-Zagier formula.

3.1 Big Hirzebruch-Zagier classes

In this section we present the construction of big Hirzebruch-Zagier classes following the recent work [DR17b]. An $\mathbf{I}_{\mathscr{G}}$-adic Hida family $\mathscr{G}$ of Hilbert cuspforms over L and an $\mathbf{I}_{\mathscr{F}}$-adic Hida family $\mathscr{F}$ of elliptic cuspforms come equipped with big Galois representation $\mathbf{V}_{\mathscr{G}}$ and $\mathbf{V}_{\mathscr{F}}$ interpolating the representations of their specializations ([Hid89a], Theorem 1).

Let $\boldsymbol{\epsilon}_{\mathscr{G}} : \Gamma_L \longrightarrow \mathbf{I}_{\mathscr{G}}^{\times}$ (resp. $\boldsymbol{\epsilon}_{\mathscr{F}} : \Gamma_{\mathbb{Q}} \longrightarrow \mathbf{I}_{\mathscr{F}}^{\times}$) denote the composition of character $\boldsymbol{\epsilon} : \Gamma_L \longrightarrow \boldsymbol{\Lambda}_L^{\times}$ (resp. $\boldsymbol{\epsilon} : \Gamma_{\mathbb{Q}} \longrightarrow \boldsymbol{\Lambda}_{\mathbb{Q}}^{\times}$) with the natural map $\boldsymbol{\Lambda}_L \to \mathbf{I}_{\mathscr{G}}$ (resp. $\boldsymbol{\Lambda}_{\mathbb{Q}} \to \mathbf{I}_{\mathscr{F}}$). Then big Hirzebruch-Zagier classes attached to the pair $(\mathscr{G},\mathscr{F})$ are Galois cohomology classes with value in the Kummer self-dual big Galois representation of $\Gamma_{\mathbb{Q}}$

$$\mathbf{V}^{\dagger}_{\mathscr{G},\mathscr{F}} = \otimes\text{-}\mathrm{Ind}_L^{\mathbb{Q}}\left[\mathbf{V}_{\mathscr{G}}\left(\boldsymbol{\epsilon}_{\mathscr{G}}^{-1/2}\right)\right](-1) \otimes \mathbf{V}_{\mathscr{F}}\left(\boldsymbol{\epsilon}_{\mathscr{F}}^{-1/2}\right). \tag{3.1}$$

The explicit realization of that Galois representation in the cohomology of a tower of certain threefolds with increasing level at p plays an important role in the construction of the classes.

Abelian varieties up to isogeny

In this section F denotes a totally real number field. Given our adelic viewpoint, it is more convenient to interpret the Shimura varieties for the algebraic group $G_F = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ as moduli problems classifying abelian varieties up to isogeny. We recall here that point of view following ([Hid04], Section 4.2).

Let A/S be an abelian scheme with real multiplication by $\mathcal{O}_F$ such that the sheaf of invariant differentials $\omega_{A/S}$ is isomorphich to $\mathfrak{d}_F^{-1} \otimes_{\mathbb{Z}} \mathcal{O}_S$ Zariski locally on S. Considering abelian schemes up to isogeny means that two polarizations $\lambda, \lambda' : A \to A^{\vee}$ are equivalent if $\lambda = \lambda \circ a$ for a totally positive $a \in F$ and we denote an equivalence class by $\bar{\lambda}$. By choosing a geometric point $s \in S$, one can consider the Tate module $\mathcal{T}(A) = \mathcal{T}_s(A) = \varprojlim_{N} A[N](k(s))$

and define

$$V_s(A) = \mathcal{T}_s(A) \otimes_{\mathbb{Z}} \mathbb{A}_{\mathbb{Q}}^{(\infty)}.$$

If we let $V\big(\mathbb{A}_{\mathbb{Q}}^{(\infty)}\big) = F^{\oplus 2} \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^{(\infty)}$, a full level structure on A is a collection of isomorphisms $\eta_s : V\big(\mathbb{A}_{\mathbb{Q}}^{(\infty)}\big) \cong V_s(A)$ of $\mathbb{A}_F^{(\infty)}$-modules, indexed by a set of chosen geometric points, one for each connected component of S. The group $G_F\big(\mathbb{A}_{\mathbb{Q}}^{(\infty)}\big)$ acts on full level structures on the right by precomposition.

Definition 3.1.1. *Let K be a compact open subgroup of $G_F\big(\mathbb{A}_{\mathbb{Q}}^{(\infty)}\big)$. A level K-structure on abelian scheme A/S with real multiplication by $\mathcal{O}_F$ is a K-orbit $\bar{\eta} = \eta K$ of a full level structure η. A K-level structure $\bar{\eta}$ is defined over S if it satisfies $\sigma \circ \bar{\eta}_s = \bar{\eta}_s$ for all the chosen geometric points $s \in S$ and $\sigma \in \pi_1^{\mathrm{et}}(S, s)$.*

If K is sufficiently small, the functor from $\mathrm{Sch}_{/\mathbb{Q}}$ to Sets, defined as

$$\mathcal{E}_K(S) = [(A, \bar{\lambda}, \bar{\eta})_{/S} \mid \bar{\eta} \text{ is a level } K\text{-structure}]$$

is representable by a $\mathbb{Q}$-scheme $\mathrm{Sh}_K(G_F)$ whose complex points are canonically isomorphic to $G_F(\mathbb{Q})\backslash(\mathfrak{H}^{\pm})^{\mathrm{I}_F} \times G_F(\mathbb{A}^{(\infty)})/K$. When $K = U(N\mathcal{O}_F)$ for some integer N it is possible to describe the Galois action of $G(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ on $\mathcal{E}_K$ as follows. Let $c \in \widehat{\mathbb{Z}}^{\times}$ and $\sigma_c \in G(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ the corresponding Galois automorphism, then for $[A, \bar{\lambda}, \bar{\eta}] \in \mathcal{E}_{U(\mathfrak{N})}(S)$ one finds ([Hid04], Section 4.2.1)

$$[A, \bar{\lambda}, \bar{\eta}]^{\sigma_c} = [A, \bar{\lambda}, \overline{(\eta \circ \gamma_c)}] \qquad \text{for} \qquad \gamma_c = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Hecke operators

Let's consider the Shimura varieties $Sh_\alpha = \mathrm{Sh}_{K(p^\alpha)}(G_F)$ and $Sh_\alpha(p) = \mathrm{Sh}_{K(p^\alpha)^g \cap K(p^\alpha)}(G_F)$, where $K(p^\alpha)^g = g^{-1}K(p^\alpha)g$ for g the diagonal matrix with non-zero entries $g_{11} = p$ and $g_{22} = 1$. There are two natural projections $\varpi_1 : Sh_{\alpha+1} \longrightarrow Sh_\alpha$, $\varpi_2 : Sh_{\alpha+1} \longrightarrow Sh_\alpha$ given by $[x, h] \mapsto [x, h]$ and $[x, hg^{-1}]$ on the complex uniformizations. These maps can be factored as

$$\begin{tikzcd} Sh_{\alpha+1} \arrow[d, "\mu"] \arrow[dr, "\varpi_1"] & \\ Sh_\alpha(p) \arrow[r, "\pi_1"'] & Sh_\alpha, \end{tikzcd} \qquad \begin{tikzcd} Sh_{\alpha+1} \arrow[d, "\mu"] \arrow[dr, "\varpi_2"] & \\ Sh_\alpha(p) \arrow[r, "\pi_2"'] & Sh_\alpha, \end{tikzcd}$$

where $\mu : Sh_{\alpha+1} \to Sh_\alpha(p)$ is the natural Galois cover with Galois group

$$[K(p^\alpha)^g \cap K(p^\alpha)]/K(p^{\alpha+1})\mathcal{O}_{F,\alpha}^{\times} = [K(p^\alpha) \cap K_0(p^{\alpha+1})]/K(p^{\alpha+1})\mathcal{O}_{F,\alpha}^{\times}$$

for $\mathcal{O}_{F,\alpha}^{\times}$ the group of global units congruent to 1 $\pmod{p^\alpha}$. Then the Hecke correspondences at p can be defined as $U^*(p) = \pi_{1*}\pi_2^*$, $U(p) = \pi_{2*}\pi_1^*$ ([Hid04], Section 4.2.5). There is also a group of diamond operators naturally acting on level structures of the tower $\{Sh_\alpha\}_{\alpha \geq 1}$, thery are given by

$$\mathbb{G}_F^\alpha(K) = K_0(p^\alpha)\mathcal{O}_F^{\times}/K(p^\alpha)\mathcal{O}_F^{\times}, \qquad \mathbb{G}_F(K) = \lim_{\leftarrow, \alpha} \mathbb{G}_F^\alpha(K).$$

If we set $Z_F^\alpha(K) = K_F\mathcal{O}_F^\times / K_F(p^\alpha)\mathcal{O}_F^\times$, for $K_F = K \cap \mathbb{A}_F^{(\infty),\times}$, then there is an isomorphism

$$\mathbb{G}_F^\alpha(K) \xrightarrow{\sim} Z_F^\alpha(K) \times (\mathcal{O}_F/p^\alpha)^\times, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, a_p^{-1}d_p). \tag{3.3}$$

Lemma 3.1.2. *Set* $\mathfrak{I}_{\alpha,F}^{\alpha+1}(K) = \ker\left(\mathbb{G}_F^{\alpha+1}(K) \to \mathbb{G}_F^\alpha(K)\right)$*, then the natural inclusion*

$$\left[K(p^\alpha) \cap K_0(p^{\alpha+1})\right] / K(p^{\alpha+1})\mathcal{O}_{F,\alpha}^\times \hookrightarrow \mathfrak{I}_F^{\alpha+1}(K)$$

is an isomorphism.

Proof. It follows applying the snake lemma and using the canonical isomorphisms

$$K(p^{\alpha+1})\mathcal{O}_F^\times / K(p^{\alpha+1})\mathcal{O}_{F,\alpha}^\times \cong \mathcal{O}_F^\times / \mathcal{O}_{F,\alpha}^\times \cong \left[K(p^\alpha) \cap K_0(p^{\alpha+1})\right]\mathcal{O}_F^\times / \left[K(p^\alpha) \cap K_0(p^{\alpha+1})\right].$$

□

3.1.1 Hirzebruch-Zagier cycles

Let $L/\mathbb{Q}$ be a real quadratic field, $K \leq G_L\left(\mathbb{A}_\mathbb{Q}^{(\infty)}\right)$ a sufficiently small compact open subgroup and $K' = K \cap G_\mathbb{Q}(\mathbb{A}_\mathbb{Q})$. For every $\alpha \geq 1$ we choose inductively a smooth toroidal compactification S_α of $\mathrm{Sh}_{K(p^\alpha)}(G_L)$ by requiring that the degeneracy maps of Section 3.1 extend. Since any rational map from a smooth projective curve to a projective variety extends uniquely to a morphism of schemes, we can choose X_α the smooth compactification of $\mathrm{Sh}_{K'(p^\alpha)}(G_\mathbb{Q})$ for every $\alpha \geq 1$, and define the codimension 2 cycle $\Delta_0 \subset S_0 \times X_0$ by

$$X_0 \longrightarrow \Delta_0 \subset S_0 \times X_0$$

$$[A, \bar{\lambda}, \bar{\eta}] \mapsto \Big(\left[A \otimes \mathcal{O}_L, \overline{\lambda \otimes 1}, \overline{\eta \otimes 1}\right], [A, \bar{\lambda}, \bar{\eta}]\Big).$$

We denote by Δ_α the pull-back of Δ_0 under $\varpi_1^2 : S_\alpha \times X_\alpha \to S_0 \times X_0$. If $\mathbb{X}(p^\alpha)$ denotes the smooth compactification of a geometrically connected component of the modular curve $\mathrm{Sh}_{K'(p^\alpha)}(G_\mathbb{Q})$ defined over $\mathbb{Q}(\zeta_{p^\alpha})$, then for each $(d_1, d_2) \in \mathbb{G}_{0,L}^\alpha(K) \times \mathbb{G}_{0,\mathbb{Q}}^\alpha(K')$ the morphism

$$\varphi_{(d_1,d_2)} : \mathbb{X}(p^\alpha) \longrightarrow \Delta_\alpha \subset S_\alpha \times X_\alpha$$

$$(A, \bar{\lambda}, \bar{\eta}) \mapsto \Big(\left(A \otimes \mathcal{O}_L, \overline{\lambda \otimes 1}, \overline{(\eta \otimes 1) \circ d_1}\right), \left(A, \bar{\lambda}, \overline{(\eta \circ d_2)}\right)\Big).$$

defines a codimension two cycle $\Delta_\alpha[d_1, d_2] := \varphi_{(d_1,d_2)}(\mathbb{X}_\alpha) \in \mathrm{CH}^2(S_\alpha \times X_\alpha)$. The diagonal embedding of the subgroup $\mathbb{G}_{0,\mathbb{Q}}^\alpha(K')^{\det}$ of matrices with determinant congruent to 1 $\pmod{p^\alpha}$ acts trivially on the cycle $\Delta_\alpha[d_1, d_2]$, because such matrices preserve $\mathbb{X}(p^\alpha)$. Hence, the cycle $\Delta_\alpha[d_1, d_2]$ depends only on the image $[d_1, d_2]$ of (d_1, d_2) in

$$\mathscr{I}_\alpha = \Delta\left(\mathbb{G}_{0,\mathbb{Q}}^\alpha(K')^{\det}\right) \backslash \left[\mathbb{G}_{0,L}^\alpha(K) \times \mathbb{G}_{0,\mathbb{Q}}^\alpha(K')\right].$$

Recall that $\mathbb{G}_{0,L}^\alpha(K) \times \mathbb{G}_{0,\mathbb{Q}}^\alpha(K') \cong Z_{0,L}^\alpha(K) \times (\mathcal{O}_L/p^\alpha)^\times \times Z_{0,\mathbb{Q}}^\alpha(K') \times (\mathbb{Z}/p^\alpha)^\times$ with the isomorphism explicitly given by

$$\left[\begin{pmatrix} a^\circ & b^\circ \\ c^\circ & d^\circ \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \mapsto \Big((a^\circ, a_p^{\circ-1}d_p^\circ), (a, a_p^{-1}d_p)\Big).$$

From now on, we will denote elements of $\mathbb{G}^{\alpha}_{0,L}(K) \times \mathbb{G}^{\alpha}_{0,\mathbb{Q}}(K')$ both as pairs of matrices $\langle a_1, a_2\rangle$ and pairs of 2-tuples $\langle (z^\circ, a^\circ), (z, a)\rangle$ according to what is most convient. Note that it is possible to describe $\mathbb{G}^{\alpha}_{0,\mathbb{Q}}(K')^{\det}$ as the kernel of the group homomorphism $Z^{\alpha}_{0,\mathbb{Q}}(K') \times (\mathbb{Z}/p^\alpha)^\times \longrightarrow (\mathbb{Z}/p^\alpha)^\times$, $(z, a) \mapsto z_p^2 a$. Now, we present two lemmas describing the behavior of Hirzebruch-Zagier cycles under Hecke operators and Galois automorphisms.

Lemma 3.1.3. *For all diamond operators* $\langle a_1, a_2\rangle \in \mathbb{G}^{\alpha}_{0,L}(K) \times \mathbb{G}^{\alpha}_{0,\mathbb{Q}}(K')$

$$\Delta_\alpha[d_1, d_2]\langle a_1, a_2\rangle = \Delta_\alpha[d_1 \cdot a_1, d_2 \cdot a_2].$$

Moreover, for all $c \in \widehat{\mathbb{Z}}^\times$ *corresponding to the Galois automorphism* $\sigma_c \in G(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q})$,

$$\Delta_\alpha[d_1, d_2]^{\sigma_c} = \Delta_\alpha[d_1 \cdot \gamma_c, d_2 \cdot \gamma_c].$$

If $\sigma_c \in G(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q}(\zeta_p))$ *then*

$$\Delta_\alpha[d_1, d_2]^{\sigma_c} = \Delta_\alpha[d_1, d_2]\langle (c, 1), (c, 1)\rangle^{1/2}.$$

Proof. The first claim follows directly from the definitions. For the second claim, we recall that the Galois action is explicitly given by $\Delta_\alpha[d_1, d_2]^{\sigma_c} = \Delta_\alpha[d_1, d_2]\langle \gamma_c, \gamma_c\rangle$, see (3.2). Furthermore, if $\sigma_c \in G(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q}(\zeta_p))$, we have

$$\langle \gamma_c, \gamma_c\rangle = \langle (c^{1/2}, c_p^{-1}), (c^{1/2}, c_p^{-1})\rangle \langle (c, 1), (c, 1)\rangle^{1/2}$$

for $(c^{1/2}, c_p^{-1}) \in \mathbb{G}^{\alpha}_{0,\mathbb{Q}}(K')^{\det}$ and the Lemma follows. $\square$

Lemma 3.1.4. *For all* $\alpha \geq 1$ *and all* $[d_1', d_2'] \in \mathscr{I}_{\alpha+1}$ *whose image in* $\mathscr{I}_\alpha$ *is* $[d_1, d_2]$,

$$(\varpi_1^2)_*\Delta_{\alpha+1}[d_1', d_2'] = p^3 \cdot \Delta_\alpha[d_1, d_2], \qquad (\varpi_2^2)_*\Delta_{\alpha+1}[d_1', d_2'] = (U(p) \times U(p)) \cdot \Delta_\alpha[d_1, d_2].$$

The cycles $\Delta_\alpha[d_1, d_2]$ *satisfy the distribution relations*

$$\sum_{\langle a_1, a_2\rangle} \Delta_{\alpha+1}[d_1' \cdot a_1, d_2' \cdot a_2] = (\varpi_1^2)^*\Delta_\alpha[d_1, d_2],$$

the sum over $\mathfrak{I}^{\alpha+1}_{\alpha,L}(K) \times \mathfrak{I}^{\alpha+1}_{\alpha,\mathbb{Q}}(K')$, *the kernel of* $\mathbb{G}^{\alpha+1}_{0,L}(K) \times \mathbb{G}^{\alpha+1}_{0,\mathbb{Q}}(K') \to \mathbb{G}^{\alpha}_{0,L}(K) \times \mathbb{G}^{\alpha}_{0,\mathbb{Q}}(K')$.

Proof. By the commutativity of the diagram

$$\begin{tikzcd} \mathbb{X}(p^{\alpha+1}) \arrow[r, "\varphi_{(d_1', d_2')}"] \arrow[d, "\varpi_1"] & S_{\alpha+1} \times X_{\alpha+1} \arrow[d, "\varpi_1^2"] \\ \mathbb{X}(p^\alpha) \arrow[r, "\varphi_{(d_1, d_2)}"] & S_\alpha \times X_\alpha \end{tikzcd}$$

it is enough to compute the degree of $\varpi_1 : \mathbb{X}(p^{\alpha+1}) \to \mathbb{X}(p^\alpha)$ which is p^3.

The map $\mu^2 \circ \varphi_{(d_1',d_2')} : \mathbb{X}(p^{\alpha+1}) \hookrightarrow S_\alpha(p) \times X_\alpha(p)$ is a closed embedding, hence the diagram

$$\begin{tikzcd} \mathbb{X}(p^{\alpha+1}) \arrow[r, "\mu^2\circ\varphi_{(d_1',d_2')}"] \arrow[d, "\varpi_1"] & S_\alpha(p) \times X_\alpha(p) \arrow[d, "\pi_1^2"] \\ \mathbb{X}(p^\alpha) \arrow[r, "\varphi_{(d_1,d_2)}"] & S_\alpha \times X_\alpha \end{tikzcd}$$

is cartesian because the horizontal maps are closed embeddings and the vertical maps have the same degree. The push-pull formula gives

$$(\mu^2)_*\Delta_{\alpha+1}[d_1', d_2'] = (\pi_1^2)^*\Delta_\alpha[d_1, d_2], \tag{3.4}$$

and by pushing forward the equality with π_2^2 we get

$$(\varpi_2^2)_*\Delta_{\alpha+1}[d_1', d_2'] = (U(p) \times U(p)) \cdot \Delta_\alpha[d_1, d_2].$$

The distribution relation follows by noting that

$$\sum_{\langle a_1, a_2\rangle} \Delta_{\alpha+1}[d_1' \cdot a_1, d_2' \cdot a_2] = (\mu^2)^*(\mu^2)_*\Delta_{\alpha+1}[d_1', d_2'] = (\varpi_1^2)^*\Delta_\alpha[d_1, d_2],$$

where the second equality comes from (3.4) and the first equality follows from the fact that (μ^2) is a Galois cover with Galois group $\mathfrak{I}_L^{\alpha+1}(K) \times \mathfrak{I}_{\mathbb{Q}}^{\alpha+1}(K')$, (Lemma 3.1.2). ☐

For any number field D, the p-adic etale Abel-Jacobi map

$$\mathrm{AJ}_p^{\mathrm{et}} : \mathrm{CH}^2(S_\alpha \times X_\alpha)_0(D) \longrightarrow \mathrm{H}^1\big(D, \mathrm{H}^3_{\mathrm{et}}\big((S_\alpha \times X_\alpha)_{\overline{\mathbb{Q}}}, \mathcal{O}(2)\big)\big)$$

sends null-homologous cycles to Galois cohomology classes. In order to make Hirzebruch-Zagier cycles null-homologous, we introduce an auxiliary Hecke operator θ_q.

Lemma 3.1.5. *Let q be a rational prime not dividing the level of K', then the operator $\theta_q = 1 \otimes (T_q - (q+1))$ annihilates the cohomology group $\mathrm{H}^4_{\mathrm{et}}\big((S_\alpha \times X_\alpha)_{\overline{\mathbb{Q}}}, \mathcal{O}(2)\big)$.*

Proof. The correspondence T_q acts as multiplication by $(q+1)$ on $\mathrm{H}^i_{\mathrm{et}}(X_{\alpha,\overline{\mathbb{Q}}}, \mathcal{O}(1))$ for $i = 0, 2$ and $\mathrm{H}^3_{\mathrm{et}}(S_{\alpha,\overline{\mathbb{Q}}}, \mathcal{O}(1)) = 0$ because the connected components of S_α are simply connected ([Gee88], Theorem 6.1, page 81). ☐

Thus, the modified Hirzebruch-Zagier cycles $\Delta_\alpha^\circ[d_1, d_2] = \theta_q\Delta_\alpha[d_1, d_2]$ are homologically trivial and give rise cohomology classes in the appropriate Kunneth component

$$\kappa_\alpha[d_1, d_2] = \mathrm{AJ}_p^{\mathrm{et}}(\Delta_\alpha^\circ[d_1, d_2]) \in \mathrm{H}^1\big(\mathbb{Q}(\zeta_{p^\alpha}), \mathrm{H}^2_{\mathrm{et}}(S_{\alpha,\overline{\mathbb{Q}}}, \mathcal{O}(1)) \otimes \mathrm{H}^1_{\mathrm{et}}(X_{\alpha,\overline{\mathbb{Q}}}, \mathcal{O}(1))\big).$$

We would like to patch all these classes together, but they are defined over the increasingly larger fields $\mathbb{Q}(\zeta_{p^\alpha})$ as α grows. We recall that Lemma 3.1.3 expressed the Galois action of $G(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q}(\zeta_p))$ on Hirzebruch-Zagier cycles in terms of diamond operators, therefore, in order to descend the field of definition, we replace the $\mathcal{O}[\mathfrak{I}^\alpha][\Gamma_{\mathbb{Q}}]$-module

$$\mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(2) := \mathrm{H}^2_{\mathrm{et}}(S_{\alpha,\overline{\mathbb{Q}}}, \mathcal{O}(1)) \otimes \mathrm{H}^1_{\mathrm{et}}(X_{\alpha,\overline{\mathbb{Q}}}, \mathcal{O}(1))$$

by a twist over $\mathbb{Q}(\zeta_{p^\alpha})$, where $\mathfrak{I}^\alpha = \ker\big(\mathbb{G}_{0,L}^\alpha(K) \times \mathbb{G}_{0,\mathbb{Q}}^\alpha(K') \to \mathbb{G}_{0,L}^1(K) \times \mathbb{G}_{0,\mathbb{Q}}^1(K')\big)$.

Definition 3.1.6. *We denote by $O[\mathfrak{I}^\alpha]^\dagger$ the free $O[\mathfrak{I}^\alpha]$-module of rank one on which the Galois group $\Gamma_{\mathbb{Q}(\zeta_p)}$ acts via its quotient $G(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q}(\zeta_p))$, the element σ_c acting as multiplication by the group-like element $\langle (c,1),(c,1)\rangle^{-1/2}$.*

There is a canonical Galois-equivariant $O[\mathfrak{I}^\alpha]$-hermitian bilinear pairing

$$\langle\!\langle\ ,\ \rangle\!\rangle_\alpha : \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(2)^\dagger \times \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(1)^\dagger \longrightarrow O[\mathfrak{I}^\alpha] \tag{3.5}$$

given by the formula

$$\langle\!\langle z, \nu \rangle\!\rangle_\alpha = \sum_{\langle a_1,a_2\rangle \in \mathfrak{I}^\alpha} \left\langle z \cdot \langle a_1,a_2\rangle, \nu \right\rangle_\alpha \cdot [\langle a_1,a_2\rangle]$$

where $\langle\ ,\ \rangle_\alpha$ is the Poincaré pairing. The pairing (3.5) identifies $\mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(2)^\dagger$ with $\mathrm{Hom}_{O[\mathfrak{I}^\alpha]}\left(\mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(1)^\dagger, O[\mathfrak{I}^\alpha]\right)$, hence we can define $\boldsymbol{\kappa}_\alpha[d_1,d_2]$ by declaring that for all $\sigma \in \Gamma_{\mathbb{Q}(\zeta_{p^\alpha})}$ and all $\xi_\alpha \in \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(1)^\dagger$,

$$\boldsymbol{\kappa}_\alpha[d_1,d_2](\sigma)(\xi_\alpha) = \langle\!\langle k_\alpha[d_1,d_2](\sigma), \xi_\alpha \rangle\!\rangle_\alpha.$$

Lemma 3.1.7. *The class $\boldsymbol{\kappa}_\alpha[d_1,d_2]$ is the restriction to $G_{\mathbb{Q}(\zeta_{p^\alpha})}$ of a class*

$$\boldsymbol{\kappa}_\alpha[d_1,d_2] \in \mathrm{H}^1\left(\mathbb{Q}(\zeta_p), \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(2)^\dagger\right).$$

Furthermore, for all $\sigma \in G(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, $\boldsymbol{\kappa}_\alpha[d_1,d_2]^{\sigma_c} = \boldsymbol{\kappa}_\alpha[d_1\cdot\gamma_c, d_2\cdot\gamma_c]$.

Proof. Let $\Delta_\alpha^\circ[[d_1,d_2]] \in \mathrm{CH}^2(S_\alpha \times X_\alpha)(\mathbb{Q}(\zeta_p))$ be the inverse image of $\Delta_1^\circ[d_1,d_2]$ in $S_\alpha \times X_\alpha$ and consider the extension of $O[\mathfrak{I}^\alpha][\Gamma_{\mathbb{Q}(\zeta_p)}]$-modules

$$0 \longrightarrow \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(2)^\dagger \longrightarrow E_\alpha \longrightarrow O[\mathfrak{I}^\alpha] \longrightarrow 0 \tag{3.6}$$

obtained from the excision sequence

$$\mathrm{H}^3_{\mathrm{et}}\left((S_\alpha \times X_\alpha)_{\overline{\mathbb{Q}}}, O(2)\right) \hookrightarrow \mathrm{H}^3_{\mathrm{et}}\left((S_\alpha \times X_\alpha)_{\overline{\mathbb{Q}}} \setminus \Delta_\alpha^\circ[[d_1,d_2]], O(2)\right) \twoheadrightarrow \mathrm{H}^0_{\mathrm{et}}(\Delta_\alpha^\circ[[d_1,d_2]], O)_0,$$

by pull back along

$$j : O[\mathfrak{I}^\alpha]^{-\dagger} \hookrightarrow \mathrm{H}^0_{\mathrm{et}}(\Delta_\alpha^\circ[[d_1,d_2]], O)_0,$$

push out along $\mathrm{H}^3_{\mathrm{et}}\left((S_\alpha \times X_\alpha)_{\overline{\mathbb{Q}}}, O(2)\right) \to \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(2)$ and twist by $\dagger$. The map j is the $\Gamma_{\mathbb{Q}(\zeta_p)}$-equivariant inclusion defined on group-like elements by

$$j(\langle a_1,a_2\rangle) = \mathrm{cl}_{\mathrm{et}}(\Delta_\alpha^\circ[d_1\cdot a_1, d_2\cdot a_2]).$$

The cohomology class $c \in \mathrm{H}^1\left(\mathbb{Q}(\zeta_p), \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(2)^\dagger\right)$ corresponding to (3.6) satisfy

$$c_{|\Gamma_{\mathbb{Q}(\zeta_{p^\alpha})}} = \boldsymbol{\kappa}_\alpha[d_1,d_2].$$

□

3.1.2 Big cohomology classes

For any $[d_1, d_2] \in \mathscr{I}_\infty = \lim_{\leftarrow,\alpha} \mathscr{I}_\alpha$ we managed to obtain a collection of classes $\{\boldsymbol{\kappa}_\alpha[d_1, d_2]\}_\alpha$ defined over the same field $\mathbb{Q}(\zeta_p)$. However, they are not compatible under the natural trace maps $(\varpi_1^2)_*$ in cohomology (Lemma 3.1.4) and the naive approach to fix the problem introduces unwanted denominators. Fortunately, Darmon and Rotger found a way to elegantly solve the problem. Consider the map $p_{\alpha+1} : O[\mathfrak{I}^{\alpha+1}] \to O[\mathfrak{I}^\alpha]$ induced by the natural map between groups.

Lemma 3.1.8. *For all $\alpha \geq 1$ let $\xi_{\alpha+1} \in \mathbb{H}^{2,1}(S_{\alpha+1} \times X_{\alpha+1})(1)^\dagger$ and $\xi_\alpha \in \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(1)^\dagger$ be compatible elements, $(\varpi_1^2)_*(\xi_{\alpha+1}) = \xi_\alpha$. Then for all $[d_1, d_2] \in \mathscr{I}_\infty$ and every $\sigma \in \Gamma_{\mathbb{Q}(\zeta_p)}$,*

$$p_{\alpha+1}\big(\boldsymbol{\kappa}_{\alpha+1}[d_1, d_2](\sigma)(\xi_{\alpha+1})\big) = \boldsymbol{\kappa}_\alpha[d_1, d_2](\sigma)(\xi_\alpha).$$

Proof. Firstly, we note that $\boldsymbol{\kappa}_{\alpha+1}[d_1, d_2]$, $\boldsymbol{\kappa}_\alpha[d_1, d_2]$ naturally give rise to cohomology classes in

$$\mathrm{H}^1(\mathbb{Q}(\zeta_p), \mathrm{Hom}_{O[\mathfrak{I}^{\alpha+1}]}(\mathbb{H}^{2,1}(S_{\alpha+1} \times X_{\alpha+1})(1)^\dagger, O[\mathfrak{I}^\alpha])).$$

For $\sigma \in \Gamma_{\mathbb{Q}(\zeta_{p^{\alpha+1}})}$ we compute

$$\begin{aligned} p_{\alpha+1}\big(\boldsymbol{\kappa}_{\alpha+1}[d_1, d_2](\sigma)(\xi_{\alpha+1})\big) &= \sum_{\langle a_1, a_2\rangle \in \mathfrak{I}^\alpha} \big\langle (\varpi_1^2)^* k_\alpha[d_1 \cdot a_1, d_2 \cdot a_2](\sigma), \xi_{\alpha+1} \big\rangle_\alpha \cdot [\langle a_1, a_2\rangle] \\ &= \sum_{\langle a_1, a_2\rangle \in \mathfrak{I}^\alpha} \big\langle k_\alpha[d_1 \cdot a_1, d_2 \cdot a_2](\sigma), (\varpi_1^2)_* \xi_{\alpha+1} \big\rangle_\alpha \cdot [\langle a_1, a_2\rangle] \\ &= \langle\langle k_\alpha[d_1, d_2](\sigma), \xi_\alpha \rangle\rangle_\alpha = \boldsymbol{\kappa}_\alpha[d_1, d_2](\sigma)(\xi_\alpha), \end{aligned}$$

where the first equality follows from the distribution relations of Lemma 3.1.4. Therefore, the lemma follows form the inflation-restriction exact sequence and the fact that the $\Gamma_{\mathbb{Q}(\zeta_{p^{\alpha+1}})}$-invariants of

$$\begin{aligned} \mathrm{Hom}_{O[\mathfrak{I}^{\alpha+1}]}\big(\mathbb{H}^{2,1}(S_{\alpha+1} \times X_{\alpha+1})(1)^\dagger, O[\mathfrak{I}^\alpha]\big) &\cong \mathbb{H}^{2,1}(S_{\alpha+1} \times X_{\alpha+1})(2)^\dagger \otimes_{O[\mathfrak{I}^{\alpha+1}]} O[\mathfrak{I}^\alpha] \\ &\cong \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(2)^\dagger \end{aligned}$$

are trivial. □

Therefore, if we denote by

$$\mathbb{H}^{2,1}(S_\infty \times Y_\infty)(1)^\dagger = \lim_{\leftarrow,\alpha} \mathbb{H}^{2,1}(S_\alpha \times X_\alpha)(1)^\dagger$$

the projective limit obtained from the traces $(\varpi_1^2)_*$, there is a class

$$\boldsymbol{\kappa}_\infty[a, b] \in \mathrm{H}^1\big(\mathbb{Q}(\zeta_p), \mathrm{Hom}_{O[\![\mathfrak{I}]\!]}(\mathbb{H}^{2,1}(S_\infty \times X_\infty)(1)^\dagger, O[\![\mathfrak{I}]\!])\big)$$

for every element $[a, b] \in \mathscr{I}_\infty$. By taking nearly-ordinary parts, a generalization of Ohta's pairing ([Oht95], Theorem 4.2.5) gives an isomorphism

$$e^*_{\mathrm{n.o.}}\mathbb{H}^{2,1}(S_\infty \times Y_\infty)(2)^\dagger \cong \mathrm{Hom}_{O[\![\mathfrak{I}]\!]}(e^*_{\mathrm{n.o.}}\mathbb{H}^{2,1}(S_\infty \times X_\infty)(1)^\dagger, O[\![\mathfrak{I}]\!])$$

giving classes $\boldsymbol{\kappa}_\infty^{\mathrm{n.o.}}[a, b] \in \mathrm{H}^1\big(\mathbb{Q}(\zeta_p), e^*_{\mathrm{n.o.}}\mathbb{H}^{2,1}(S_\infty \times X_\infty)(2)^\dagger\big)$ for all $[a, b] \in \mathscr{I}_\infty$.

Remark 3.1.9. *The key point is that the pairing used to define the classes $\boldsymbol{\kappa}_\alpha[d_1, d_2]$ at finite level is not compatible with push-forwards along ϖ_1, while the generalization of Ohta's pairing is.*

Lemma 3.1.10. *Let $[a,b] \in \mathscr{I}_\infty$ and $\chi : \mathbb{G}_L(K)_{\mathrm{tor}} \to \overline{\mathbb{Q}}_p^\times$, $\psi : \mathbb{G}_{\mathbb{Q}}(K')_{\mathrm{tor}} \to \overline{\mathbb{Q}}_p^\times$ be characters such that $\chi_{|\mathbb{Q}} \cdot \psi \equiv 1$, then*

$$\boldsymbol{\kappa}_\infty^{\mathrm{n.o.}}[a,b]_{\chi,\psi} = \sum_{(a_1,a_2)\in \mathbb{G}_L(K)_{\mathrm{tor}} \times \mathbb{G}_{\mathbb{Q}}(K')_{\mathrm{tor}}} \chi(a_1)^{-1}\psi(a_2)^{-1}\boldsymbol{\kappa}_\infty^{\mathrm{n.o.}}[a\cdot a_1, b\cdot a_2]$$

belongs to $\mathrm{H}^1\big(\mathbb{Q}, e^*_{\mathrm{n.o.}}\mathbb{H}^{2,1}(S_\infty \times X_\infty)(2)^\dagger\big)$.

Proof. The order of $G(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is invertible in O, thus

$$\mathrm{H}^1\big(\mathbb{Q}, e^*_{\mathrm{n.o.}}\mathbb{H}^{2,1}(S_\infty \times X_\infty)(2)^\dagger\big) = \mathrm{H}^1\big(\mathbb{Q}(\zeta_p), e^*_{\mathrm{n.o.}}\mathbb{H}^{2,1}(S_\infty \times X_\infty)(2)^\dagger\big)^{G(\mathbb{Q}(\zeta_p)/\mathbb{Q})}.$$

The claim follows by computing that for $\sigma_c \in G(\mathbb{Q}(\zeta_p)/\mathbb{Q})$

$$\begin{aligned}
\big(\boldsymbol{\kappa}_\infty^{\mathrm{n.o.}}[a,b]_{\chi,\psi}\big)^{\sigma_c} &= \sum_{(a_1,a_2)} \chi(a_1)^{-1}\psi(a_2)^{-1}\boldsymbol{\kappa}_\infty^{\mathrm{n.o.}}[a\cdot a_1\gamma_c, b\cdot a_2\gamma_c] \\
&= \sum_{(a_1,a_2)} \chi(a_1\gamma_c)^{-1}\psi(a_2\gamma_c)^{-1}\boldsymbol{\kappa}_\infty^{\mathrm{n.o.}}[a\cdot a_1\gamma_c, b\cdot a_2\gamma_c] \\
&= \boldsymbol{\kappa}_\infty^{\mathrm{n.o.}}[a,b]_{\chi,\psi}.
\end{aligned}$$

□

Finally, let $\mathscr{G}$ be an $\mathbf{I}_{\mathscr{G}}$-adic Hida family of Hilbert cuspforms over L and $\mathscr{F}$ an $\mathbf{I}_{\mathscr{F}}$-adic Hida family of elliptic cuspforms of respective characters $\chi : \mathbb{G}_L(K)_{\mathrm{tor}} \to \overline{\mathbb{Q}}_p^\times$, $\psi : \mathbb{G}_{\mathbb{Q}}(K')_{\mathrm{tor}} \to \overline{\mathbb{Q}}_p^\times$ satisfying $\chi_{|\mathbb{Q}} \cdot \psi \equiv 1$.

Definition 3.1.11. *For every $[a,b] \in \mathscr{I}_\infty$ we define the big Hirzebruch-Zagier class*

$$\boldsymbol{\kappa}_{\mathscr{G},\mathscr{F}}[a,b] \in \mathrm{H}^1\big(\mathbb{Q}, \mathbf{V}^\dagger_{\mathscr{G},\mathscr{F}}\big)$$

attached to the pair $(\mathscr{G},\mathscr{F})$ of Hida families as the projection of $\boldsymbol{\kappa}_\infty^{\mathrm{n.o.}}[a,b]_{\chi,\psi}$ under the natural map $e^*_{\mathrm{n.o.}}\mathbb{H}^{2,1}(S_\infty \times X_\infty)(2)^\dagger \to \mathbf{V}^\dagger_{\mathscr{G},\mathscr{F}}$.

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