Department of Mathematics and Statistics
McGill University, Montreal

Metaplectic stacks and vector-valued modular forms of half-integral weight

Author:
Luca Candelori

Supervisor:
Prof. Henri Darmon

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Doctor in Philosophy.

Submitted: March 2014

Abstract

In this thesis we give a geometric theory of vector-valued modular forms attached to Weil representations, and in particular of modular forms of half-integral weight. More specifically, we construct over the integers a 'metaplectic' stack of elliptic curves and vector bundles $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ over it, $k \in \mathbb{Z}$, whose sections over the complex numbers give weight $k/2$ vector-valued modular forms attached to rank 1 lattices with quadratic form $x \mapsto mx^2/2$, for $m \in 2\mathbb{Z}_{>0}$.

The metaplectic stack is the stack of elliptic curves endowed with a non-degenerate rank one quadratic form. It is canonically endowed with a square root $\underline{\omega}^{1/2}$ of the Hodge bundle $\underline{\omega}$ of the moduli stack of elliptic curves. The vector bundles $\mathcal{V}_m$ are obtained from the Schrödinger representations of Heisenberg groups of elliptic curves: though the $\mathcal{V}_m$'s do not exist over the moduli stack of elliptic curves, we show that they can be defined over the metaplectic stack. We can then define q-expansions of vector-valued modular forms by pulling back sections of $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ to Tate curves endowed with a quadratic form.

We then find a canonical isomorphism between $\mathcal{V}_m \otimes \underline{\omega}^{-1/2}$ and the 'geometric' representations $\mathcal{J}_m$ of Heisenberg groups, given by the sections of totally symmetric invertible sheaves of degree m on elliptic curves. Using this isomorphism, we are able to give entirely geometric constructions of the classical level m single-variable theta functions and their theta constants, and prove that they are indeed vector-valued modular forms of half-integral weight, in our algebro-geometric sense. We also compute their q-expansions and show that they agree with the classical analytic q-expansions. In the case $m = 2$, we obtain a geometric theory of modular forms of half-integral weight, as defined by Shimura.

Finally we show that over the category of analytic spaces, the previous constructions recover the usual notions of vector-valued modular forms and modular forms of half-integral weight. In particular, the canonical isomorphism between $\mathcal{V}_m \otimes \underline{\omega}^{-1/2}$ and $\mathcal{J}_m$ turns into a well-known theorem of Eichler and Zagier relating vector-valued modular forms to Jacobi forms, and can be used to give a purely geometric proof of the analytic transformation laws of single-variable theta functions.

Abrégé

Dans cette thèse, nous présentons une théorie géometrique des formes modulaires à valeurs dans un fibré vectoriel attachées aux représentations de Weil et nous nous intéressons plus particulièrement aux formes modulaires de poids demi-entier. Plus précisément, nous construisons un champs 'métaplectique' et des fibrés vectoriels $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ ($k \in \mathbb{Z}$) au dessus de ce champs, dont les sections sur $\mathbb{C}$ nous donnent des formes modulaires à à valeurs dans un fibré vectoriel de poids $k/2$ attachées aux réseaux de rang 1 munis d'une forme quadratique $x \mapsto mx^2/2$ avec $m \in 2\mathbb{Z}_{>0}$.

Le champs métaplectique est un champs de courbes elliptiques dotées d'une forme quadratique non-dégénérée de rang 1. Il est canoniquement muni d'une racine carrée $\underline{\omega}^{1/2}$ du fibré de Hodge $\underline{\omega}$ du champs de modules des courbes elliptiques. Les fibrés vectoriels $\mathcal{V}_m$ proviennent des représentations de Schrödinger des groupes d'Heisenberg des courbes elliptiques: ces $\mathcal{V}_m$ n'existent cependant pas au dessus du champs de modules des courbes elliptiques mais nous montrons qu'ils peuvent être construits au dessus du champs métaplectique. Ceci nous permet ensuite de définir les q-expansions des formes modulaires à valeurs dans un fibré vectoriel en considérant les sections sur les courbes de Tate.

Nous trouvons ensuite un isomorphisme canonique entre les $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ et les représentations géometriques J_m des groupes d'Heisenberg données par les sections des faisceaux inversibles totalement symétriques de degré m sur les courbes elliptiques. Grâce à cet isomorphisme, nous sommes capables de donner des constructions géometriques des fonctions theta classiques de niveau m à une varable et de leur constante theta et nous prouvons qu'elles sont véritablement des formes modulaires à valeurs dans un fibré vectoriel de poids demi-entier dans le sens algebrico-géometrique que nous avons donné précedement.

Finalement, lorsque nous nous plaçons au dessus de la catégorie des espaces analytiques, les constructions précédentes nous redonnent les notions classiques de formes modulaires à valeurs dans un fibré vectoriel, de fonctions theta, de constantes theta et de formes modulaires à poids demi-entier. En particulier, l'isomorphisme canonique entre $\mathcal{V}_m \otimes \underline{\omega}^{-1/2}$ et $\mathcal{J}_m$ correspond à un théorème bien connu d'Eichler-Zagier reliant les formes modulaires à valeurs dans un fibré vectoriel et les formes de Jacobi et peut être utilisé pour donner une preuve purement géometrique des lois de transformations analytiques des fonctions theta à une variable.

Acknowledgments

I would like to thank my doctoral adviser, Henri Darmon, for his consistent support over the years. At a mathematical level, he has taught me the philosophy of considering modular forms as purely geometric objects, which is what this entire thesis is based on. At a personal level I have been inspired by his mentorship style, always spurring me on towards bold and daring new ideas, while letting me develop independently as a mathematician. For giving me the opportunity of pursuing my doctoral studies at McGill, and for always believing in me, I thank him very much.

I would also like to thank Niky Kamran for his teachings in complex and differential geometry, and especially for his kindness, for his availability and for his interest in this work.

Outside of McGill, I would like to acknowledge the support I have received over the years from Ken Ono. His work has been an endless source of inspiration for me. His interest has guided and fueled the development of this work and of my earlier graduate work on harmonic weak Maass forms.

I would also like to thank my earlier undergraduate mentors: Samit Dasgupta, for introducing me early on to Henri's circle of ideas and for teaching me how to write mathematics; and Barry Mazur, for being an inspiration inside and outside the seminar rooms of academia. I would also like to thank Chung Pang Mok for first introducing me to the classical theory of theta functions when I was still an undergraduate, and years later for suggesting to me that the theory of theta characteristics could be a starting point for a geometric theory of modular forms of half-integral weight.

From a financial point of view, I would like to thank the McGill Department of Mathematics and Statistics, McGill University, the ISM and the Schulich Family for funding my graduate studies, thus making the development of this work possible.

Finally, I would like to thank Eyal Goren and Vinayak Vatsal for helping me improve earlier drafts of this thesis and for the many interesting insights and comments about this work.

Contents

Introduction

The purpose of this work is to give a geometric interpretation to the theory of vector-valued modular forms. To be more precise, to construct vector bundles over the moduli stack of elliptic curves whose sections over the complex numbers correspond to the analytic notion of vector-valued modular forms. We thus want to liberate these objects from the formation of automorphic quotients, and instead give them a purely moduli-theoretic definition. The advantage of doing so is that we are then able to give purely algebraic definitions (for example, integral or mod p) of vector-valued modular forms, and in particular of modular forms of half-integral weight.

Vector-valued modular forms are a natural generalization of integral weight modular forms, at least from the point of view of embeddings of elliptic curves into projective space. In particular, let $E_\tau = \mathbb{C}/\langle \tau, 1\rangle$ be an elliptic curve over $\mathbb{C}$, with $\tau \in \mathfrak{h} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$, the complex upper half-plane. It is well-known that E_τ admits an embedding

$$E_\tau \hookrightarrow \mathbb{P}^2,$$

cut out in homogeneous coordinates $[X_0, X_1, X_2]$ by the equation

$$X_2 X_1^2 = 4X_0^3 - g_2(\tau)X_0X_2 - g_3(\tau)X_2^3.$$

As functions of τ, it is not hard to show that the coefficients $g_2(\tau)$ and $g_3(\tau)$ are holomorphic on $\mathfrak{h}$ and satisfy a functional equation of the form

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{1}$$

for $k = 4$ and $k = 6$ respectively. These are the prototypical examples of modular forms of integral weight, essentially corresponding to the Eisenstein series E_4 and E_6. The functional equation (1) is then taken as the *definition* of modular forms of integral weight: a modular form of integral weight $k \in \mathbb{Z}$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ that transforms like (1).

What about embeddings in projective spaces of higher dimension? For any $n \geq 3$, there are indeed embeddings

$$E_\tau \hookrightarrow \mathbb{P}^{n-1}.$$

The equations cutting out the image of E_τ in $\mathbb{P}^{n-1}$ are very classical and go back to 19th century, to Bianchi and Klein (see e.g. [34], §1 for a beautiful account). For example, for $m = 2n \geq 4$ a positive even integer, these equations are given in homogeneous coordinates $[X_\mu]_{\mu\in\mathbb{Z}/m\mathbb{Z}}$ by quadratic expressions of the form

$$s_{\alpha-\beta}(\tau)s_{\gamma-\delta}(\tau)X_{\alpha+\beta}X_{\gamma+\delta} + s_{\alpha-\gamma}(\tau)s_{\delta-\beta}(\tau)X_{\alpha+\gamma}X_{\delta+\beta} + s_{\alpha-\delta}(\tau)s_{\beta-\gamma}(\tau)X_{\alpha+\delta}X_{\beta-\gamma} = 0,$$

where $(\alpha, \beta, \gamma, \delta)$ are either all in $\mathbb{Z}/m\mathbb{Z}$ or all in $1/2 + \mathbb{Z}/m\mathbb{Z}$ ([34], 1.2). The vector of functions $(s_\mu(\tau))_{\mu\in\mathbb{Z}/m\mathbb{Z}}$ is holomorphic on $\mathfrak{h}$. It is the prototypical example of a vector-valued modular form, essentially corresponding to the vector of theta constants $\theta_{\mathrm{null},m}(\tau)$, whose μ-component is given by:

$$\theta_{m,\mu}(\tau) = \sum_{\substack{n\in\mathbb{Z} \\ n\equiv\mu \mod m}} e^{\pi i\tau n^2/m}.$$

The functional equations of $(s_\mu(\tau))_{\mu\in\mathbb{Z}/m\mathbb{Z}}$ with respect to linear fractional transformations are more involved than (1). In particular, we have ([14], §5, (6) and (8)):

$$\begin{aligned} s_\mu(\tau+1) &= e^{\pi i\mu^2/m}\, s_\mu(\tau) \\ s_\mu\left(-\frac{1}{\tau}\right) &= \sqrt{\tau/mi} \sum_{\nu\in\mathbb{Z}/m\mathbb{Z}} e^{-2\pi i\mu\nu/m}\, s_\nu(\tau), \end{aligned} \tag{2}$$

where $\sqrt{\ }$ is the principal value of the square root, with $-\pi/2 < \arg(\sqrt{\ }) \leq \pi/2$. The functional equations (2) are taken as the *definition* of vector-valued modular forms of weight 1/2: a vector-valued modular form of weight 1/2 is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}^m$ such that its components $(f_\mu(\tau))_{\mu\in\mathbb{Z}/m\mathbb{Z}}$ transform according to (2). Vector-valued modular forms of weight $k/2$, for $k \in \mathbb{Z}$, are similarly defined by replacing $\sqrt{\tau}$ with $\sqrt{\tau}^k$ in the second line of (2). These natural generalizations of modular forms of integral weight were first introduced by Eichler and Zagier ([14]), and the theory was further developed by Borcherds (e.g. [4], [5], [3]) in the context of theta lifts, automorphic infinite products and Gross-Kohnen-Zagier-type formulas. Today, they appear prominently in the work of Bruinier and Ono (e.g. [6], [7]) as the overarching framework for the arithmetic and combinatorial study of q-series, especially those arising from modular forms of half-integral weight.

Vector-valued modular forms can alternatively be viewed as a 'theta-function free' way of packaging and studying the transformation laws of single-variable theta functions. This approach was pioneered by Shimura ([31]) in his theory of *modular forms of half-integral weight*, which indeed arise as a special case of vector-valued modular forms. In particular, consider the theta series

$$\theta_0(\tau) := \theta_{2,0}(\tau) = \sum_{\substack{n\in\mathbb{Z} \\ n\equiv 0 \mod 2}} q^{n^2/4} = \sum_{n\in\mathbb{Z}} q^{n^2}, \quad q = e^{2\pi i\tau},$$

which is the first component of $\theta_{\mathrm{null},2}(\tau)$, a vector-valued modular form of weight $1/2$. Then Shimura defines:

Definition ([31]). Let $k \in \mathbb{Z}$. A modular form of half-integral weight $k/2$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that:

$$f(\tau)\theta_0(\tau)^{-k}$$

is a weight 0 modular function.

Modular forms of half-integral weight are extremely useful as generating series for arithmetic data arising from quadratic extensions. For example, consider

$$f(q) := \theta_0(q)^3 = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + \dots,$$

a modular form of half-integral weight $3/2$. The coefficients of $f(q)$ essentially correspond to class numbers of imaginary quadratic fields (e.g. [28], §8.2). Understanding $f(q)$ thus provides insights into the arithmetic of imaginary quadratic fields. More generally, the whole family of modular forms of half-integral weight contains many other q-series of arithmetic interest, for example generating series for real quadratic class numbers, generating series for special L-values of quadratic twists of elliptic curves and so on. For a survey of these results, the reader may consult [28], especially Sections 8 and 9.

Now the theory of modular forms of integral weight has a very meaningful geometric interpretation, first pioneered by Shimura and fully developed later by authors such as Deligne ([11]) and Katz ([19]). This geometric interpretation is the cornerstone for the construction and classification of 2-dimensional Galois representations, one of the central topics of modern number theory. The goal of this thesis is to lay the foundations for a similar geometric theory of vector-valued modular forms.

To better explain what we mean by a 'geometric theory', let's recall briefly the geometric theory of modular forms of integral weight, which will serve as a model in all that follows. The first observation is that the transformation law (1) of modular forms of integral weight k defines a cocycle

$$j_k : \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathcal{O}_{\mathfrak{h}}^*$$
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (c\tau + d)^k.$$

This cocycle defines a line bundle $\underline{\omega}^k$ over the orbifold (in the sense of [17]) $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$: modular forms can then be viewed as holomorphic global sections of $\underline{\omega}^k$. Now the automorphic quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ has a moduli-theoretic interpretation as the moduli stack $\mathcal{M}_1^{\mathrm{an}}$ of elliptic curves over an analytic space. Under this interpretation, $\underline{\omega}^k$ corresponds to the invertible sheaf on $\mathcal{M}_1^{\mathrm{an}}$ given by the functor

$$\{E \to S\} \longmapsto \Gamma(S, \underline{\omega}^k_{E/S}), \tag{3}$$

that to each elliptic curve $E \to S$ over an analytic space assigns the $\Gamma(S, \mathcal{O}_S)$-module of global sections of tensor powers of its Hodge bundle.

The functor (3) also makes sense over the moduli stack $\mathcal{M}_1$ of elliptic curves $E \to S$ over a scheme S. As such, it defines an invertible sheaf $\underline{\omega}^k$ over $\mathcal{M}_1$ whose global sections are the algebraic incarnations of modular forms of integral weight. We thus can speak, for example, of *modular forms mod p*, by restricting $\underline{\omega}^k$ to the moduli stack of elliptic curves over a $\mathbb{F}_p$-scheme. We can also restrict $\underline{\omega}^k$ to the Tate elliptic curve, defined over the power series ring $\mathbb{Z}(\!(q)\!)$: the sections so obtained are power series in q, and when we substitute $q = e^{2\pi i\tau}$ for $\tau \in \mathfrak{h}$ we obtain the classical q-expansions of modular forms ([11], [13], [19]).

This is what we mean by a 'geometric theory' of modular forms: a theory that unifies the arithmetic, combinatorial and analytic aspects of modular forms by replacing the study of cocycles over automorphic quotients by that of vector bundles over the moduli space of elliptic curves. It is the goal of this thesis to formulate a similar theory for vector-valued modular forms. In particular, we would like to understand in what sense q-series of the form

$$\theta_0(q) = \sum_{n\in\mathbb{Z}} q^{n^2}, \quad f(q) = \theta_0(q)^3 \in \mathbb{Z}[\![q]\!],$$

are modular forms of half-integral weight, *without* having to plug in $q = e^{2\pi i\tau}$ and appeal to the analytic theory. Yet in other words, to interpret the functional equations satisfied by single-variable theta functions as an intrinsic property of their definition as elements in $\mathbb{Z}[\![q]\!]$, where q is an abstract variable, and *not* as a property of the analytic functions they define when letting $q = e^{2\pi i\tau}$. The impatient reader might wish to jump directly to the very end of Chapter 2 to see how this is accomplished.

For the patient reader, we would now like to give an overview of our constructions and of our results.

The first complication that arises when considering vector-valued modular forms is the appearance of square roots in the functional equation (2). Namely, we have only defined the transformations laws with respect to generators of $\mathrm{SL}_2(\mathbb{Z})$. To extend the transformation laws (2) to a representation of all of $\mathrm{SL}_2(\mathbb{Z})$ we must specify compatible choices of square roots. In particular, following Shimura [31], define the *metaplectic group* $\mathrm{Mp}_2(\mathbb{Z})$ as the group of pairs

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi(\tau)\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),\ \phi(\tau) \in \mathcal{O}_{\mathfrak{h}}^*,\ \phi^2(\tau) = c\tau + d,$$

with multiplication given by:

$$(A_1, \phi_1(\tau)) \cdot (A_2, \phi_2(\tau)) = (A_1 A_2, \phi_1(A_2\tau)\phi_2(\tau)).$$

The metaplectic group is the unique non-trivial central extension of $\mathrm{SL}_2(\mathbb{Z})$ by $\mu_2 = \{\pm 1\}$. It is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right), \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right).$$

Let now $m \in 2\mathbb{Z}_{>0}$ be a positive even integer, and let $V_m := \mathbb{C}[\mathbb{Z}/m\mathbb{Z}]$ be the vector space of $\mathbb{C}$-valued functions on $\mathbb{Z}/m\mathbb{Z}$. We can define a representation ρ_m of $\mathrm{Mp}_2(\mathbb{Z})$ on V_m by the formulas

$$\rho_m(T)(\delta_\mu) = e^{\pi i \mu^2/m}\, \delta_\mu$$
$$\rho_m(S)(\delta_\mu) = \frac{1}{\sqrt{im}} \sum_{\nu \in \mathbb{Z}/m\mathbb{Z}} e^{-2\pi i \mu\nu/m}\, \delta_\nu,$$

where by $\{\delta_\mu\}_{\mu \in \mathbb{Z}/m\mathbb{Z}}$, we denote the basis of delta functions of V_m, such that δ_μ takes the value 1 at μ and 0 everywhere else. This is the *Weil representation* ρ_m attached to rank 1 lattices $(\mathbb{Z}, x \mapsto mx^2/2)$ (e.g. [4]). It is a not-so-distant relative of certain Hilbert-space representations appearing in quantum mechanics: it was Weil ([36]) who first discovered arithmetic analogs of these physical phenomena which could explain, among other things, generalized quadratic reciprocity laws.

Using the Weil representation we can make a more rigorous definition of vector-valued modular forms:

Definition. Let $k \in \mathbb{Z}$ and $m \in 2\mathbb{Z}_{>0}$. A weight $k/2$, ρ_m-valued modular form is a holomorphic function $f : \mathfrak{h} \to V_m$ satisfying the transformation law:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \phi^k \rho_m(\gamma) f(\tau), \quad \forall\, \gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) \in \mathrm{Mp}_2(\mathbb{Z}).$$

In particular, plugging in S and T in the above definition gives back the transformation laws (2).

The ‘geometrization’ of vector-valued modular forms begins as in the integral weight case.

We can indeed consider them as holomorphic global sections of the vector bundle

$$\mathcal{V}_m \otimes \underline{\omega}^{k/2}$$

over the orbifold $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$, where $\mathrm{Mp}_2(\mathbb{Z})$ acts on $\mathfrak{h}$ via the map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$, i.e. via linear-fractional transformations of the underlying matrix, and:

(i) $\mathcal{V}_m$ is the local system over $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ corresponding to the representation ρ_m,

(ii) $\underline{\omega}^{k/2}$ is the line bundle over $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ corresponding to the cocycle:

$$\begin{aligned} j_{k/2} : \mathrm{Mp}_2(\mathbb{Z}) &\longrightarrow \mathcal{O}_{\mathfrak{h}}^* \\ \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) &\longmapsto \phi^k. \end{aligned}$$

The problem now is to give these objects an interpretation in terms of moduli of elliptic curves. As in the geometric theory of modular forms of integral weight, the starting point is to give an algebraic description of the automorphic quotient $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$. We show that this can indeed be viewed as the moduli stack $\mathcal{M}_{1/2}^{\mathrm{an}}$ of elliptic curves $E \to S$ over an analytic space equipped with a non-degenerate rank 1 quadratic form:

$$q : \mathcal{Q} \longrightarrow \underline{\omega}_{E/S},$$

i.e. an invertible sheaf $\mathcal{Q}$ and a map of abelian sheaves q such that q induces a $\mathcal{O}_S$-module isomorphism $\mathcal{Q}^{\otimes 2} \simeq \underline{\omega}_{E/S}$. This interpretation leads to the definition of the *metaplectic stack* $\mathcal{M}_{1/2}$ (rigorously defined in Definition 2.1.2):

Definition. The *metaplectic stack* $\mathcal{M}_{1/2}$ is the moduli stack of pairs $(E/S, \mathcal{Q})$ of an elliptic curve $E \to S$ over a scheme equipped with a quadratic form $q : \mathcal{Q} \to \underline{\omega}_{E/S}$.

The metaplectic stack is canonically endowed with an invertible sheaf $\underline{\omega}^{1/2}$ given by the functor

$$\{(E/S, \mathcal{Q})\} \longmapsto \Gamma(S, \mathcal{Q}),$$

and such that

$$p^*\underline{\omega} \simeq (\underline{\omega}^{1/2})^{\otimes 2},$$

where $p : \mathcal{M}_{1/2} \to \mathcal{M}_1$ is the 'forget the quadratic form' functor. Over the category of analytic spaces the two definitions of $\underline{\omega}^{1/2}$ via the cocycle $j_{1/2}$ and via this algebraic definition coincide. Thus, the invertible sheaf $\underline{\omega}^{1/2}$ over $\mathcal{M}_{1/2}$ can be viewed as a purely algebraic analog of the line bundle over $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ defined by $j_{1/2}$, and similarly for the higher tensor powers $\underline{\omega}^{k/2}$.

The construction of vector bundles $\mathcal{V}_m$ over $\mathcal{M}_{1/2}$ which give the Weil representation over $\mathcal{M}_{1/2}^{\mathrm{an}} = \mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ is a bit more involved, but it is what gives life to the theory. These vector bundles naturally stem out of Mumford's theory of algebraic theta functions ([22], [23]), a foundational piece of mathematics that has inspired the development of this thesis. In fact it is Mumford, in the introduction to [22], who states that

> "There are several interesting topics which I have not gone into in this paper, but which can be investigated in the same spirit: for example, [...] a discussion of the transformation theory of theta-functions".

The idea is to consider, for any elliptic curve $E \to S$ over a scheme and $m \in 2\mathbb{Z}_{>0}$ a positive even integer, the *level m Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$* (see Section 1.3.2 below for definitions), a flat affine group scheme over S. We can construct weight 1, rank m representations $\mathcal{V}_H$ of $\mathcal{G}_E(\mathcal{L}_m)$ starting from lagrangian subgroups $H \subset \mathcal{G}_E(\mathcal{L}_m)$. These are called *Schrödinger representations* of $\mathcal{G}_E(\mathcal{L}_m)$, in analogy with their analytic relatives in quantum mechanics. From the general theory of Heisenberg groups, if there is a weight 1, rank m representation $\mathcal{V}_H$ of $\mathcal{G}_E(\mathcal{L}_m)$ it must be unique up to tensoring with an invertible sheaf over S. Thus the functor:

$$\{E \to S\} \longmapsto \mathrm{End}_{\mathcal{O}_S}(\mathcal{V}_H)$$

is independent of the choice of H and defines a sheaf $\mathcal{A}_m$ of $\mathcal{O}_{\mathcal{M}_1}$-algebras over the moduli stack $\mathcal{M}_1$. This is an Azumaya algebra ([16]), whose order in the Brauer group $H^2(\mathcal{M}_1, \mathbb{G}_m)$ is two. From Giraud's general theory of torsor lifting (see Section 1.1.3), we can then find a locally free sheaf $\mathcal{V}_m$ of rank m over the metaplectic stack $\mathcal{M}_{1/2}$ such that:

$$p^*\mathcal{A}_m \simeq \mathrm{End}(\mathcal{V}_m).$$

Over the analytic category, $\mathcal{V}_m$ (or rather its dual $\mathcal{V}_m^\vee$, but conventions vary) is precisely the local system corresponding to the Weil representation ρ_m defined above.

We can now define:

Definition. Let $k \in \mathbb{Z}$ and let $m \in 2\mathbb{Z}_{>0}$. A weight $k/2$, $\mathcal{V}_m$-valued modular form is a global section of

$$\mathcal{V}_m \otimes \underline{\omega}^{k/2}$$

over $\mathcal{M}_{1/2}$.

This gives a purely algebraic definition of vector-valued modular forms. For example, we could restrict $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ to the moduli stack of elliptic curves over a $\mathbb{F}_p$-scheme to obtain a mod p theory of vector-valued modular forms. Or we could restrict to the Tate curve to obtain the q-expansions of vector-valued modular forms in a purely algebraic way.

An interesting phenomenon is that when k is odd the sheaf $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ descends to the modular stack $\mathcal{M}_1$ (Theorem 2.2.14). In fact, we can prove that there is a canonical isomorphism (Theorem 2.2.20):

$$\mathcal{V}_m \otimes \underline{\omega}^{-1/2} \simeq \mathcal{J}_m$$

where $\mathcal{J}_m$ is the sheaf on $\mathcal{M}_1$ associated to the functor

$$\{\pi : E \to S\} \longmapsto \Gamma(S, \pi_* \mathcal{L}_m),$$

where $\mathcal{L}_m = \mathcal{O}_E(m\, e) \otimes (\Omega^1_{E/S})^{\otimes m}$ is the unique totally symmetric invertible sheaf of degree m on E, normalized along e. Over the analytic category, sections of $\mathcal{J}_m$ essentially correspond to weight 0, index $m/2$ Jacobi forms up to a simple factor. In particular, by tensoring both sides with $\underline{\omega}^k$ we obtain an isomorphism

$$\mathcal{V}_m \otimes \underline{\omega}^{k-1/2} \simeq \mathcal{J}_m \otimes \underline{\omega}^k, \tag{4}$$

between weight $k - 1/2$, $\mathcal{V}_m$-valued modular forms and weight k, index $m/2$ Jacobi forms. This is an algebraic analog of the isomorphism discovered by Eichler and Zagier in [14].

The reader fluent in geometric quantization theory might start to see an analogy between our theory and that of projectively flat connections and the metaplectic correction (e.g. [18]). Indeed, we show in Theorem 2.2.19 that the vector bundle $\mathcal{V}_m$ is flat for the étale topology of $\mathcal{M}_{1/2}$. Thus the sheaf $\mathcal{J}_m \otimes \underline{\omega}^{1/2}$ can be endowed with a canonical integrable connection ∇. This connection essentially corresponds to the heat equations satisfied by theta functions ([18], [37]).

Finally, we would like to mention how to obtain an algebraic interpretation of the functional equations of single-variable theta functions, and in particular of Shimura's modular forms of half-integral weight.

The geometric construction of theta constants is well-known. It follows from the fact that over an elliptic curve E/S the sheaf $\mathcal{L}_m$ defined above is normalized along the identity e, i.e. $e^* \mathcal{L}_m \simeq \mathcal{O}_S$. Hence the map e^* defines an 'evaluation at e' element

$$\mathrm{ev}_e \in \Gamma(S, \mathcal{L}_m^\vee).$$

The assignment

$$\{E \to S\} \longmapsto \mathrm{ev}_e \in \Gamma(S, \mathcal{L}_m^\vee)$$

gives a section of $\mathcal{J}_m^\vee$ over $\mathcal{M}_1$. By our theorem (4), it must also define a section

$$\theta_{\mathrm{null},m} \in \Gamma(\mathcal{M}_1, \mathcal{V}_m^\vee \otimes \underline{\omega}^{1/2}),$$

i.e. theta constants *are* modular forms of half-integral weight, in our algebro-geometric sense, as they should be. To make the connection with q-expansions, we pull-back $\theta_{\mathrm{null},m}$ to the Tate curve and compute it as a vector of power series in q (Section 2.3.2) to recover the classical q-expansions of theta constants. This is a standard computation in the theory of modular forms. Our contribution is that we can now speak of q-series defining theta constants as vector-valued modular forms of half-integral weight without having to plug in $q = e^{2\pi i\tau}$ and appealing to the analytic theory.

The case $m = 2$ is of particular interest. In Section 2.3.4 we define an invertible subsheaf

$$\mathcal{L}_{\mathrm{Shi}} \subseteq \mathcal{V}_2^{\vee} \otimes \underline{\omega}^{1/2}$$

over the moduli stack $\mathcal{M}_0(4)$, and a section $\theta_0 \in \Gamma(\mathcal{M}_0(4), \mathcal{L}_{\mathrm{Shi}})$ such that the q-expansion over the Tate curve is given by:

$$\theta_0(q) = \sum_{n\in\mathbb{Z}} q^{n^2} \in \mathbb{Z}[\![q]\!].$$

We thus have a completely algebraic theory of modular forms of half-integral weight, by considering sections of tensor powers $\mathcal{L}_{\mathrm{Shi}}^{\otimes k}$. In particular, one could construct the generating series $f(q)$ for quadratic imaginary class numbers and consider it as a section of $\mathcal{L}_{\mathrm{Shi}}^{\otimes 3}$ modulo a prime p. This could be a possible approach for studying divisibility properties of class numbers.

The above theory is worked out in the following order, though more details on how the thesis is structured can be found in the introductions to each chapter.

Chapter 1 contains general tools and techniques needed for the construction of algebraic vector-valued modular forms. Perhaps the key tool is the 'torsor-lifting' theory of Giraud ([15]), which measures the obstruction to extending the structure group of a torsor in terms of gerbes. In Section 1.3 we also review the theory of Heisenberg groups and their Schrödinger representations, mainly following Moret-Bailly ([20]) and Mumford ([24]).

In Chapter 2 we turn to the construction of a geometric theory of vector-valued modular forms as outlined above. In Section 2.1.1 we construct the metaplectic stack $\mathcal{M}_{1/2}$. In Section 2.2.2 we construct the bundle $\mathcal{V}_m$ of Schrödinger representations and then we proceed in proving (4) in Sections 2.2.4 and 2.2.5. We then define q-expansions of vector-valued modular forms (Section 2.2.7) and compute the q-expansions of theta constants (Section 2.3.1). The theory of half-integral weight modular forms and their q-expansions is given in 2.3.4.

In Chapter 3, we work out the theory of Chapter 2 in the analytic category, to recover the usual notions of vector-valued modular forms, Jacobi forms and modular forms of half-integral weight. This chapter was written, and was meant to be read, in parallel with Chapter 2: the reader will notice that the sections in each chapter mirror each other.

Finally, in Chapter 4 we draw some philosophical conclusions about the geometric study of vector-valued modular forms, and speculate about directions for future investigations in the subject.

A note about terminology Throughout this work, by a 'scheme' S we mean a scheme which is separated and locally noetherian. On the other hand, we almost always invert m, where m is a positive even integer. In particular, we almost always invert 2 in our base schemes.

Chapter 1

Background

In this chapter we lay out the basic tools needed for the development of a geometric theory of vector-valued modular forms. Perhaps the most important concept of the chapter is that of 'torsor-lifting': roughly speaking, this is a technique that given a central exact sequence of groups

$$0 \to A \to B \to C \to 0,$$

it measures the obstruction to lifting a C-torsor over a 'space' X (a topological space, a scheme, a stack, a site...) to a B-torsor in terms of $H^2(X, A)$. It can be viewed as a vast generalization of the theory of lifting projective representations to linear representations via Schur multipliers. Our main reference for this torsor-lifting theory is Giraud ([15]). In this work, among many other ideas, it is shown that the obstruction vanishes if one is willing to move from the base space X to an *A-gerbe* $\mathcal{K} \to X$, some kind of higher-categorical version of a torsor. In later chapters, this technique will be used (a) to construct canonical square roots of line bundles over the moduli stack of elliptic curves, and (b) to lift projective bundles over the moduli stack of elliptic curves to vector bundles. These projective bundles will arise from the canonical projective representations attached to *level m Heisenberg groups*, which are central extensions of the m-torsion of elliptic curves.

The torsor-lifting theory of Giraud is summarized in Section 1.1. Since this theory makes essential use of the language of stacks and gerbes, we also recall the basic definitions related to these concepts following the open-source reference [33].

In Section 1.2, we make use of Giraud's theory to study the problem of existence of rank one quadratic forms over Deligne-Mumford stacks, and the problem of lifting vector bundles defined up to ± 1 over Deligne-Mumford stacks to honest vector bundles. These problems are motivated, respectively, to (a) and (b) above.

Finally, in Section 1.3 we summarize the theory of Heisenberg group schemes following [20], §5, and [24], §23. We depart a bit from our sources in Section 1.3.5, where we construct canonical Azumaya algebras of order 2 in the Brauer group associated to *symmetric* Heisenberg group schemes.

1.1 Stacks, gerbes and torsor-lifting

1.1.1 Deligne-Mumford stacks

Let $\mathcal{C}$ be a category and let $p : \mathcal{S} \to \mathcal{C}$ be a functor from a category $\mathcal{S}$. For any $U \in \mathrm{Ob}(\mathcal{C})$ let $\mathcal{S}_U$ be the *fiber category* over U ([33], Tag 02XH).

DEFINITION 1.1.1. ([33], Tag 003T) The category $\mathcal{S}$ is *fibered in groupoids* over $\mathcal{C}$ if

(a) For every morphism $f : V \to U$ in $\mathcal{C}$ and $y \in \mathrm{Ob}(\mathcal{S}_U)$ there is a morphism $\phi : y \to x$ in $\mathcal{S}$ such that $p(\phi) = f$.

(b) For every pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.

Properties (a) and (b) above imply that for each morphism $f : V \to U$ there is a 'pull-back' functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$.

Suppose now that the category $\mathcal{C}$ has the further structure of a *site* ([33], Tag 00VH). For any $U \in \mathrm{Ob}(\mathcal{C})$ denote by $\mathcal{C}/U$ the site whose objects are morphisms $V \to U$ ([33], Tag 00XZ).

DEFINITION 1.1.2 ([33], Tag 02ZI). Let $\mathcal{C}$ be a site. A *stack in groupoids*, or simply a *stack*, is a category $\mathcal{S}$ equipped with a functor $p : \mathcal{S} \to \mathcal{C}$ such that:

(a) $\mathcal{S}$ is fibered in groupoids over $\mathcal{C}$.

(b) For any triple $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}(\mathcal{S}_U)$, the functor on $\mathcal{C}/U$ that to any $f : V \to U$ it associates $\mathrm{Isom}_{\mathcal{S}_V}(f^*x, f^*y)$ is a sheaf on $\mathcal{C}/U$.

(c) For any covering $\{f_i : U_i \to U\}$ of the site $\mathcal{C}$, any descent datum in $\mathcal{S}$ relative to the f_i is effective.

Property (c) in the definition of a stack means that if objects $x_i \in \mathrm{Ob}(\mathcal{S}_{U_i})$ are given together with gluing isomorphisms over the products $U_i \times U_j$, satisfying a cocycle condition over triple products, then the x_i descend to define an object $x \in \mathrm{Ob}(\mathcal{S}_U)$ such that $f_i^* x = x_i$ ([33], Tag 02ZC).

The category of stacks $p : \mathcal{S} \to \mathcal{C}$ over a fixed base forms a 2-category where the 1-morphisms are functors commuting with the projection functor p to $\mathcal{C}$, and whose 2-morphisms are natural transformations of functors ([33], Tag 02ZG). If $\mathcal{X} \to \mathcal{S}$ and $\mathcal{Y} \to \mathcal{S}$ are 1-morphisms in the category of stacks over $\mathcal{C}$, then the 2-fiber product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ exists ([33], Tag 026G).

For any $U \in \mathrm{Ob}(\mathcal{C})$, the category $\mathcal{C}/U \to \mathcal{C}$ is a stack which we simply denote by U. If $p : \mathcal{S} \to \mathcal{C}$ is a stack, the 2-Yoneda Lemma ([33], Tag 004B) says that there is an equivalence of categories:

$$\mathcal{S}_U = \mathrm{Hom}_{Stacks/\mathcal{C}}(U, \mathcal{S}) \tag{1.1}$$

for any $U \in \mathrm{Ob}(\mathcal{C})$, viewed as a stack.

Let now $Sch_{étale}$ be the site given by the category of schemes endowed with the big étale topology, i.e. coverings of an object $S \in \mathrm{Ob}(Sch_{étale})$ are collections of étale morphisms of schemes $\{U_\alpha \to S\}$ with $\coprod U_\alpha \to S$ surjective. As above, for any scheme S we will also denote by S the corresponding stack $S \to Sch_{étale}$.

Definition 1.1.3 ([33], Tags 026O, 03YO). A *Deligne-Mumford stack* is a stack $\mathcal{S} \to Sch_{étale}$ such that:

(i) For any triple $S \in \mathrm{Ob}(Sch_{étale})$ and $X, Y \in \mathrm{Ob}(\mathcal{S}_S)$, the functor on $Sch_{étale}/U$ that to any $f : U \to S$ it associates $\mathrm{Isom}_{\mathcal{S}_U}(f^*X, f^*Y)$ is representable.

(ii) There exists a scheme U and a 1-morphism of stacks $U \to \mathcal{S}$ that is étale and surjective, i.e. for any other scheme Y and 1-morphism $Y \to \mathcal{S}$, the morphism of schemes $U \times_{\mathcal{S}} Y \to Y$ is étale and surjective.

Note in condition (ii) that $U \times_{\mathcal{S}} Y$ is a scheme by property (i).

Remark 1.1.4. Compared to the definitions in [33], we have replaced the fppf topology by the étale topology, and in (i) we require $\mathrm{Isom}_{\mathcal{S}_U}(f^*X, f^*Y)$ to be representable by a scheme and not by an algebraic space.

The *étale topology* of a Deligne-Mumford stack $p : \mathcal{S} \to Sch_{étale}$ is the topology inherited from the étale topology of $Sch_{étale}$ ([33], Tag 06NU). Explicitly:

Definition 1.1.5. Let $\mathcal{S} \to Sch_{étale}$ be a Deligne-Mumford stack. The *étale site* $\mathcal{S}_{ét}$ of $\mathcal{S}$ is the site whose underlying category is $\mathcal{S}$ and whose coverings of an object $X \in \mathrm{Ob}(\mathcal{S})$ are families of morphisms $\{X_i \to X\}$ such that $\{p(X_i) \to p(X)\}$ is a covering family in $Sch_{étale}$. Equivalently (by (1.1)), it is the category whose objects are 1-morphisms $x : X \to \mathcal{S}$ with X a scheme, and whose morphisms $(X, x) \to (Y, y)$ are morphisms of schemes $f : X \to Y$ plus a natural transformation between x and y, and whose coverings $(X_i, x_i) \to (X, x)$ are morphisms whose underlying morphisms of schemes are coverings in $Sch_{étale}$.

A *sheaf* on a Deligne-Mumford stack $\mathcal{S}$ is a contravariant functor $\mathcal{F} : \mathcal{S}_{ét} \to \mathcal{A}$ into some category $\mathcal{A}$ (sets, abelian groups, rings...) satisfying the usual descent properties of a sheaf with respect to the étale topology of $\mathcal{S}$ ([33], Tag 06TR). In particular, the *structure sheaf* $\mathcal{O}_{\mathcal{S}}$ of $\mathcal{S}$ is the sheaf of rings defined by the functor ([33], Tag 06TV):

$$\{X \in \mathrm{Ob}(\mathcal{S})\} \longmapsto \Gamma(p(X), \mathcal{O}_{p(X)}).$$

Thus the pair $(\mathcal{S}_{\acute{e}t}, \mathcal{O}_{\mathcal{S}})$ has the structure of a *ringed site*. The notions of *quasi-coherent* $\mathcal{O}_{\mathcal{S}}$-module, *coherent* $\mathcal{O}_{\mathcal{S}}$-module, *locally free* $\mathcal{O}_{\mathcal{S}}$-module and so on are defined as for any ringed site ([33], Tags 06WG, 03DL).

The categories of sheaves of abelian groups and of sheaves of $\mathcal{O}_{\mathcal{S}}$-modules have enough injectives, hence if $\mathcal{F}$ is a sheaf in any of these two categories, the étale cohomology groups $H^i_{\acute{e}t}(\mathcal{S}, \mathcal{F})$ are well-defined ([33], Tag 075E).

1.1.2 Torsors and gerbes

Let $\mathcal{C}$ be a site.

Definition 1.1.6 ([33], Tag 03AH). Let G be a sheaf of groups on $\mathcal{C}$. A *G-torsor* over $\mathcal{C}$ is a sheaf of sets $\mathcal{F}$ together with an action $G \times \mathcal{F} \to \mathcal{F}$ such that:

(i) Whenever $U \in \mathrm{Ob}(\mathcal{C})$ is such that $\mathcal{F}(U)$ is non-empty, the action:

$$\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$$

is simply transitive.

(ii) For every $U \in \mathrm{Ob}(\mathcal{C})$, there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}(U_i)$ is non-empty for all i.

A morphism of G-torsors is a morphism of sheaves commuting with the G-action. The *trivial* G-torsor is the sheaf G endowed with the left action on itself. We denote by:

$$H^1(\mathcal{C}, G)$$

the set of isomorphism classes of G-torsors over the site $\mathcal{C}$ ([15], III.2.4.2). The class of the trivial torsor gives a canonical element in this set, hence $H^1(\mathcal{C}, G)$ has the structure of a *pointed set*.

Intuitively, a G-torsor is an object that locally on $\mathcal{C}$ looks like the group G acting on itself. The next natural step is to construct categories that locally on $\mathcal{C}$ look like the category of G-torsors. This is accomplished by the notion of a *G-gerbe*:

Definition 1.1.7 ([33] Tag 06NZ, [15] IV.2.2.2). Let G be an abelian group. A stack $\mathcal{K} \to \mathcal{C}$ is a *G-gerbe* over $\mathcal{C}$ if it satisfies:

(a) For every $U \in \mathrm{Ob}(\mathcal{C})$, there exists an open covering $\{U_i \to U\}$ such that $\mathcal{K}_{U_i}$ is non-empty.

(b) For any triple $U \in \mathrm{Ob}(\mathcal{C})$, $x, y \in \mathrm{Ob}(\mathcal{K}_U)$ there exists an open covering $\{\varphi_i : U_i \to U\}$ such that $\varphi_i^* x \simeq \varphi_i^* y$ in $\mathcal{K}_{U_i}$.

(c) For any pair $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathcal{K}_U$, the sheaf on $\mathcal{C}/U$ that to each $f : V \to U$ it associates $\mathrm{Aut}_{\mathcal{K}_V}(f^*x)$ is isomorphic to the constant sheaf of fiber G.

Two G-gerbes $\mathcal{K}_1, \mathcal{K}_2$ over $\mathcal{C}$ are *equivalent* if they are equivalent as categories over $\mathcal{C}$. The *trivial* G-gerbe is the category $\mathrm{Tor}(\mathcal{C}, G)$ of G-torsors over $\mathcal{C}$ ([15], IV.3.1.1.2). For any two G-gerbes $\mathcal{K}_1, \mathcal{K}_2$ over $\mathcal{C}$, the product category $\mathcal{K}_1 \times \mathcal{K}_2$ is a $G \times G$-gerbe over $\mathcal{C}$. The *contracted product* $\mathcal{K}_1 \times^G \mathcal{K}_2$ ([15], IV.2.4) is the G-gerbe over $\mathcal{C}$ whose objects are the same as those of $\mathcal{K}_1 \times \mathcal{K}_2$ and the morphisms $X = (X_1, X_2) \to Y = (Y_1, Y_2)$ of objects over $U \in \mathrm{Ob}(\mathcal{C})$ are the quotient:

$$\mathrm{Hom}_{\mathcal{K}_1 \times^G \mathcal{K}_{2,U}}(X, Y) := \mathrm{Hom}_{\mathcal{K}_{1,U}}(X_1, Y_1) \times^G \mathrm{Hom}_{\mathcal{K}_{2,U}}(X_2, Y_2)$$

by the diagonal action of G. Denote by

$$H^2(\mathcal{C}, G)$$

the set of equivalence classes of G-gerbes over $\mathcal{C}$ ([15], IV.3.1.1). This is a group under contracted product with identity the class of the trivial gerbe $\mathrm{Tor}(\mathcal{C}, G)$.

1.1.3 Lifting of torsors over a site

Let $\mathcal{C}$ be a site and consider a central extension:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{1.2}$$

of sheaves of groups on $\mathcal{C}$. Given a B-torsor $\mathcal{Q}$ on $\mathcal{C}$, we denote by $\mathcal{Q}/A$ the C-torsor given by the presheaf:

$$U \in \mathrm{Ob}(\mathcal{C}) \mapsto \mathcal{Q}(U)/A.$$

In this section we examine to what extent this construction can be reversed, namely:

Question 1.1.8. Given a C-torsor $\mathcal{P}$ over $\mathcal{C}$, is there a B-torsor $\mathcal{Q}$ over $\mathcal{C}$ such that $\mathcal{Q}/A \simeq \mathcal{P}$ as C-torsors?

The question can be approached via the *gerbe of lifts* of $\mathcal{P}$, constructed as follows.

Definition 1.1.9 ([15], IV.2.5.8). For $\mathcal{P}$ a C-torsor over $\mathcal{C}$, let $\mathcal{K}(\mathcal{P}) \to \mathcal{C}$ be the category such that over each object $U \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{K}(\mathcal{P})_U$ is the category whose objects are all pairs $(\mathcal{Q}_U, \alpha)$ of a B-torsor $\mathcal{Q}_U$ over $\mathcal{C}/U$ together with an isomorphism of C-torsors

$$\alpha : \mathcal{Q}_U/A \simeq \mathcal{P}|_U,$$

where $\mathcal{P}|_U$ is the *restriction* of the sheaf $\mathcal{P}$ to $\mathcal{C}/U$ ([33], Tag 00Y0). The morphisms

$\psi : (\mathcal{Q}_{U,1}, \alpha_1) \to (\mathcal{Q}_{U,2}, \alpha_2)$ in $\mathcal{K}(\mathcal{P})_U$ are isomorphisms $\psi : \mathcal{Q}_{U,1} \xrightarrow{\simeq} \mathcal{Q}_{U,2}$ of B-torsors over $\mathcal{C}/U$ such that the following diagram:

$$\begin{array}{ccc} \mathcal{Q}_{U,1}/A & \xrightarrow{\quad\psi\quad} & \mathcal{Q}_{U,2}/A \\ & {\scriptstyle\alpha_1}\searrow \quad \swarrow{\scriptstyle\alpha_2} & \\ & \mathcal{P}|_U & \end{array}$$

is commutative.

The gerbe $\mathcal{K}(\mathcal{P})$ is in fact an A-gerbe ([15], IV.2.5.8 (i)). Its class:

$$[K(\mathcal{P})] \in H^2(\mathcal{C}, A)$$

is precisely the obstruction to lifting $\mathcal{P}$ to a B-torsor $\mathcal{Q}$, in the following sense:

Theorem 1.1.10 ([15], IV.2.5.8 (ii)). *Let $\mathcal{P}$ be a C-torsor over $\mathcal{C}$. Then $\mathcal{P}$ can be lifted to a B-torsor $\mathcal{Q}$ such that $\mathcal{Q}/A \simeq \mathcal{P}$ if and only if the class $[K(\mathcal{P})] \in H^2(\mathcal{C}, A)$ is trivial.*

By analogy with cohomology with abelian coefficients, it is helpful to 'visualize' the theorem as saying that there exists an exact sequence of pointed sets:

$$H^1(\mathcal{C}, B) \longrightarrow H^1(\mathcal{C}, C) \xrightarrow{\delta_2} H^2(\mathcal{C}, A), \tag{1.3}$$

where the first arrow is $[\mathcal{Q}] \mapsto [\mathcal{Q}/A]$ and the second arrow is

$$\delta_2(\mathcal{P}) := [\mathcal{K}(\mathcal{P})].$$

Remark 1.1.11. Note that essentially by definition, the A-gerbe $p : \mathcal{K}(\mathcal{P}) \to \mathcal{C}$ is canonically equipped with a B-torsor $\mathcal{Q}$ such that:

$$\mathcal{Q}/A \simeq p^{-1}\mathcal{P}$$

as C-torsors over $\mathcal{K}(\mathcal{P})$, where $p^{-1}\mathcal{P}$ is the *pull-back* of $\mathcal{P}$ to $\mathcal{K}(\mathcal{P})$ ([33], Tag 00X0). Explicitly, this torsor is given by the presheaf of sets:

$$\begin{aligned} \mathcal{Q} : \mathcal{K}(\mathcal{P}) &\longrightarrow Sets \\ (\mathcal{Q}_U, \alpha) &\longmapsto \Gamma(U, \mathcal{Q}_U) \end{aligned}$$

that to each torsor Q over $\mathcal{C}/U$ it assigns its corresponding set of global sections over U. By Theorem 1.1.10, this torsor over $\mathcal{K}(\mathcal{P})$ descends to a torsor $\mathcal{Q}$ over $\mathcal{C}$ with $\mathcal{Q}/A \simeq \mathcal{P}$ if and only if the class $\delta_2(\mathcal{P})$ is trivial in $H^2(\mathcal{C}, A)$.

In light of the remark, we can say that the obstruction to lifting a C-torsor $\mathcal{P}$ to a B-torsor can be 'eliminated' by passing from the site $\mathcal{C}$ to the A-gerbe $\mathcal{K}(\mathcal{P})$.

1.2 Existence problems over Deligne-Mumford stacks

1.2.1 Existence of rank 1 quadratic forms

Let $\mathcal{L}$ be an invertible sheaf over a Deligne-Mumford stack $p : \mathcal{S} \to Sch[1/2]$.

DEFINITION 1.2.1. A *non-degenerate rank 1 quadratic form* $(\mathcal{N}, \mathcal{L}, q)$ is an invertible sheaf $\mathcal{N}$ together with a map of abelian sheaves $q : \mathcal{N} \to \mathcal{L}$ which factors as:

$$q : \mathcal{N} \xrightarrow{\Delta} \mathcal{N}^{\otimes 2} \simeq \mathcal{L},$$

where Δ is the diagonal map.

Thus, a rank 1 non-degenerate quadratic form can be thought of as a *square root* of $\mathcal{L}$.

DEFINITION 1.2.2. A *similitude* between rank 1 non-degenerate quadratic forms $(\mathcal{N}_1, \mathcal{L}_1, q_1)$ and $(\mathcal{N}_2, \mathcal{L}_2, q_2)$ is a pair of $\mathcal{O}_{\mathcal{S}}$-module isomorphisms $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ and $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ such that the following diagram:

$$\begin{array}{ccc} \mathcal{N}_1 & \xrightarrow{\phi} & \mathcal{N}_2 \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} \\ \mathcal{L}_1 & \xrightarrow{\varphi} & \mathcal{L}_2 \end{array}$$

is commutative. A similitude with $\mathcal{L}_1 = \mathcal{L}_2$ and $\varphi = \mathrm{id}$ is called an *isometry.*

In this section we want to examine the question:

QUESTION 1.2.3. Given a Deligne-Mumford stack $p : \mathcal{S} \to Sch[1/2]$ and an invertible sheaf $\mathcal{L}$ on $\mathcal{S}$, is there an invertible sheaf $\mathcal{N}$ such that $\mathcal{N}^{\otimes 2} \simeq \mathcal{L}$? Equivalently, is there a $\mathcal{L}$-valued non-degenerate rank 1 quadratic form over $\mathcal{S}$?

Of course the question has a negative answer in general, even in the case when $\mathcal{L}$ is trivial over an affine scheme. In fact, this problem is a special instance of the 'torsor lifting' problem of Section 1.1.3. In particular, let $\mathcal{O}^*_{\mathcal{S}}$ be the sheaf of groups associated to the presheaf:

$$X \in \mathrm{Ob}(\mathcal{S}) \longmapsto \Gamma(p(X), \mathcal{O}^*_{p(X)}).$$

The squaring map $\lambda^2 : \mathcal{O}_{\mathcal{S}}^* \to \mathcal{O}_{\mathcal{S}}^*$ gives an exact sequence of étale sheaves over $\mathcal{S}$

$$0 \to \mu_{2/S} \to \mathcal{O}_{\mathcal{S}}^* \to \mathcal{O}_{\mathcal{S}}^* \to 0,$$

where μ_2 is the constant sheaf of square roots of unity. This is called the *Kummer sequence* for μ_2. It is a simple instance of a central exact sequence of the type (1.2). In the corresponding exact sequence

$$H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}^*) \xrightarrow{\lambda^2} H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}^*) \xrightarrow{\delta_2} H^2(\mathcal{S}, \mu_2),$$

given by 1.3, we have

$$H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}^*) \simeq \mathrm{Pic}(\mathcal{S})$$

since $\mathcal{O}_{\mathcal{S}}^*$-torsors correspond to invertible sheaves and vice-versa ([33], Tag 09NU). We can then associate to the isomorphism class of $\mathcal{L}$ in $\mathrm{Pic}(\mathcal{S})$ a class $\delta_2(\mathcal{L}) \in H^2(\mathcal{S}, \mu_2)$. By Theorem 1.1.10, this class represents the obstruction to finding a square root of $\mathcal{L}$, in the following sense:

Proposition 1.2.4. *Let $\mathcal{L}$ be an invertible sheaf over a Deligne-Mumford stack $\mathcal{S} \to Sch[1/2]$. The class $\delta_2(\mathcal{L})$ is trivial in $H^2(\mathcal{S}, \mu_2)$ if and only if there exists an invertible sheaf $\mathcal{N}$ over $\mathcal{S}$ such that $\mathcal{N}^{\otimes 2} \simeq \mathcal{L}$.*

Proof. This is just a restatement of Theorem 1.1.10 in terms of invertible sheaves instead of torsors. In fact, if $\mathcal{P}(\mathcal{L})$ is the $\mathcal{O}_{\mathcal{S}}^*$-torsor attached to $\mathcal{L}$ ([33], Tag 09NU), then the theorem says that an $\mathcal{O}_{\mathcal{S}}^*$-torsor $\mathcal{Q}(\mathcal{L})$ with $\psi : \mathcal{Q}(\mathcal{L})/\mu_2 \simeq \mathcal{P}(\mathcal{L})$ exists if and only if $\delta_2(\mathcal{L})$ is trivial in $H^2(\mathcal{S}, \mu_2)$. Now the invertible sheaf $\mathcal{N}$ associated to $\mathcal{Q}(\mathcal{L})/\mu_2$ has the property that $\psi : \mathcal{N}^{\otimes 2} \simeq \mathcal{L}$. □

By Remark 1.1.11, the obstruction to finding $\mathcal{N}$ can be lifted by passing from $\mathcal{S}$ to the μ_2-gerbe of lifts of $\mathcal{L}$ of Definition 1.1.9. Unpacking Definition 1.1.9, we obtain:

Definition 1.2.5. For an invertible sheaf $\mathcal{L}$ over $\mathcal{S}$, denote by $\mathcal{K}_{1/2}(\mathcal{L}) \to \mathcal{S}$ the category whose objects above $X \in \mathrm{Ob}(\mathcal{S})$ are non-degenerate, rank 1, $\mathcal{L}|_X$-valued quadratic forms $(\mathcal{N}, \mathcal{L}|_X, q)$ over X and whose morphisms above $X_1 \to X_2$ are similitudes between $(\mathcal{N}_1, \mathcal{L}|_{X_1}, q_1)$ and $(\mathcal{N}_2, \mathcal{L}|_{X_2}, q_2)$. Here we view X as the Deligne-Mumford stack $j_X : \mathcal{S}/X \to \mathcal{S}$ and $\mathcal{L}|_X$ is the restriction $j_X^*\mathcal{L}$ of $\mathcal{L}$ to X.

Remark 1.2.6. By Remark 1.1.11, $p : \mathcal{K}_{1/2}(\mathcal{L}) \to \mathcal{S}$ is endowed with a canonical square root of $p^*\mathcal{L}$. In particular, consider the presheaf given by:

$$(\mathcal{N}, \mathcal{L}|_X, q) \longmapsto \Gamma(X, \mathcal{N})$$

that to each rank 1, $\mathcal{L}|_X$-valued quadratic form over X it associates the global sections of the underlying invertible sheaf $\mathcal{N}$. This is an invertible sheaf over the gerbe $\mathcal{K}_{1/2}(\mathcal{L})$, with the property that:

$$p^*\mathcal{L} \simeq \mathcal{N}^{\otimes 2}$$

as invertible sheaves over $\mathcal{K}_{1/2}(\mathcal{L})$.

Definition 1.2.7. The invertible sheaf over $\mathcal{K}_{1/2}(\mathcal{L})$ constructed above is the *square-root* of $\mathcal{L}$, denoted by $\mathcal{L}^{1/2}$.

In passing, note that we have:

Proposition 1.2.8. *The μ_2-gerbe $\mathcal{K}_{1/2}(\mathcal{L}) \to \mathcal{S}$ is a Deligne-Mumford stack.*

Proof. Let $\{u_i : U_i \to \mathcal{S}\}$ be a collection of étale morphisms from schemes U_i such that $\coprod_i U_i \to \mathcal{S}$ is surjective and such that $u_i^*\mathcal{L}$ is trivial for each i. It suffices to show that the product $\mathcal{K}_{1/2}(\mathcal{L}) \times_{\mathcal{S}} U_i$ is a Deligne-Mumford stack for each i. But since $u_i^*\mathcal{L}$ is trivial, $\mathcal{K}_{1/2}(\mathcal{L}) \times_{\mathcal{S}} U_i$ is equivalent to the trivial μ_2-gerbe $\mathrm{Tor}(U_i, \mu_2)$, which is a Deligne-Mumford stack. □

1.2.2 Existence of locally free sheaves with given endomorphisms

Let $\mathcal{S}$ be a Deligne-Mumford stack.

Definition 1.2.9 ([16],[15], V.4). An *Azumaya algebra* of rank r^2 over $\mathcal{S}$ is a sheaf $\mathcal{A}$ of $\mathcal{O}_{\mathcal{S}}$-algebras of rank r^2 such that there exists a collection $\{u_i : U_i \to \mathcal{S}\}$ of étale morphisms from schemes U_i with $\coprod_i U_i \to \mathcal{S}$ surjective with the property that for every i,

$$u_i^*\mathcal{A} \simeq M_r(\mathcal{O}_{U_i})$$

as $\mathcal{O}_{U_i}$-algebras, where M_r is the algebra of $r \times r$ matrices, for some positive integer r.

If $\mathcal{V}$ is a locally free $\mathcal{O}_{\mathcal{S}}$-module of finite rank, the $\mathcal{O}_{\mathcal{S}}$-algebra $\mathrm{End}_{\mathcal{O}_{\mathcal{S}}}(\mathcal{V})$ is an Azumaya algebra. Define an equivalence relation on Azumaya algebras by declaring that $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if

$$\mathcal{A}_1 \otimes \mathrm{End}(\mathcal{V}_1) \simeq \mathcal{A}_2 \otimes \mathrm{End}(\mathcal{V}_2),$$

for some locally free $\mathcal{O}_{\mathcal{S}}$-modules $\mathcal{V}_1, \mathcal{V}_2$.

Definition 1.2.10. ([16], 1.2) The *Brauer group* of $\mathcal{S}$, denoted by $\mathrm{Br}(\mathcal{S})$, is the group of equivalence classes of Azumaya algebras on $\mathcal{S}$ under tensor product.

Definition 1.2.11. A *2-torsion datum* for an Azumaya algebra $\mathcal{A}$ is a triple $(\mathcal{A}, \mathcal{W}, \iota)$ of an Azumaya algebra $\mathcal{A}$ of rank r^2 over $\mathcal{S}$, a locally free sheaf $\mathcal{W}$ of rank r^2 over $\mathcal{S}$ and an

$\mathcal{O}_{\mathcal{S}}$-algebra isomorphism

$$\iota : \mathcal{A} \otimes \mathcal{A} \simeq \mathrm{End}(\mathcal{W}).$$

In particular, $\mathcal{A}$ is of order 2 in the Brauer group. An *isomorphism* of 2-torsion data $(\mathcal{A}_1, \mathcal{W}_1, \iota_1)$ and $(\mathcal{A}_2, \mathcal{W}_2, \iota_2)$ is a pair (ϕ, φ) of an $\mathcal{O}_{\mathcal{S}}$-algebra isomorphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ and an $\mathcal{O}_{\mathcal{S}}$-module isomorphism $\varphi : \mathcal{W}_1 \to \mathcal{W}_2$ such that the following diagram

$$\begin{CD} \mathcal{A}_1 \otimes \mathcal{A}_1 @>{\phi^{\otimes 2}}>> \mathcal{A}_2 \otimes \mathcal{A}_2 \\ @V{\iota_1}VV @VV{\iota_2}V \\ \mathrm{End}(\mathcal{W}_1) @>{\varphi}>> \mathrm{End}(\mathcal{W}_2) \end{CD}$$

is commutative.

In this section we would like to study the question:

QUESTION 1.2.12. Given an Azumaya algebra $\mathcal{A}$ of rank r^2 over a Deligne-Mumford stack $\mathcal{S}$ together with a 2-torsion datum $(\mathcal{A}, \mathcal{W}, \iota)$, is there a locally free sheaf $\mathcal{V}$ of rank r and isomorphisms $\phi : \mathcal{A} \simeq \mathrm{End}\,\mathcal{V}$ and $\psi : \mathcal{W} \simeq \mathcal{V} \otimes \mathcal{V}$, such that the following diagram

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\phi^{\otimes 2}} & \mathrm{End}(\mathcal{V} \otimes \mathcal{V}) \\ & {\scriptstyle\iota}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathrm{End}(\mathcal{W}) & \end{array}$$

commutes?

The question is again an instance of the 'torsor lifting problem' of Section 1.1.3. In particular, consider the central exact sequence of étale sheaves

$$0 \to \mathbb{G}_m \to \mathrm{GL}_r \to \mathrm{PGL}_r \to 0$$

over a Deligne-Mumford stack $\mathcal{S} \to Sch[1/r]$. This exact sequence is an instance of the central exact sequence (1.2). In the corresponding long exact sequence (1.3) of pointed sets:

$$H^1(\mathcal{S}, \mathrm{GL}_r) \to H^1(\mathcal{S}, \mathrm{PGL}_r) \xrightarrow{\delta_2} H^2(\mathcal{S}, \mathbb{G}_m)$$

we have ([16], 1.1)

$$H^1(\mathcal{S}, \mathrm{GL}_r) = \{\text{isomorphism classes of locally free } \mathcal{O}_{\mathcal{S}}\text{-modules of rank } r\},$$

and ([16], 1.1)

$$H^1(\mathcal{S}, \mathrm{PGL}_r) = \{\text{isomorphism classes of Azumaya algebras of rank } r^2\},$$

the map between the first set and the second being given by $\mathcal{V} \mapsto \mathrm{End}(\mathcal{V})$. Similarly, if $\mathcal{S} \to Sch[1/2r]$ is a Deligne-Mumford stack, we have a central exact sequence of étale sheaves

$$0 \to \mu_2 \to \mathrm{GL}_r \to \mathrm{GL}_r/\mu_2 \to 0,$$

and an associated exact sequence of pointed sets

$$H^1(\mathcal{S}, \mathrm{GL}_r) \to H^1(\mathcal{S}, \mathrm{GL}_r/\mu_2) \xrightarrow{\delta_2} H^2(\mathcal{S}, \mu_2),$$

with

$$H^1(\mathcal{S}, \mathrm{GL}_r/\mu_2) = \{\text{isomorphism classes of Azumaya algebras of rank } r^2 \text{ with 2-torsion data}\},$$

the map between the first set and the second being given by $\mathcal{V} \mapsto (\mathrm{End}(\mathcal{V}), \mathcal{V} \otimes \mathcal{V}, \mathrm{id})$.

Theorem 1.1.10 in this case reads:

Proposition 1.2.13. *Let $(\mathcal{A}, \mathcal{W}, \iota)$ be an Azumaya algebra of rank r^2 over a Deligne-Mumford stack $\mathcal{S} \to Sch[1/2r]$, together with 2-torsion data. The class*

$$\delta_2(\mathcal{A}) \in H^2(\mathcal{S}, \mu_2)$$

is trivial if and only if there exists a locally free sheaf $\mathcal{V}$ of rank r and isomorphisms $\phi : \mathcal{A} \simeq \mathrm{End}\,\mathcal{V}$ and $\psi : \mathcal{W} \simeq \mathcal{V} \otimes \mathcal{V}$, such that the following diagram

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\phi^{\otimes 2}} & \mathrm{End}(\mathcal{V} \otimes \mathcal{V}) \\ & {\scriptstyle \iota} \searrow \quad \swarrow {\scriptstyle \psi} & \\ & \mathrm{End}(\mathcal{W}) & \end{array}$$

commutes.

Hence $\delta_2(\mathcal{A})$ can be viewed as the obstruction to finding a positive answer to Question 1.2.12. By Remark 1.1.11, we can lift this obstruction by passing to the appropriate μ_2-gerbe of lifts constructed in Definition 1.1.9, which in this case it reads:

Definition 1.2.14. For an Azumaya algebra $\mathcal{A}$ together with 2-torsion data $(\mathcal{A}, \mathcal{W}, \iota)$ over $\mathcal{S}$, denote by $\mathcal{K}(\mathcal{A}, \iota) \to \mathcal{S}$ the category whose objects above $X \in \mathrm{Ob}(\mathcal{S})$ are triples $(\mathcal{V}, \phi, \psi)$

of locally free $\mathcal{O}_{\mathcal{S}}$-modules $\mathcal{V}$ over X together with isomorphisms $\phi : \mathcal{A}|_X \xrightarrow{\simeq} \mathrm{End}_{\mathcal{O}_X}(\mathcal{V})$, $\psi : \mathcal{W}|_X \simeq \mathcal{V} \otimes \mathcal{V}$, commuting with ι as above. The morphisms in this category are $\mathcal{O}_{\mathcal{S}}$-isomorphisms making the obvious diagrams commute. Here we view X as the Deligne-Mumford stack $j_X : X/\mathcal{S} \to \mathcal{S}$ and denote by $\mathcal{A}|_X$ the restriction $j_X^*\mathcal{A}$.

Note that as in Proposition 1.2.8, the stack $\mathcal{K}(\mathcal{A}, \iota)$ is a Deligne-Mumford stack.

Remark 1.2.15. By Remark 1.1.11, the μ_2-gerbe $p : \mathcal{K}(\mathcal{A}, \iota) \to \mathcal{S}$ is canonically equipped with a locally free $\mathcal{O}_{\mathcal{K}(\mathcal{A},\iota)}$-module $\mathcal{V}$ of rank r and isomorphisms $\phi : p^*\mathcal{A} \simeq \mathrm{End}\,\mathcal{V}$ and $\psi : p^*\mathcal{W} \simeq \mathcal{V} \otimes \mathcal{V}$ commuting with ι.

1.3 Heisenberg group schemes

1.3.1 Representations of group schemes

Let S be a scheme and let X be an affine scheme over S. Write $X = \underline{\mathrm{Spec}}(\mathcal{A}(X))$ where $\mathcal{A}(X)$ is a quasi-coherent $\mathcal{O}_S$-module of algebras. We identify $\mathcal{A}(X)$ with the sheaf given by:

$$\{T \to S\} \longmapsto \mathrm{Hom}(X_T(T), \mathbb{A}^1_T(T))$$

so that a section of $\mathcal{A}(X)$ can be represented by a family of functions f_T on the set $X(T)$, functorial with respect to base change.

For $\mathcal{N}$ a quasi-coherent $\mathcal{O}_S$-module, let $\underline{\mathrm{Aut}}_{\mathcal{O}_S}(\mathcal{N})$ be the group scheme over S representing the functor:

$$\{T \to S\} \longmapsto \mathrm{Aut}(\Gamma(T, \mathcal{N}_T)).$$

Definition 1.3.1. ([20], V.1.1, V.2.3) A *linear representation* of a group scheme G/S on a quasi-coherent $\mathcal{O}_S$-module $\mathcal{M}$ is a homomorphism of group schemes:

$$G \longrightarrow \underline{\mathrm{Aut}}_{\mathcal{O}_S}(\mathcal{M}).$$

A quasi-coherent $\mathcal{O}_S$-module $\mathcal{M}$ equipped with a representation of G is called a *G-module*. A G-module $\mathcal{M}$ is *irreducible* if the only G-submodules of $\mathcal{M}$ are of the form $\mathcal{I}\mathcal{M}$ for an ideal sheaf $\mathcal{I} \subset \mathcal{O}_S$.

For any group scheme G/S, the quasi-coherent module $\mathcal{A}(G)$ has a natural structure of $G \times G$-module, given by the formula ([20], V.1.3)

$$(g_1, g_2)f(g) = f(g_2^{-1} g g_1).$$

We denote by $\mathcal{A}(G)_+$ (resp. $\mathcal{A}(G)_-$), the G-module obtained by restricting the $G \times G$-module structure of $\mathcal{A}(G)$ to $\{1\} \times G$ (resp. $G \times \{1\}$).

If $\mathcal{M}$ is a G-module, we denote by $\mathcal{M}^G$ the subsheaf of G-invariants ([20], V.1.2).

1.3.2 Heisenberg groups

Let K be a finite commutative locally free group scheme over S.

DEFINITION 1.3.2 ([24], §23 Definition). A *Heisenberg group* (or *theta group*) $\mathcal{G}$ is a central extension

$$0 \to \mathbb{G}_m \to \mathcal{G} \to K \to 0$$

of group schemes over S.

The group $\mathcal{G}$ is flat and affine, being a $\mathbb{G}_m$-torsor. It is generally not commutative, as the commutator pairing:

$$e_{\mathcal{G}} : K \times K \to \mathbb{G}_m$$

maybe non-trivial. Whenever this commutator pairing is perfect we say that $\mathcal{G}$ is *non-degenerate*. In this case $\mathbb{G}_m$ is precisely the center of $\mathcal{G}$ ([24], §23 Corollary). We will assume throughout that this is always the case.

By the assumptions on K the exact sequence defining $\mathcal{G}$ splits Zariski-locally as a sequence of S-schemes ([24], §23), and $\mathcal{G} \simeq \mathbb{G}_m \times K$ locally as an S-scheme. Once a splitting is chosen, locally all the possible group structures are classified by the classes of 2-cocyles $f \in H^2(K, \mathbb{G}_m)$. Explicitly, the group law on $\mathcal{G}$ is given by

$$(\alpha_1, k_1) \cdot (\alpha_2, k_2) = (\alpha_1\, \alpha_2\, f(k_1, k_2), k_1 + k_2)$$

for a morphism

$$f : K \times K \to \mathbb{G}_m$$

satisfying

$$f(k_1 + k_2, k_3) f(k_1, k_2) = f(k_1, k_2 + k_3) f(k_2, k_3)$$

and normalized so that $f(0, 0) = 1$ ([24], §23).

Locally, the commutator pairing $e_{\mathcal{G}}$ can be expressed in terms of the cocycle f as:

$$e_{\mathcal{G}}(k_1, k_2) = f(k_1, k_2)/f(k_2, k_1).$$

Several operations can be performed on a Heisenberg group $\mathcal{G}$ over K:

(i) The *inverse* of $\mathcal{G}$ is the central extension

$$0 \to \mathbb{G}_m \to \mathcal{G}^{-1} \to K \to 0$$

determined by pushing forward $\mathcal{G}$ along the inversion map $[-1] : \mathbb{G}_m \to \mathbb{G}_m$.

(ii) The *opposite* of $\mathcal{G}$ is the central extension:

$$0 \to \mathbb{G}_m \to \mathcal{G}^{\mathrm{op}} \to K \to 0$$

determined by switching the order of multiplication.

(iii) If $n \in \mathbb{Z}$, and $[n] : K \to K$ denotes the addition map, then the Heisenberg group $[n]^*\mathcal{G}$ is the central extension of K by $\mathbb{G}_m$ determined by pulling-back $\mathcal{G}$ along $[n] : K \to K$.

1.3.3 The Stone-Von Neumann Theorem

Suppose now that $\mathcal{G}$ is a Heisenberg group where K has rank m^2 over S. If $\mathcal{M}$ is a $\mathcal{G}$-module, the action of $\mathbb{G}_m \subset \mathcal{G}$ decomposes $\mathcal{M}$ into 'weights':

$$\mathcal{M} = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}^{(i)}$$

characterized by the fact that the scalar λ acts on $\mathcal{M}$ by λ^i. If $\mathcal{M} = \mathcal{M}^{(i)}$ we say that $\mathcal{M}$ has *weight* i ([20], V.2.1).

Theorem 1.3.3 (Stone-Von Neumann, [20] V.2.4.2, [32] Theorem 1.1). *Let $\mathcal{V}$ be $\mathcal{G}$-module of weight one which is locally free of rank m as an $\mathcal{O}_S$-module. Then*

(a) $\mathcal{V}$ is irreducible.

(b) If $\mathcal{V}'$ is another weight one $\mathcal{G}$-module which is locally free of rank m, then there is an isomorphism of $\mathcal{G}$-modules:

$$\mathcal{V}' \simeq \mathcal{V} \otimes_{\mathcal{O}_S} \mathcal{L}_{\mathcal{V},\mathcal{V}'}$$

where $\mathcal{L}_{\mathcal{V},\mathcal{V}'}$ is an invertible $\mathcal{O}_S$-module equipped with trivial $\mathcal{G}$-action.

(c) The isomorphism of part (b) is unique up to multiplication by $\Gamma(S, \mathcal{O}_S^*)$.

Proof. Part (a) is [20] V.2.4.2 (ii). Also by [20] V.2.4.3, we know that there is an equivalence of categories between quasi-coherent $\mathcal{O}_S$-modules and $\mathcal{G}$-modules of weight 1 given by $\mathcal{F} \mapsto \mathcal{V} \otimes \mathcal{F}$, where the quasi-coherent module $\mathcal{F}$ is given the trivial $\mathcal{G}$-action. Therefore $\mathcal{V}' \simeq \mathcal{V} \otimes \mathcal{F}_{V,V'}$ for some quasi-coherent module $\mathcal{F}_{V,V'}$, and since both $\mathcal{V}$ and $\mathcal{V}'$ are locally free of rank m, we must have that $\mathcal{F}_{V,V'}$ is locally free of rank 1 over $\mathcal{O}_S$, which proves (b). Finally, if ψ and ϕ are two isomorphisms $\psi, \phi : \mathcal{V}' \otimes \mathcal{L}_{V,V'}^{-1} \overset{\simeq}{\to} \mathcal{V}$, then $\psi \circ \phi^{-1}$ is in $\mathrm{Aut}_{\mathcal{G}-mod}(\mathcal{V})$. By the equivalence of categories above stated, this is equal to $\mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}_S) = \Gamma(S, \mathcal{O}_S^*)$. □

REMARK 1.3.4. By part (b) of Theorem 1.3.3, the isomorphism class of the *projective* $\mathcal{G}$-module $\mathcal{PV} := \mathbb{P}(\mathcal{V})$ is canonically associated to $\mathcal{G}$. Equivalently, the Azumaya algebra:

$$\mathcal{A}_{\mathcal{G}} := \mathrm{End}_{\mathcal{O}_S}(\mathcal{V})$$

is canonically associated to $\mathcal{G}$, since for any other weight 1, rank m representation $\mathcal{V}'$ of $\mathcal{G}$ we have:

$$\mathrm{End}_{\mathcal{O}_S}(\mathcal{V}') = \mathrm{End}_{\mathcal{O}_S}(\mathcal{V} \otimes \mathcal{L}_{\mathcal{V},\mathcal{V}'}) = \mathrm{End}_{\mathcal{O}_S}(\mathcal{V}).$$

1.3.4 Representations coming from lagrangian subgroups

By Theorem 1.3.3, if a weight one $\mathcal{G}$-module $\mathcal{V}$ locally free of rank m over S exists, it is unique up to tensoring by an invertible sheaf. In this section we turn to the problem of constructing such $\mathcal{V}$'s explicitly.

DEFINITION 1.3.5 ([20], V.2.5.1). A *level subgroup* $H \subset \mathcal{G}$ is a subgroup scheme $H \subset \mathcal{G}$, finite and locally free over S, such that $H \cap \mathbb{G}_m = \{1\}$. A level subgroup is called *lagrangian* if it is of rank m over S.

In particular, a level subgroup H is isomorphic to its image $H_1 \subset K$ under the projection map $\mathcal{G} \to K$. Hence it is commutative and it is therefore isotropic for the commutator pairing $e_{\mathcal{G}}$. A lagrangian subgroup is maximal isotropic for $e_{\mathcal{G}}$.

The quotient $H\backslash\mathcal{G}$ does not make sense as a group scheme, but we can still view it as the $\mathbb{G}_m$-torsor over K/H_1 of H-invariant functions on $\mathcal{G}$. Since K/H_1 is finite, hence affine, the torsor $H\backslash\mathcal{G}$ is affine as well and its algebra of functions $\mathcal{A}(H\backslash\mathcal{G})$ (notation as in the beginning of Section 1.3.1) has a natural structure of $\mathcal{G}$-module.

PROPOSITION 1.3.6 ([20], Prop. V.2.5.2). *Let $H \subset \mathcal{G}$ be a level subgroup of rank m', and let*

$$\mathcal{V}_H := \mathcal{A}^{(1)}(H\backslash\mathcal{G}),$$

with the notation of the beginning of Section 1.3.1. Then $\mathcal{V}_H$ is a sub $\mathcal{G}$-module of $\mathcal{A}_+^{(1)}(\mathcal{G})$, locally free of rank m^2/m'.

COROLLARY 1.3.7. *For any lagrangian subgroup $H \subset \mathcal{G}$, the $\mathcal{G}$-module $\mathcal{V}_H$ is of weight one and locally free of rank m as $\mathcal{O}_S$-module.*

Therefore from any lagrangian subgroup $H \subset \mathcal{G}$ we can construct an explicit *model* of the representation of Theorem 1.3.3. Explicitly, $\mathcal{V}_H$ can be identified with the functions $f \in \mathcal{A}(\mathcal{G})$ such that:

(i) $f(hg) = f(g), \quad \forall\, h \in H$

(ii) $f(\lambda g) = \lambda f(g), \quad \forall \lambda \in \mathbb{G}_m$

just by unwinding the definitions ([20], V.3.3.3).

Definition 1.3.8. A *Schrödinger* representation of an Heisenberg group $\mathcal{G}$ is any representation of the form $\mathcal{V}_H$, for $H \subset \mathcal{G}$ a lagrangian subgroup.

1.3.5 Symmetric Heisenberg Groups

Recall from section 1.3.2 that if $\mathcal{G}$ is a Heisenberg group and $[-1] : K \to K$ is the inversion map in K, then $[-1]^*\mathcal{G}$ is the Heisenberg group over K obtained by pulling back $\mathcal{G}$ along $[-1] : K \to K$.

Definition 1.3.9 ([29], §1). A Heisenberg group $\mathcal{G}$ over K is *symmetric* if there is an isomorphism

$$\mathcal{G} \simeq [-1]^*\mathcal{G}$$

.

Symmetric Heisenberg groups $\mathcal{G}$ are remarkable because the product $\mathcal{G} \times \mathcal{G}$ has a *canonical* rank m^2, weight 1 representation $\mathcal{V}_\Delta$. To construct this representation, consider the Heisenberg group:

$$0 \to \mathbb{G}_m \to \mathcal{G} \times \mathcal{G}^{-1,\mathrm{op}} \to K \times K \to 0.$$

Locally, if f is the 2-cocycle in $H^2(K \times K, \mathbb{G}_m)$ giving the group law in $\mathcal{G}$, then the group law of $\mathcal{G} \times \mathcal{G}^{-1,\mathrm{op}}$ is given by the 2-cocyle

$$g((k_1, k_2), (k_3, k_4)) = f(k_1, k_2) f(k_4, k_3)^{-1}.$$

In particular, the extension $\mathcal{G} \times \mathcal{G}^{-1,\mathrm{op}}$ splits above the diagonal embedding $\Delta : K \to K \times K$, since g is the commutator pairing above Δ and K is commutative. Thus we can find a canonical lagrangian subgroup H_Δ by lifting the image of Δ to $\mathcal{G} \times \mathcal{G}^{-1,\mathrm{op}}$ along the splitting. The weight 1, rank m^2 representation $\mathcal{V}_\Delta$ constructed from H_Δ is then canonically associated to $\mathcal{G} \times \mathcal{G}^{-1,\mathrm{op}}$.

Lemma 1.3.10. *If $\mathcal{G}$ is symmetric, we have:*

$$\mathcal{G}^{-1,op} \simeq \mathcal{G},$$

hence $\mathcal{G} \times \mathcal{G}$ has a canonical weight 1, rank m^2 Schrödinger representation induced by the lagrangian subgroup $\Delta : K \to K \times K$.

Proof. From the above discussion, only the first statement needs proof. Now for any

Heisenberg group $\mathcal{G}$, not necessarily symmetric, we claim that there is an isomorphism $\mathcal{G}^{-1} \simeq [-1]^*\mathcal{G}^{\text{op}}$. To show this, note that if $f \in H^2(K, \mathbb{G}_m)$ determines the group law of $\mathcal{G}$ locally, then the 2-cocycles $f(k_1, k_2)^{-1}$ and $f(-k_2, -k_1)$ determine the group laws of $\mathcal{G}^{-1}$ and $[-1]^*\mathcal{G}^{\text{op}}$, respectively. Now we want to show that these two cocycles are cohomologous in $H^2(K, \mathbb{G}_m)$ or, equivalently, that the 2-cocycle

$$g(k_1, k_2) := f(k_1, k_2) f(-k_2, -k_1)$$

is trivial in $H^2(K, \mathbb{G}_m)$. Now the commutator pairing corresponding to g is trivial, since:

$$\begin{aligned} g(k_1, k_2) g(k_2, k_1)^{-1} &= f(k_1, k_2) f(-k_2, -k_1) f(k_2, k_1)^{-1} f(-k_1, -k_2)^{-1} \\ &= e_{\mathcal{G}}(k_1, k_2) e_{\mathcal{G}}(-k_2, -k_1) \\ &= e_{\mathcal{G}}(k_1, k_2) e_{\mathcal{G}}(-k_1, -k_2)^{-1} \\ &= e_{\mathcal{G}}(k_1, k_2) e_{\mathcal{G}}(-k_1, k_2) \\ &= e_{\mathcal{G}}(0, k_2) = 1. \end{aligned}$$

and therefore the central extension corresponding to the cocycle g is commutative. But commutative Heisenberg groups are trivial extensions ([24], §23 Lemma 1 (i)). Therefore $\mathcal{G}^{-1} \simeq [-1]^*\mathcal{G}^{\text{op}}$, or, equivalently:

$$\mathcal{G}^{-1,\text{op}} \simeq [-1]^*\mathcal{G}$$

canonically. The isomorphism must the hold globally by descent. Finally, if $\mathcal{G}$ is symmetric, we have $[-1]^*\mathcal{G} \simeq \mathcal{G}$, hence $\mathcal{G}^{-1} \simeq \mathcal{G}^{-1,\text{op}} \simeq \mathcal{G}$. □

In particular, to any symmetric Heisenberg group $\mathcal{G}$ and any integer m we can canonically attach an Azumaya algebra of rank m^2 together with a 2-torsion datum for it, in the sense of Section 1.2.2. In fact, by Remark 1.3.4 the Azumaya algebra

$$\mathcal{A} := \text{End}(\mathcal{V}),$$

where $\mathcal{V}$ is *any* weight 1, rank m representation of $\mathcal{G}$, is canonically attached to $\mathcal{G}$. Moreover, by Lemma 1.3.10 we can also canonically attach to $\mathcal{G}$ a locally free sheaf $\mathcal{W} := \mathcal{V}_\Delta$ of rank m^2 together with an isomorphism

$$\iota : \text{End}(\mathcal{W}) \simeq \mathcal{A}^{\otimes 2}.$$

In fact, note that $\mathcal{V} \otimes \mathcal{V}$ is also a weight 1, rank m^2 representation of $\mathcal{G} \times \mathcal{G}$. Thus, by Theorem 1.3.3, we have

$$\mathcal{V} \otimes \mathcal{V} \simeq \mathcal{W} \otimes \mathcal{L},$$

for some invertible $\mathcal{O}_S$-module. In turn, this gives

$$\mathrm{End}(\mathcal{W}) \simeq \mathrm{End}(\mathcal{W} \otimes \mathcal{L}) \simeq \mathrm{End}(\mathcal{V} \otimes \mathcal{V}) \simeq \mathrm{End}(\mathcal{V})^{\otimes 2} = \mathcal{A}^{\otimes 2}.$$

The triple $(\mathcal{A}, \mathcal{W}, \iota)$ is thus a 2-torsion datum for the Azumaya algebra $\mathcal{A}$ canonically associated to the Heisenberg group $\mathcal{G}$.

Chapter 2

Algebraic Theory

In this chapter we work out a geometric construction of modular forms of half-integral weight with values in a vector bundle $\mathcal{V}_m$ of Schrödinger representations. These modular forms are algebro-geometric analogs of the complex analytic notion of vector-valued modular forms with values in the Weil representation attached to rank 1 lattices with quadratic form $x \mapsto mx^2/2$ (m will always be a positive even integer), a notion first introduced by Eichler and Zagier in [14] and further developed by Borcherds (e.g. [4], [5]).

Geometrically, modular forms of integral weight are sections of the Hodge bundle $\underline{\omega}^k$, $k \in \mathbb{Z}$, over the moduli stack $\mathcal{M}_1$ of elliptic curves, the *modular stack*. The first task is thus to construct invertible sheaves $\underline{\omega}^{k/2}$, whose sections should correspond to 'modular forms of half-integral weight'. This is done in Section 2.1.1. The point is that these sheaves do not exists over $\mathcal{M}_1$, but rather over the *metaplectic stack* $\mathcal{M}_{1/2}$ (Definition 2.1.2), which we construct as in Section 1.2.1. In Proposition 2.1.4, we compute the Picard group of $\mathcal{M}_{1/2}$ and we show that it is cyclic of order 24, generated by $\underline{\omega}^{1/2}$. The sheaves $\underline{\omega}^{k/2}$ are the tensor powers of $\underline{\omega}^{1/2}$. Since the term 'modular forms of half-integral weight' has already been chosen to designate the modular forms studied by Shimura ([31]), we call sections of $\underline{\omega}^{k/2}$ over $\mathcal{M}_{1/2}$ *metaplectic forms* of weight $k/2$ (Definition 2.1.5).

Next, in Section 2.2.2 we construct the vector bundle of Schrödinger representations $\mathcal{V}_m$. The idea is to patch together all the Schrödinger representations of Heisenberg groups of elliptic curves of level m (Definition 2.2.4). This vector bundle cannot exist over the modular stack $\mathcal{M}_1$, essentially by the Stone-Von-Neumann Theorem (Theorem 1.3.3): there is no hope of choosing a Schrödinger representation for each elliptic curve *functorially* with respect to morphisms of elliptic curves, since these representations will always differ by an invertible sheaf. However, following ideas of Polishchuk ([29]), the endomorphism algebras of Schrödinger representations over elliptic curves do patch functorially (Proposition 2.2.8), and they define an Azumaya algebra together with 2-torsion data, in the sense of Definition 1.2.11. We can thus apply the theory of Section 1.2.2 to find a μ_2-gerbe over $\mathcal{M}_1$ over which $\mathcal{V}_m$ is properly defined (Definition 2.2.10). This gerbe is essentially the metaplectic stack

(Remark 2.2.15), though we only make this identification when it is strictly necessary.

We can then define a *vector-valued modular form* of weight $k/2$ (Definition 2.2.11) as a section of $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ over the metaplectic stack $\mathcal{M}_{1/2}$ (more rigorously, over an appropriate contracted product of μ_2-gerbes). We prove that the bundles $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$, for k odd, do in fact descend to $\mathcal{M}_1$ (2.2.14), thus some 'miracolous' cancellation occurs when choosing the ambiguity of square roots of $\underline{\omega}$ and the ambiguity in choosing a Schrödinger representation over an elliptic curve.

In Section 2.2.4 we show (Theorem 2.2.16) that there is a canonical isomorphism

$$\mathcal{V}_m \otimes \underline{\omega}^{-1/2} \simeq \mathcal{J}_m, \tag{2.1}$$

defined up to a multiplication by an invertible function on $\mathcal{M}_1$, where $\mathcal{J}_m$ is the vector bundle over $\mathcal{M}_1$ obtained by attaching to each elliptic curve the global sections of its unique totally symmetric sheaf $\mathcal{L}_m$ of degree m, normalized along the identity section, in the sense of Mumford ([23]). We call this bundle the sheaf of *geometric representations* of the level m Heisenberg groups of elliptic curves, since by [22] and [23] we know that the global sections of $\mathcal{L}_m$ carry Heisenberg representations.

In Section 2.2.5 we show that $\mathcal{V}_m$ is locally constant for the étale topology of $\mathcal{M}_{1/2}$ (Theorem 2.2.19). Thus the sheaf $\mathcal{J}_m \otimes \underline{\omega}^{1/2}$ is also locally constant, i.e. it is endowed with a canonical integrable connection ∇. By work of Welters [37], this connection is given by algebraic analogs of heat operators. The horizontal sections of $\mathcal{J}_m \otimes \underline{\omega}^{1/2}$, defined over an étale cover of $\mathcal{M}_{1/2}$, are thus algebraic analogs of the classical level m theta functions. A striking consequence of the flatness of $\mathcal{V}_m$ is also that the isomorphism (2.1) can be normalized so that it is defined up to a constant (Theorem 2.2.20).

Next, we give a geometric definition of q-expansions of vector-valued modular forms. Following the geometric definition of q-expansions of modular forms of integral weight (e.g. [19]), q-expansions of vector-valued modular forms are obtained by constructing canonical trivializations of the sheaves $\mathcal{V}_m$ and $\underline{\omega}^{k/2}$ over Tate curves. This is done in Sections 2.2.6 and 2.2.7.

In Section 2.3.1, we define the prototypical examples of vector-valued modular forms, the vectors $\theta_{\mathrm{null},m}$ of theta constants (Definition 2.3.1), and compute their q-expansions in Section 2.3.2. We show that these q-expansions are exactly the same as the classical analytic q-expansions of theta constants. We then briefly discuss the relationship between our theory and that of Mumford's theta structures ([22],[23]) in Section 2.3.3.

Finally, in Section 2.3.4 we construct algebro-geometric analogs of *modular forms of half-integral weight* (Definition 2.3.11), in the proper sense of Shimura ([31]). We define q-expansions for them (Definition 2.3.14) and recover the classical q-expansion of $\theta_0(q) = \sum q^{n^2}$ as the q-expansion of an (algebraic) modular form of weight $1/2$ and level 4.

2.1 The metaplectic stack and metaplectic forms of weight $k/2$

2.1.1 The metaplectic stack $\mathcal{M}_{1/2}$

DEFINITION 2.1.1. An *elliptic curve* over a scheme S is a pair $(E/S, e)$, where $\pi : E \to S$ is a proper morphism, smooth of relative dimension 1 and whose geometric fibers are connected curves of genus 1, together with a section $e : S \to E$.

The classifying stack of elliptic curves is the category $\mathcal{M}_1 \to Sch$ whose objects above a scheme S are elliptic curves $(E/S, e)$ and whose morphisms $E_1/S_1 \to E_2/S_2$ above $\varphi : S_1 \to S_2$ are pairs (ϕ, φ) of morphisms of schemes fitting in the cartesian diagram:

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow & & \downarrow \\ S_1 & \xrightarrow{\varphi} & S_2, \end{array}$$

where $E_1 \simeq E_2 \times_{S_2} S_1$ and $\phi \circ e_1 = e_2 \circ \varphi$. There is a functor $\mathcal{M}_1 \to Sch$ sending $E \to S$ to S and (ϕ, φ) to φ which makes $\mathcal{M}_1$ a category fibered in groupoids over Sch: the fiber $\mathcal{M}_{1,S}$ above a scheme S is the category of all elliptic curves $E \to S$ over a fixed base scheme S with morphisms being isomorphisms of S-schemes. In [21] and [12], it is shown that this is a Deligne-Mumford stack over $Sch_{ét}$, called the *modular stack*.

Let now $\pi : E \to S$ be an elliptic curve over a scheme S, and let $\Omega^1_{E/S}$ be the sheaf of relative differentials. Let

$$\underline{\omega}_{E/S} := \pi_* \Omega^1_{E/S},$$

which is an invertible sheaf over S. The assignment

$$\{\pi : E \to S\} \longmapsto \Gamma(S, \underline{\omega}_{E/S})$$

defines an invertible sheaf $\underline{\omega}$ on the modular stack $\mathcal{M}_1$, called the *Hodge bundle* of $\mathcal{M}_1$. For $k \in \mathbb{Z}$ an integer, sections of $\underline{\omega}^k$ are *modular forms* of weight k.

We would now like to apply the theory of Section 1.2.1 to the problem of finding a square root of the invertible sheaf $\underline{\omega}$ over the modular stack $\mathcal{M}_1$. Recall that the strategy is to construct a μ_2-gerbe $p : \mathcal{M}_{1/2} \to \mathcal{M}_1$ canonically endowed with an invertible sheaf $\underline{\omega}^{1/2}$ such that

$$p^* \underline{\omega} = (\underline{\omega}^{1/2})^{\otimes 2}.$$

The stack $\mathcal{M}_{1/2}$ is the stack $\mathcal{M}_{1/2} := \mathcal{K}_{1/2}(\underline{\omega})$ of Definition 1.2.5 of Section 1.2.1. In particular:

DEFINITION 2.1.2. The *metaplectic stack* $\mathcal{M}_{1/2} \to Sch[1/2]$ is the category whose objects above a scheme $S \in \mathrm{Ob}(Sch[1/2])$ are pairs $(E/S, \mathcal{Q})$ of an elliptic curve $\pi : E \to S$ and a non-degenerate rank one $\underline{\omega}_{E/S}$-valued quadratic form $q : \mathcal{Q} \to \underline{\omega}_{E/S}$, and whose morphisms are similitudes lying above morphisms of elliptic curves.

The metaplectic stack is a μ_2-gerbe $p : \mathcal{M}_{1/2} \to \mathcal{M}_1$, the functor p being given by 'forget the quadratic form'. By Proposition 1.2.8, $\mathcal{M}_{1/2}$ is a Deligne-Mumford stack. By Remark 1.2.6, $\mathcal{M}_{1/2}$ is endowed with a canonical square-root of $\underline{\omega}$, i.e. an invertible sheaf $\underline{\omega}^{1/2}$ such that:

$$p^*\underline{\omega} \simeq \left(\underline{\omega}^{1/2}\right)^{\otimes 2}.$$

Explicitly, the invertible sheaf $\underline{\omega}^{1/2}$ is given by the presheaf:

$$(E/S, \mathcal{Q}) \longmapsto \Gamma(S, \mathcal{Q}).$$

The geometry of $\mathcal{M}_{1/2}$ is closely related to that of $\mathcal{M}_1$. For example, we have:

PROPOSITION 2.1.3. *Let $\mathcal{M}_{1/2} \to Sch[1/6]$ be the metaplectic stack over the category of schemes where 2 and 3 are invertible. Let k be an algebraically closed field of characteristic $\neq 2, 3$ and let*

$$\tilde{s} : \mathrm{Spec}(k) \longrightarrow \mathcal{M}_{1/2}$$

be a geometric point, classifying a pair $(E/k, q)$ of an elliptic curve E/k together with a non-degenerate rank 1 quadratic form:

$$q : \Gamma(\mathrm{Spec}(k), \tilde{s}^*\underline{\omega}^{1/2}) \to \Gamma(\mathrm{Spec}(k), \underline{\omega}_{E/k}).$$

Then:

(i) $\mathrm{Aut}(E/k, q) \simeq \mathbb{Z}/8\mathbb{Z}$*, in the case when $E = E_1$ is the elliptic curve*

$$E_1 : \quad y^2 = x^3 - x = x(x+1)(x-1).$$

(ii) $\mathrm{Aut}(E/k, q) \simeq \mathbb{Z}/12\mathbb{Z}$*, in the case when $E = E_2$ is the elliptic curve*

$$E_2 : \quad y^2 = x^3 - 1 = (x-1)(x-\zeta)(x-\zeta^2)$$

where $\zeta \in k$ is a primitive 3rd root of unity.

(iii) $\mathrm{Aut}(E/k, q) \simeq \mathbb{Z}/4\mathbb{Z}$ *in all other cases.*

Proof. Suppose E is given by the Weierstrass equation

$$y^2 = x^3 + a_4 x + a_6, \quad a_4, a_6 \in k.$$

Then there is a canonical choice of generator

$$\Gamma(\mathrm{Spec}(k), \underline{\omega}_{E/k}) = \Gamma(E, \Omega^1_{E/k}) \simeq k \cdot \omega,$$

where $\omega = dx/y$. Now any automorphism $\alpha \in \mathrm{Aut}(E/k)$ acts by functoriality on $\Gamma(\mathrm{Spec}(k), \underline{\omega}_{E/k})$ via multiplication by a scalar $\omega \mapsto \alpha_{\underline{\omega}} \cdot \omega$, and any automorphism $\widetilde{\alpha} \in \mathrm{Aut}(E/k, q)$ acts on q by a similitude lying above α, i.e. via multiplication by a scalar $\widetilde{\alpha}_{\underline{\omega}^{1/2}}$ such that:

$$\widetilde{\alpha}^2_{\underline{\omega}^{1/2}} = \alpha_{\underline{\omega}}.$$

Thus for any geometric point $\tilde{s} : \mathrm{Spec}(k) \to \mathcal{M}_{1/2}$, the group of automorphisms $\mathrm{Aut}(E/k, q)$ of the pair $(E/k, q)$ classified by $\tilde{s}$ is a group extension:

$$\begin{gathered} 0 \to \mu_2 \to \mathrm{Aut}(E/k, q) \to \mathrm{Aut}(E/k) \to 0 \\ (\alpha, \alpha_{\underline{\omega}^{1/2}}) \longmapsto \alpha. \end{gathered}$$

There are three possibilities for $\mathrm{Aut}(E/k, q)$, each one corresponding to the three well-known possibilities for $\mathrm{Aut}(E/k)$:

(i) The geometric point $\tilde{s}_1 : \mathrm{Spec}(k) \to \mathcal{M}_{1/2}$, classifying the elliptic curve:

$$E_1 : \quad y^2 = x^3 - x = x(x+1)(x-1),$$

together with a rank 1 non-degenerate quadratic form

$$q : \Gamma(\mathrm{Spec}(k), s_1^* \underline{\omega}^{1/2}) \to \Gamma(\mathrm{Spec}(k), \underline{\omega}_{E_1/k}).$$

In this case

$$\mathrm{Aut}(E_1/k) \simeq \mathbb{Z}/4\mathbb{Z},$$

generated by the automorphism

$$\begin{aligned} \sigma &: x \mapsto -x \\ \sigma &: y \mapsto iy, \quad i^2 = -1. \end{aligned} \tag{2.2}$$

Thus the group $\mathrm{Aut}(E_1/k, q)$ is of order 8. Now σ acts on $\omega = dx/y$ by:

$$\sigma_{\underline{\omega}} = i$$

and therefore $\mathrm{Aut}(E_1/k, q)$ must contain an element generating a cyclic subgroup of order 8, namely the automorphism

$$\widetilde{\sigma} := (\sigma, \sqrt{i}) \tag{2.3}$$

for some choice of $\sqrt{i} \in k^\times$. Therefore

$$\mathrm{Aut}(E_1/k, q) \simeq \mathbb{Z}/8\mathbb{Z}.$$

(ii) The geometric point $\tilde{s}_2 : \mathrm{Spec}(k) \to \mathcal{M}_{1/2}$ classifying the elliptic curve

$$E_2 : \quad y^2 = x^3 - 1 = (x-1)(x-\zeta)(x-\zeta^2)$$

together with a rank 1 non-degenerate quadratic form q as above. In this case

$$\mathrm{Aut}(E_2/k) \simeq \mathbb{Z}/6\mathbb{Z},$$

generated by the automorphism:

$$\begin{aligned} \tau &: x \mapsto \zeta x \\ \tau &: y \mapsto -y. \end{aligned} \tag{2.4}$$

This automorphism acts on ω by:

$$\tau_{\underline{\omega}} = -\zeta$$

therefore

$$\mathrm{Aut}(E_2/k, q) \simeq \mathbb{Z}/12\mathbb{Z},$$

generated by

$$\tilde{\tau} := (\tau, \sqrt{-\zeta}), \tag{2.5}$$

for some choice of $\sqrt{-\zeta} \in k^\times$.

(iii) For any other elliptic curve E, we have:

$$\mathrm{Aut}(E/k) = \mathbb{Z}/2\mathbb{Z},$$

generated by the 'inversion' automorphism $[-1]$:

$$\begin{aligned} x &\mapsto x \\ y &\mapsto -y. \end{aligned} \tag{2.6}$$

This automorphism acts on ω via:

$$[-1]_{\underline{\omega}} = -1$$

thus

$$\mathrm{Aut}(E/k, q) \simeq \mathbb{Z}/4\mathbb{Z}$$

generated by the automorphism:

$$z = ([-1], i) \tag{2.7}$$

for some choice of $i = \sqrt{-1} \in k^\times$.

□

The computation of $\mathrm{Aut}(E/k, q)$ for E/k over an algebraically closed field k leads directly to the computation of $\mathrm{Pic}(\mathcal{M}_{1/2})$, following the direct method discovered by Mumford to compute the Picard group of $\mathcal{M}_1$ ([21], §6).

PROPOSITION 2.1.4. *Let* $\mathrm{Pic}(\mathcal{M}_{1/2})$ *be the group of isomorphism classes of invertible sheaves over the metaplectic stack* $\mathcal{M}_{1/2} \to Sch[1/6]$*. Then there is a canonical isomorphism*

$$\mathrm{Pic}(\mathcal{M}_{1/2}) \simeq \mathbb{Z}/24\mathbb{Z}$$

given by $[\underline{\omega}^{1/2}] \mapsto 1 \mod 24$.

Proof. Consider an invertible sheaf $\mathcal{L}$ over $\mathcal{M}_{1/2}$ and let $\tilde{s} : \mathrm{Spec}(k) \to \mathcal{M}_{1/2}$ be a geometric point classifying a pair $(E/k, q)$ of an elliptic curve E/k over an algebraically closed field of characteristic $\neq 2, 3$ together with a rank 1 non-degenerate quadratic form $q : \Gamma(\mathrm{Spec}(k), \tilde{s}^*\underline{\omega}^{1/2}) \to \Gamma(\mathrm{Spec}(k), \underline{\omega}_{E/k})$. Any automorphism of $\mathrm{Aut}(E/k, q)$ acts by functoriality on $\tilde{s}^*\mathcal{L}$ via a scalar. In particular:

(i) For $E = E_1$, the automorphism $\widetilde{\sigma}$ generating $\mathrm{Aut}(E_1/k, q)$ given by (2.3) acts via an 8-th root of unity $\widetilde{\sigma}_{\mathcal{L}}$.

(ii) For $E = E_2$, the automorphism $\widetilde{\tau}$ generating $\mathrm{Aut}(E_2/k, q)$ given by (2.5) acts via a 12-th root of unity $\widetilde{\tau}_{\mathcal{L}}$.

(iii) For all other pairs $(E/k, q)$, the automorphism z generating $\mathrm{Aut}(E/k, q)$ given by (2.7) acts via a 4-th root of unity $z_{\mathcal{L}}$.

Moreover, a similar argument to that of [21], §6, shows that in case (iii) the 4-th root of unity $z_{\mathcal{L}}$ must be the same for all pairs $(E/k, q)$. Thus we can uniquely attach to $\mathcal{L}$ a 24-th root of unity $\eta_{\mathcal{L}}$ defined by:

$$\eta_{\mathcal{L}}^3 = \widetilde{\sigma}_{\mathcal{L}}, \quad \eta_{\mathcal{L}}^2 = \widetilde{\tau}_{\mathcal{L}}, \quad \eta_{\mathcal{L}}^6 = z_{\mathcal{L}}.$$

This assignment gives a group homomorphism:

$$\begin{aligned} \mathrm{Pic}(\mathcal{M}_{1/2}) &\longrightarrow \mathbb{Z}/24\mathbb{Z} \\ \mathcal{L} &\longmapsto \eta_{\mathcal{L}}, \end{aligned}$$

which can be normalized by the requirement that

$$\underline{\omega}^{1/2} \longmapsto 1 \mod 24,$$

since $\eta_{\underline{\omega}^{1/2}}$ is a *primitive* 24-th root of unity, as can be seen by (2.3) and (2.5).

In particular, the homomorphism must be surjective since the tensor powers of $\underline{\omega}^{1/2}$ will map to all the residue classes mod 24. But it is also injective, for if $\eta_{\mathcal{L}} = 1$, then $\mathcal{L}$ descends to an invertible sheaf over $\mathcal{M}_1$ with $\sigma_{\mathcal{L}} = \tau_{\mathcal{L}} = 1$. Now by [21] §6, $\mathcal{L}$ must further descend to the affine line over $\mathbb{Z}[1/6]$ under the map

$$\mathcal{M}_{1/2} \xrightarrow{p} \mathcal{M}_1 \xrightarrow{j} \mathbb{A}^1_{\mathbb{Z}[1/6]},$$

where j is the usual j-function, and thus be trivial. $\square$

2.1.2 Metaplectic forms

A modular form of weight $k \in \mathbb{Z}$ is a global section of $\underline{\omega}^k$ over $\mathcal{M}_1$. Thus, it is a rule f that to each elliptic curve $E \to S$ assigns an element $f(E/S) \in \Gamma(S, \underline{\omega}^k_{E/S})$ such that $\varphi^* f(E_2/S_2) = f(E_1/S_1)$ for any morphism of elliptic curves $\varphi : E_1/S_1 \to E_2/S_2$. This is precisely the definition of [19]. We define a similar notion of *metaplectic forms*, i.e. sections of

$$\underline{\omega}^{k/2} := (\underline{\omega}^{1/2})^{\otimes k}$$

over the metaplectic stack $\mathcal{M}_{1/2}$.

Definition 2.1.5. A *metaplectic form of weight* $k/2$, with $k \in \mathbb{Z}$, is a global section of $\underline{\omega}^{k/2}$ over $\mathcal{M}_{1/2}$. In other words, it is a rule f that to each pair $(E/S, \mathcal{Q})$ of an elliptic curve over a scheme $S \in \mathrm{Ob}(Sch[1/2])$ and a non-degenerate rank 1 $\underline{\omega}_{E/S}$-valued quadratic form $q : \mathcal{Q} \to \underline{\omega}_{E/S}$, assigns an element

$$f(E/S, \mathcal{Q}) \in \Gamma(S, \mathcal{Q}^k),$$

functorially with respect to similitudes lying above morphisms of elliptic curves.

2.2 Vector-valued modular forms

2.2.1 Geometric representations of Heisenberg groups

Let $\pi : E \to S$ be an elliptic curve with identity section $e : S \to E$, and let $\mathcal{L}$ be an invertible sheaf on E.

Definition 2.2.1. ([23], §6) $K_E(\mathcal{L})$ is the group of sections $P : S \to E$ of π such that:

$$T_P^*(\mathcal{L}) \simeq \mathcal{L} \otimes \pi^* \mathcal{L}_0$$

for some invertible sheaf $\mathcal{L}_0$ over S. $K_{E,0}(\mathcal{L})$ is the subgroup of $K_E(\mathcal{L})$ given by those sections P with:

$$T_P^*(\mathcal{L}) \simeq \mathcal{L}.$$

$\mathcal{G}_E(\mathcal{L})$ is the group of pairs (P, φ) with $P \in K_{0,E}(\mathcal{L})$ and $\varphi : T_P^*(\mathcal{L}) \simeq \mathcal{L}$.

For any S-scheme T, the assignments

$$\begin{aligned} \underline{K}_E(\mathcal{L}) &: T \longmapsto K_E(\mathcal{L} \times_S T) \\ \underline{\mathcal{G}}_E(\mathcal{L}) &: T \longmapsto \mathcal{G}_E(\mathcal{L} \times_S T) \end{aligned} \tag{2.8}$$

define functors from the category of S-schemes to the category of groups. If $\mathcal{L}$ is relatively ample over S, then these functors are representable by group schemes which are flat and of finite presentation over S (in fact the first functor is actually a finite subgroup scheme of E). If we denote these schemes again by $K_E(\mathcal{L})$ and $\mathcal{G}_E(\mathcal{L})$, respectively, then by ([23], Prop. 1) there is an exact sequence of group schemes over S:

$$0 \to \mathbb{G}_m \to \mathcal{G}_E(\mathcal{L}) \to K_E(\mathcal{L}) \to 0,$$

which makes $\mathcal{G}_E(\mathcal{L})$ into a Heisenberg group. In particular, the finite flat group scheme $K_E(\mathcal{L})$ is always of order d^2, for some integer d.

Definition 2.2.2 ([23], §6). An invertible sheaf $\mathcal{L}$ on E/S is *symmetric* if $[-1]^*\mathcal{L} \simeq \mathcal{L}$, where $[-1] : E \to E$ is the inversion map. It is *totally symmetric* if there is an isomorphism $\varphi : \mathcal{L} \xrightarrow{\simeq} [-1]^*\mathcal{L}$ which restricts to the identity on $\mathcal{L} \otimes \mathcal{O}_{E[2d]}$, where d^2 is the order of $K_E(\mathcal{L})$ and $E[2d]$ is the subgroup scheme of $2d$-torsion. The sheaf $\mathcal{L}$ is *normalized* if $e^*\mathcal{L} \simeq \mathcal{O}_S$.

Let now $m \in 2\mathbb{Z}_{>0}$ be a positive even integer and suppose we work over the category $Sch[1/m]$ of schemes where m is invertible. For any elliptic curve $E \to S$ with $S \in \mathrm{Ob}(Sch[1/m])$, consider the invertible sheaf

$$\mathcal{L}_m := \mathcal{O}_E(m\, e) \otimes (\Omega^1_{E/S})^{\otimes m}, \tag{2.9}$$

where $\Omega^1_{E/S}$ is the sheaf of relative differentials on E.

Proposition 2.2.3. *Let $m \in 2\mathbb{Z}_{>0}$. For any elliptic curve $\pi : E \to S$, $S \in \mathrm{Ob}(Sch[1/m])$, the invertible sheaf $\mathcal{L}_m$ is relatively ample, totally symmetric and normalized along e.*

Proof. The sheaf $\mathcal{L}_m$ is relatively ample since π is proper and $\mathcal{L}_m$ is ample at each geometric fiber. It is symmetric since $[-1]^*\Omega^1_{E/S} \simeq \Omega^1_{E/S}$ by functoriality of the sheaf of differentials and since:

$$[-1]^*\mathcal{O}_E(m\, e) = \mathcal{O}_E(m\,(-e)) \simeq \mathcal{O}_E(m\, e).$$

Write $m = 2m'$. Then:

$$\mathcal{L}_m \simeq \mathcal{L}_{m'}^{\otimes 2}$$

and $\mathcal{L}_{m'}$ is symmetric, thus we can pick an isomorphism $\varphi : \mathcal{L}_{m'} \simeq [-1]^*\mathcal{L}_{m'}$. But then:

$$\varphi^{\otimes 2} : \mathcal{L}_m \xrightarrow{\simeq} [-1]^*\mathcal{L}_m$$

is the identity over $\mathcal{L}_m \otimes \mathcal{O}_{E[2m]}$, thus $\mathcal{L}_m$ is totally symmetric.

To show that $\mathcal{L}_m$ is normalized, consider the exact sequence of sheaves:

$$0 \to \mathcal{O}_E \to \mathcal{O}_E(e) \to e_*e^*\mathcal{O}_E(e) \to 0.$$

Taking the long exact sequence of derived functors of π_*, we get an exact sequence:

$$0 \to \pi_*\mathcal{O}_E \to \pi_*\mathcal{O}_E(e) \to e^*\mathcal{O}_E(e) \to R^1\pi_*\mathcal{O}_E \to R^1\pi_*\mathcal{O}_E(e) \to \dots$$

But $R^1\pi_*\mathcal{O}_E(e) = 0$, as can be verified on the geometric fibers, hence:

$$e^*\mathcal{O}_E(e) \simeq R^1\pi_*\mathcal{O}_E,$$

since all the nonzero sheaves appearing in the sequence are locally free of rank 1. Now by Serre duality

$$R^1\pi_*\mathcal{O}_E \simeq \underline{\omega}^{-1}_{E/S}$$

and thus:

$$e^*\mathcal{L}_m \simeq (e^*\mathcal{O}(e))^{\otimes m} \otimes \underline{\omega}^m_{E/S} \simeq \mathcal{O}_S$$

so that $\mathcal{L}_m$ is indeed normalized along e.

□

For $\mathcal{L}_m$ as above the corresponding Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$ is an extension:

$$0 \to \mathbb{G}_m \to \mathcal{G}_E(\mathcal{L}_m) \to E[m] \to 0,$$

where $E[m]$ is the m-torsion subgroup scheme of E, and whose commutator pairing is the Weil pairing:

$$e_m : E[m] \times E[m] \to \mu_m.$$

DEFINITION 2.2.4. The group $\mathcal{G}_E(\mathcal{L}_m)$ is called the *level m Heisenberg group* of an elliptic curve $E \to S$.

PROPOSITION 2.2.5. *For $m \in 2\mathbb{Z}_{>0}$ a positive even integer, the level m Heisenberg group of an elliptic curve $E \to S$, $S \in \mathrm{Ob}(Sch[1/m])$, is a symmetric Heisenberg group.*

Proof. The sheaf $\mathcal{L}_m$ is symmetric. In particular, if $\psi : \mathcal{L}_m \xrightarrow{\simeq} [-1]^*\mathcal{L}_m$ is any isomorphism then we can define an isomorphism $\delta_{-1} : \mathcal{G}_E(\mathcal{L}_m) \xrightarrow{\simeq} [-1]^*\mathcal{G}_E(\mathcal{L}_m)$ given by ([22], Definition on p. 308)

$$\delta_{-1}(P, \varphi) = (-P, (T^*_{-P}\psi)^{-1} \circ ([-1]^*\varphi) \circ \psi).$$

□

The Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$ acts on sections of $\pi_*\mathcal{L}_m$ via

$$U_{(P,\varphi)}(s) = T^*_{-P}(\varphi^{-1}(s)). \qquad (2.10)$$

With respect to this action, we have:

PROPOSITION 2.2.6. *Let $m \in 2\mathbb{Z}_{>0}$. For any elliptic curve $\pi : E \to S$, $S \in \mathrm{Ob}(Sch[1/m])$, the sheaf $\pi_*\mathcal{L}_m$ is a weight 1, rank m representation of the Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$.*

Proof. This follows directly from [23] p.81, since by Proposition 2.2.3 the invertible sheaf $\mathcal{L}_m$ is relatively ample, totally symmetric and normalized along e. □

Now the formation of $\pi_*\mathcal{L}_m$ is functorial with respect to morphisms of elliptic curves over schemes $S \in \mathrm{Ob}(Sch[1/m])$, thus the rule:

$$\{\pi : E \to S\} \longmapsto \Gamma(S, \pi_*\mathcal{L}_m)$$

defines a rank m locally free $\mathcal{O}_{\mathcal{M}_1}$-module $\mathcal{J}_m$ over the modular stack $\mathcal{M}_1 \to Sch[1/m]$.

DEFINITION 2.2.7. The sheaf $\mathcal{J}_m$ over $\mathcal{M}_1 \to Sch[1/m]$ is the *geometric representation* of the level m Heisenberg group of elliptic curves.

2.2.2 Schrödinger representations of Heisenberg groups

As in the previous section, we let $m \in 2\mathbb{Z}_{>0}$ be a positive even integer and we work over the category $Sch[1/m]$ of schemes over $\mathbb{Z}[1/m]$. For an elliptic curve $E \to S$, $S \in \mathrm{Ob}(Sch[1/m])$, let $\mathcal{V}_H$ be a weight 1, rank m Schrödinger representation (Definition 1.3.8) of the level m Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$ over E . This representation will in general depend on a choice of lagrangian subgroup $H \subset \mathcal{G}_E(\mathcal{L}_m)$, so its formation is *not* functorial with respect to morphism of elliptic curves. Thus there can be no 'universal' Schrödinger representation over $\mathcal{M}_1$. However, following the approach of Polishchuk ([29], §2) we have:

PROPOSITION 2.2.8. *Let $m \in 2\mathbb{Z}_{>0}$ be a positive even integer. The rule*

$$\{E \to S\} \longmapsto \Gamma(S, \mathrm{End}_{\mathcal{O}_S}(\mathcal{V}_H)),$$

that to each elliptic curve $E \to S$, $S \in \mathrm{Ob}(Sch[1/m])$, assigns the endomorphism algebra of a Schrödinger representation of $\mathcal{G}_E(\mathcal{L}_m)$, defines an Azumaya algebra with 2-torsion datum (Definition 1.2.11) $(\mathcal{A}_m, \mathcal{W}_m, \iota)$ over the modular stack $\mathcal{M}_1 \to Sch[1/m]$.

Proof. The formation of $\mathcal{G}_E(\mathcal{L}_m)$ is compatible under base-change, therefore if (ϕ, φ) : $E_1/S_1 \to E_2/S_2$ is a morphism of elliptic curves, equipped with Schrödinger representations $\mathcal{V}_{H_1}$ and $\mathcal{V}_{H_2}$, we have $\mathcal{G}_{E_1}(\mathcal{L}_m) \simeq \mathcal{G}_{E_2}(\mathcal{L}_m)$ and thus:

$$\varphi^* \mathcal{V}_{H_2} \simeq \mathcal{V}_{H_1} \otimes_{\mathcal{O}_{S_1}} \mathcal{L}$$

for some invertible sheaf $\mathcal{L}$, by Theorem 1.3.3. But then

$$\mathrm{End}_{\mathcal{O}_{S_1}}(\mathcal{V}_{H_1}) \simeq \mathrm{End}_{\mathcal{O}_{S_1}}(\varphi^* \mathcal{V}_{H_2}),$$

hence the rule defining $\mathcal{A}_m$ gives a sheaf of $\mathcal{O}_{\mathcal{M}_1}$-algebras on $\mathcal{M}_1$. To show that it is an Azumaya algebra, choose a collection of étale morphisms $\{u_i : U_i \to \mathcal{M}_1\}$ from schemes $U_i \in \mathrm{Ob}(Sch[1/m])$ such that $\coprod_i U_i \to \mathcal{M}_1$ is surjective. Then by refining the cover if necessary, each of the elliptic curves $E_i \to U_i$ possesses a Lagrangian subgroup $H_i \subset \mathcal{G}_{E_i}(\mathcal{L}_m)$, since these always exists after étale base-change (as follows from example from the theory of theta structures, [23], §6, or Section 2.3.3 below). But then by Corollary 1.3.7 we can construct a Schrödinger representation $\mathcal{V}_{H_i}$ of $\mathcal{G}_{E_i}(\mathcal{L}_m)$, and thus $\mathcal{A}_m$ is an Azumaya algebra, since

$$\mathcal{A}_m|_{E_i} \simeq \mathrm{End}_{\mathcal{O}_{U_i}}(\mathcal{V}_{m,E_i})$$

by definition. Next, note that for any elliptic curve E/S the Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$ is symmetric by Proposition 2.2.5. Hence by Lemma 1.3.10 the product $\mathcal{G}_E(\mathcal{L}_m) \times \mathcal{G}_E(\mathcal{L}_m)$ has a canonical Schrödinger representation $\mathcal{W}_{E/S}$ of rank m^2. The assignment

$$\{E \to S\} \mapsto \Gamma(S, \mathcal{W}_{E/S})$$

is functorial with respect to morphisms of elliptic curves, since $\mathcal{W}_{E/S}$ corresponds to the lagrangian subgroup $\Delta : E[m] \to E[m] \times E[m]$, whose formation is clearly functorial with respect to morphisms of elliptic curves. Thus the assignment defines a locally free sheaf $\mathcal{W}_m$ of rank m^2 over $\mathcal{M}_1$ together with an isomorphism

$$\iota : \mathcal{W}_m \simeq \mathcal{A}_m^{\otimes 2},$$

as explained at the end of Section 1.3.5. □

We now apply the theory of Section 1.2.2 to find a locally free sheaf $\mathcal{V}_m$ of rank m such that $\mathrm{End}(\mathcal{V}_m) \simeq \mathcal{A}_m$ and $\mathcal{V}_m^{\otimes 2} \simeq \mathcal{W}_m$. Namely, we construct a μ_2-gerbe

$$p : \mathcal{K}(\mathcal{A}_m, \iota) \to \mathcal{M}_1,$$

canonically endowed with a locally free $\mathcal{O}_{\mathcal{K}(\mathcal{A}_m,\iota)}$-module $\mathcal{V}_m$ of rank m such that:

$$p^*\mathcal{A}_m \simeq \mathrm{End}_{\mathcal{O}_{\mathcal{K}(\mathcal{A}_m,\iota)}}(\mathcal{V}_m),$$

and

$$p^*\mathcal{W}_m \simeq \mathcal{V}_m^{\otimes 2}.$$

Definition 2.2.9. The gerbe $p : \mathcal{K}(\mathcal{A}_m, \iota) \to \mathcal{M}_1$ is called the *Schrödinger gerbe* of level m. It is the category whose objects above E/S are locally free $\mathcal{O}_S$-modules $\mathcal{V}_m$ of rank m such that $\mathcal{A}_m|_E \simeq \mathrm{End}(\mathcal{V}_m)$, $\mathcal{W}_m|_E \simeq \mathcal{V}_m^{\otimes 2}$, and whose morphisms are isomorphisms making the obvious diagrams commute.

Definition 2.2.10. The sheaf $\mathcal{V}_m$ over $\mathcal{K}(\mathcal{A}_m, \iota) \to Sch[1/m]$ is the *Schrödinger representation* of the level m Heisenberg group of elliptic curves.

2.2.3 Vector-valued modular forms

Again let $m \in 2\mathbb{Z}_{>0}$ be a positive even integer and consider the contracted product ([15], §IV.2.4):

$$\mathcal{K}(\mathcal{A}_m, \iota) \times^{\mu_2}_{\mathcal{M}_1} \mathcal{M}_{1/2} \to \mathcal{M}_1$$

in the category of μ_2-gerbes over $\mathcal{M}_1 \to Sch[1/m]$. This is a μ_2-gerbe over $\mathcal{M}_1$ classifying triples $(E/S, \mathcal{Q}, \mathcal{V}_H)$ of an elliptic curve over a scheme $S \in Sch[1/m]$, a non-degenerate, rank 1, $\underline{\omega}_{E/S}$-valued quadratic form $\mathcal{Q}_H$ and a Schrödinger representation $\mathcal{V}_H$ of the level

m Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$. For any integer $k \in \mathbb{Z}$, the sheaf $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ is defined over $\mathcal{K}(\mathcal{A}_m, \iota) \times^{\mu_2}_{\mathcal{M}_1} \mathcal{M}_{1/2}$.

Definition 2.2.11. A weight $k/2$, *$\mathcal{V}_m$-valued modular form* is a global section of $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ over the stack $\mathcal{K}(\mathcal{A}_m, \iota) \times^{\mu_2}_{\mathcal{M}_1} \mathcal{M}_{1/2}$. In other words, it is a rule that to each triple $(E/S, \mathcal{Q}, \mathcal{V}_H)$ of an elliptic curve over a scheme $S \in Sch[1/m]$, a non-degenerate, rank 1, $\underline{\omega}_{E/S}$-valued quadratic form $\mathcal{Q}$ and a Schrödinger representation $\mathcal{V}_H$ of the level m Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$ it attaches an element

$$f(E/S, \mathcal{Q}, \mathcal{V}_H) \in \Gamma(S, \mathcal{V}_H \otimes \mathcal{Q}^k),$$

functorially with respect to similitudes of $\mathcal{Q}$ and $\mathcal{V}_H$ lying above morphisms of elliptic curves.

Remark 2.2.12. We would also like to consider vector-valued modular forms with coefficients in $\mathcal{V}_m^\vee$ where $\mathcal{V}_m^\vee$ is the dual of $\mathcal{V}_m$. In particular, a weight $k/2$ *$\mathcal{V}_m^\vee$-valued modular form* is a global section of $\mathcal{V}_m^\vee \otimes \underline{\omega}^{k/2}$.

Definition 2.2.13. Let $m \in 2\mathbb{Z}_{>0}$ and let R be a ring containing $1/m$. A rank 1, index $m/2$, weight $k/2$ vector-valued modular form is *defined over* R if it is a section of the sheaf $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ over the stack $\mathcal{K}(\mathcal{A}_m, \iota) \times^{\mu_2}_{\mathcal{M}_1} \mathcal{M}_{1/2} \to Sch/R$.

The adjectives *weakly holomorphic, level one* should also be added to our definition of vector-valued modular forms. However, since in this work we do not discuss neither growth conditions at the cusps nor extensions to higher level, these adjectives will be omitted to lighten the terminology.

The reason why we use the term 'modular form', is the following theorem, which shows that, at least for k odd, vector-valued modular forms are indeed defined over the modular stack $\mathcal{M}_1$.

Theorem 2.2.14. *Let $m \in 2\mathbb{Z}_{>0}$ be a positive even integer and let $k \in \mathbb{Z}$ be odd. Then the sheaf $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$, defined over $\mathcal{K}(\mathcal{A}_m, \iota) \times^{\mu_2}_{\mathcal{M}_1} \mathcal{M}_{1/2}$, descends to a sheaf, also denoted by $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$, over the modular stack $\mathcal{M}_1$.*

Proof. The automorphism group of the map $\mathcal{K}(\mathcal{A}_m, \iota) \times^{\mu_2}_{\mathcal{M}_1} \mathcal{M}_{1/2} \to \mathcal{M}_1$ is μ_2, generated by an element $[-1]^*$ which acts diagonally on $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$. We must compute the action of this automorphism on both $\mathcal{V}_m$ and $\underline{\omega}^{1/2}$. For the latter, note that μ_2 acts on $\underline{\omega}^{1/2}$ via isometries of the quadratic form $\underline{\omega}^{1/2} \to \underline{\omega}$, i.e. as automorphisms of $\underline{\omega}^{1/2}$ preserving the quadratic form. Since automorphisms of $\underline{\omega}^{1/2}$ are given by multiplication by a scalar, we have that:

$$[-1]^*\underline{\omega}^{1/2} = a \cdot \underline{\omega}^{1/2}$$

where $a \in \Gamma(\mathcal{M}_{1/2}, \mathcal{O}^*_{\mathcal{M}_{1/2}})$ such that $a^2 = 1$. To compute a, it suffices to compute on geometric points $\tilde{s} : \mathrm{Spec}(k) \to \mathcal{M}_{1/2}$, classifying pairs (E, q) of an elliptic curve E/k over an algebraically closed field of characteristic zero or not dividing m, together with a non-degenerate rank 1 quadratic form $q : \Gamma(\mathrm{Spec}(k), \tilde{s}^*\underline{\omega}^{1/2}) \to \Gamma(\mathrm{Spec}(k), \underline{\omega}_{E/S})$. In this case, if $a = 1$, then the action of the group $\mathrm{Aut}(E/k, q)$ on q factors through that of $\mathrm{Aut}(E/k)$. But in the proof of Proposition 2.1.4 we showed that the action of $\mathrm{Aut}(E/k, q)$ on q is faithful, so $a = -1$ and therefore:

$$[-1]^*\underline{\omega}^{k/2} = (-1)^k\underline{\omega}^{k/2}.$$

For $\mathcal{V}_m$, note that the group μ_2 of automorphisms of the map $\mathcal{K}(\mathcal{A}_m, \iota) \to \mathcal{M}_1$ acts on $\mathcal{V}_m$ via transformations B in $\mathrm{GL}(\mathcal{V}_m)$ which fix $\mathcal{V}_m^{\otimes 2}$, i.e. $B^2 = I$, and whose inner action fixes $\mathrm{End}(\mathcal{V}_m)$, i.e. B is a scalar. Thus $B = \pm I$. To compute B, again it suffices to look at geometric points $\tilde{s} : \mathrm{Spec}(k) \to \mathcal{K}(\mathcal{A}_m, \iota)$ classifying pairs $(E/k, V_m)$ of an elliptic curve over an algebraically closed field of characteristic zero or not dividing m, together with a Schrödinger representation of its level m Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$. If $B = I$, then the action of $\mathrm{Aut}(E/k, V_m)$ on V_m would factor through that of $\mathrm{Aut}(E/k)$. Now the action of $\mathrm{Aut}(E/k, V_m)$ on V_m is given by the Weil representation [36], for which there are explicit formulas. In particular, if $\{\delta_r\}_{r \in \mathbb{Z}/m\mathbb{Z}}$ is a basis of delta functions for V_m, and if we denote by z a lift of the element $-1 \in \mathrm{Aut}(E/k)$ to $\mathrm{Aut}(E/k, V_m)$, then the action of z is ([27], Satz 2):

$$z(\delta_r) = i \cdot \delta_{-r},$$

where $i \in k^\times$ is a *primitive* 4-th root of unity. In particular, the action of $\mathrm{Aut}(E/k, V_m)$ on V_m is faithful, so $B = -I$ and thus:

$$[-1]^*\mathcal{V}_m = (-1) \cdot \mathcal{V}_m.$$

Combining the two computations, we see that when k is odd we must have:

$$[-1]^*\left(\mathcal{V}_m \otimes \underline{\omega}^{k/2}\right) \simeq \mathcal{V}_m \otimes \underline{\omega}^{k/2}$$

and thus $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ descends to a locally free sheaf of rank m on $\mathcal{M}_1$. □

Therefore, even if neither $\mathcal{V}_m$ nor $\underline{\omega}^{k/2}$ (k odd) descend to $\mathcal{M}_1$, *their product does!* Hence when k is odd (which is usually the interesting case) sections of $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ deserve to be called *modular* forms.

Theorem 2.2.14 perhaps also justifies why we tensor sections of $\mathcal{V}_m$ by $\underline{\omega}^{k/2}$, and do not instead consider sections of $\mathcal{V}_m$ alone over the stack $\mathcal{K}(\mathcal{A}_m, \iota)$. In particular, if we introduce level structures to rigidify $\mathcal{M}_1$ into a scheme, then we can speak of sections of $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ over a scheme: this cannot happen with sections of $\mathcal{V}_m$ or $\underline{\omega}^{k/2}$ alone, since the stacks $\mathcal{K}(\mathcal{A}_m, \iota)$ and $\mathcal{M}_{1/2}$ cannot be rigidified into schemes using level structures.

REMARK 2.2.15. It should be noted that it many cases there is a canonical equivalence

$$\mathcal{M}_{1/2} \simeq \mathcal{K}(\mathcal{A}_m, \iota)$$

as μ_2-gerbes over $\mathcal{M}_{1/2}$. This is because these two stacks are classified by the cocycles

$$\delta_2(\underline{\omega}), \delta_2(\mathcal{A}_m, \iota) \in H^2(\mathcal{M}_1, \mu_2),$$

respectively. Now suppose that $S \in \mathrm{Ob}(Sch[1/m])$ and let $\mathcal{M}_{1,S}$ be the moduli stack of elliptic curves over a scheme $T \to S$. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(\mathbb{A}^1_S, j_* R^q \mu_2) \Longrightarrow H^{p+q}(\mathcal{M}_{1,S}, \mu_2)$$

of the map $j : \mathcal{M}_{1,S} \to \mathbb{A}^1_S$. The two cocycles $\delta_2(\underline{\omega})$ and $\delta_2(\mathcal{A}_m, \iota)$ must agree over the fibers of j, since, as shown in the proof of Theorem 2.2.14, the cocycles are always nontrivial and

$$H^2(\mathrm{Aut}(E/k), \{\pm 1\}) = \mathbb{Z}/2\mathbb{Z},$$

for all elliptic curves E over an algebraically closed field of characteristic zero or not dividing m. Thus the product $\delta_2(\underline{\omega}) \cdot \delta_2(\mathcal{A}_m, \iota)^{-1}$ always descends to

$$H^2(\mathbb{A}^1_S, \mu_2) = \mathrm{Br}(\mathbb{A}^1_S)[2],$$

the 2-torsion of the Brauer group of the affine line $\mathbb{A}^1_S$. This vanishes in many cases: for example if $S = \mathrm{Spec}(R)$ is the spectrum of a ring, then

$$\mathrm{Br}(\mathbb{A}^1_R) = \mathrm{Br}(R),$$

which vanishes when R is an algebraically closed field or a finite field.

2.2.4 Comparison between geometric and Schrödinger representations

Consider the sheaf $\mathcal{V}_m \otimes \underline{\omega}^{-1/2}$ of weight -1/2 $\mathcal{V}_m$-valued modular forms. By Theorem 2.2.14, this sheaf is defined over $\mathcal{M}_1$. The 'Main Theorem' below shows that $\mathcal{V}_m \otimes \underline{\omega}^{-1/2}$ is canonically isomorphic to the sheaf $\mathcal{J}_m$ of geometric representations of the level m Heisenberg group of elliptic curves (Definition 2.2.7).

THEOREM 2.2.16 (Main Theorem). *Let $m \in 2\mathbb{Z}_{>0}$ and let $\mathcal{M}_1 \to Sch[1/3m]$ be the modular stack of elliptic curves over schemes where 3 and m are invertible. Then there is a canonical isomorphism*

$$\mathcal{V}_m \otimes \underline{\omega}^{-1/2} \simeq \mathcal{J}_m$$

of locally free $\mathcal{O}_{\mathcal{M}_1}$-modules of rank m over $\mathcal{M}_1$, defined up to multiplication by an element in $\Gamma(\mathcal{M}_1, \mathcal{O}_{\mathcal{M}_1}^\times)$.

Proof. Let $s : S \to \mathcal{M}_1$ be a morphism from a scheme S and let $\pi : E \to S$ be the corresponding elliptic curve. By Proposition 2.2.6, the sheaf $s^*\mathcal{J}_m \simeq \pi_*\mathcal{L}_m$ is a weight 1, rank m representation of the level m symmetric Heisenberg group $\mathcal{G}_E(\mathcal{L}_m)$. The same is true for $s^*(\mathcal{V}_m \otimes \underline{\omega}^{-1/2})$, if we endow $\underline{\omega}^{-1/2}$ with the trivial action. Thus by Theorem 1.3.3, there is an invertible sheaf $\mathcal{L}_S$ on S such that:

$$s^*(\mathcal{V}_m \otimes \underline{\omega}^{-1/2}) \simeq s^*\mathcal{J}_m \otimes_{\mathcal{O}_S} \mathcal{L}_S$$

as $\mathcal{G}_E(\mathcal{L}_m)$-representations. This defines an invertible sheaf $\mathcal{L}_S$ for every elliptic curve $E \to S$. Moreover, for any morphism of elliptic curves $f : E'/S' \to E/S$ we must have $f^*\mathcal{L}_S \simeq \mathcal{L}_{S'}$, since the above isomorphism is an isomorphism of $\mathcal{G}_E(\mathcal{L}_m)$-modules ([23], p.82). Thus we have constructed an invertible sheaf $\mathcal{L}$ over $\mathcal{M}_1$ such that:

$$\mathcal{V}_m \otimes \underline{\omega}^{-1/2} \simeq \mathcal{J}_m \otimes \mathcal{L}. \tag{2.11}$$

We want to show that $\mathcal{L} \simeq \mathcal{O}_{\mathcal{M}_1}$, that is, the class of $\mathcal{L}$ in $\mathrm{Pic}(\mathcal{M}_1)$ is trivial. To prove it, we again employ the direct method of [21] §6, as in our computation of $\mathrm{Pic}(\mathcal{M}_{1/2})$ (Proposition 2.1.4). Namely, let k be an algebraically closed field of characteristic 0 or not dividing $3m$, and consider the two geometric points

$$s_1, s_2 : \mathrm{Spec}(k) \to \mathcal{M}_1,$$

classifying the two elliptic curves over k:

$$\begin{aligned} E_1 : &\quad y^2 = x^3 - x = x(x+1)(x-1) \\ E_2 : &\quad y^2 = x^3 - 1 = (x-1)(x-\zeta)(x-\zeta^2), \end{aligned}$$

for $\zeta \in k^\times$ a primitive 3rd root of unity. These elliptic curves are endowed with the special automorphisms σ of (2.2) and τ of (2.4), respectively. Now σ acts on $s_1^*\mathcal{L}$ by functoriality via a scalar $\sigma_{\mathcal{L}}$ and τ acts on $s_2^*\mathcal{L}$ via a scalar $\tau_{\mathcal{L}}$. We need to show that:

$$\sigma_{\mathcal{L}} = 1, \quad \tau_{\mathcal{L}} = 1.$$

In turn, to compute $\sigma_{\mathcal{L}}$ (resp. $\tau_{\mathcal{L}}$) we must compare the action of σ (resp. τ) on $s_1^*\mathcal{J}_m$ (resp. $s_2^*\mathcal{J}_m$) to the action of $\widetilde{\sigma}$ (resp. $\widetilde{\tau}$) on $\tilde{s}_1^*\mathcal{V}_m$ (resp. $\tilde{s}_2^*\mathcal{V}_m$), where

$$\tilde{s}_1, \tilde{s}_2 : \mathrm{Spec}(k) \to \mathcal{K}(\mathcal{A}_m, \iota)$$

are geometric points classifying E_1 (resp. E_2), together with a Schrödinger representation V_m of the level m Heisenberg group, and $\widetilde{\sigma}$ (resp. $\widetilde{\tau}$) is a generator for $\mathrm{Aut}(E_1, V_m)$ (resp. $\mathrm{Aut}(E_2, V_m)$).

(i) *Action of σ, τ on $\mathcal{J}_m$*: For any elliptic curve E classified by $s : \mathrm{Spec}(k) \to \mathcal{M}_1$, we have:

$$s^* \mathcal{J}_m \simeq \Gamma(E, \mathcal{O}_E(me) \otimes (\Omega^1_E)^{\otimes m}).$$

This is a m-dimensional vector space spanned by

$$\{\omega^{\otimes m}, x\omega^{\otimes m}, y\omega^{\otimes m}, x^2\omega^{\otimes m}, xy\omega^{\otimes m}, \ldots, x^{m/2-2}y\omega^{\otimes m}, x^{m/2}\omega^{\otimes m}\} \tag{2.12}$$

where E is given by a Weierstrass equation of the form:

$$y^2 = x^3 + a_4 x + a_6, \quad a_4, a_6 \in k,$$

and $\omega = dx/y$.

In particular, for $E = E_1$ the matrix of the action of σ on $s_1^* \mathcal{J}_m$ with respect to this basis is given by:

$$\sigma_{\mathcal{J}_m} = i^m \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & (-1)^{m/2-2} \cdot i & \\ & & & & (-1)^{m/2} \end{pmatrix}.$$

as follows from (2.2). Thus $\sigma_{\mathcal{J}_m}$ is a diagonal matrix with entries $1, -1, i, -i$, whose multiplicities depend on whether $m \equiv 0, 2 \mod 4$ and can be computed as in Table 2.1 using a simple induction argument.

Table 2.1: Multiplicities of the eigenvalues of $\sigma_{\mathcal{J}_m}$

m	1	-1	i	$-i$
$4k+2$	$k+1$	$k+1$	k	k
$4k$	$k+1$	k	k	$k-1$

For $E = E_2$, the matrix of $\tau_{\mathcal{J}_m}$ with respect to the basis (2.12) is given by:

$$\tau_{\mathcal{J}_m} = (-\zeta)^m \begin{pmatrix} 1 & & & & \\ & \zeta & & & \\ & & \ddots & & \\ & & & -\zeta^{m/2-2} & \\ & & & & \zeta^{m/2} \end{pmatrix},$$

as follows from (2.4). A simple induction argument shows that

$$\mathrm{Tr}(\tau_{\mathcal{J}_m}) = -\zeta \tag{2.13}$$

for all m.

(ii) <u>*Action of* $\widetilde{\sigma}, \widetilde{\tau}$ *on* $\mathcal{V}_m$</u>:

Let $\tilde{s} : \mathrm{Spec}(k) \to \mathcal{K}(\mathcal{A}_m, \iota)$ be a geometric point classifying a pair (E, V_m) of an elliptic curve E/k and a Schrödinger representation V_m of the level m Heisenberg group. The automorphism group $\mathrm{Aut}(E/k, V_m)$ is a central extension:

$$0 \to \mu_2 \to \mathrm{Aut}(E/k, V_m) \to \mathrm{Aut}(E/k) \to 0,$$

which is nontrivial by the proof of Theorem 2.2.14. Thus, for $E = E_1$ the group $\mathrm{Aut}(E_1/k, V_m)$ is cyclic of order 8, generated by a lift $\tilde{\sigma}$ of the automorphism $\sigma \in \mathrm{Aut}(E_1/k)$ of (2.2), and for $E = E_2$ the group $\mathrm{Aut}(E_2/k, V_m)$ is cyclic of order 12, generated by a lift $\tilde{\tau}$ of the automorphism $\tau \in \mathrm{Aut}(E_2/k)$ of (2.4). We need to compute the action of these generators on the k-vector spaces V_m. This action is given by the Weil representation ([36]), which we now define.

Consider the subgroup $E[2m] \subset E$ of $2m$-torsion. The group $\mathrm{Aut}(E[2m], V_m)$ is a central extension:

$$0 \to \mu_2 \to \mathrm{Aut}(E[2m], V_m) \to \mathrm{Sp}(E[2m]) \to 0,$$

where we view $E[2m]$ as a rank 2 symplectic $\mathbb{Z}/2m\mathbb{Z}$-module with symplectic form given by the Weil pairing $e_{2m} : E[2m] \times E[2m] \to \mu_{2m}$. There is a natural embedding:

$$\mathrm{Aut}(E/k, V_m) \hookrightarrow \mathrm{Aut}(E[2m], V_m)$$

given by the faithful action of the automorphisms of E on $E[2m]$. In particular, the extension $\mathrm{Aut}(E[2m], V_m)$ is non-trivial: the automorphism $[-1]$ acts on $E[2m]$ via the scalar matrix $-I$, thus $\mathrm{Aut}(E[2m], V_m)$ contains an element Z which lifts $-I$ and is of order 4, as we showed in the proof of Theorem 2.2.14. But if the extension were trivial, Z would have to be contained in a subgroup isomorphic to the Klein 4 group, which is a contradiction.

Now it is well-known that $E[2m] \simeq \mathbb{Z}/2m\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$, thus

$$\mathrm{Sp}(E[m]) \simeq \mathrm{SL}_2(\mathbb{Z}/2m\mathbb{Z}).$$

The central extensions of $\mathrm{SL}_2(\mathbb{Z}/2m\mathbb{Z})$ by μ_2 are classified by the cohomology group $H^2(\mathrm{SL}_2(\mathbb{Z}/2m\mathbb{Z}), \mu_2)$, for which we have ([2]):

$$H^2(\mathrm{SL}_2(\mathbb{Z}/2m\mathbb{Z}), \mu_2) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Since $\mathrm{Aut}(E[2m], V_m)$ is a non-trivial extension, we must have:

$$\mathrm{Aut}(E[2m], q) \simeq \mathrm{Mp}_2(\mathbb{Z}/2m\mathbb{Z}),$$

where the group on the right is the *metaplectic cover of* $\mathrm{SL}_2(\mathbb{Z}/2m\mathbb{Z})$, the unique nontrivial central extension of $\mathrm{SL}_2(\mathbb{Z}/2m\mathbb{Z})$ by μ_2. In particular, we see that the isomorphism class of $\mathrm{Aut}(E[2m], V_m)$, just like that of $\mathrm{Sp}(E[2m])$, does not depend on the elliptic curve E.

Now the vector space V_m is the space of functions

$$f : H \to k$$

where $H \subset E[m]$ is a lagrangian subgroup. This space is a weight 1, rank m representation

$$U : \mathcal{G}_E(\mathcal{L}_m) \to \mathrm{GL}(V_m)$$

of the level m Heisenberg group over E/k. By Weil ([36]), there is a representation

$$\rho_m : \mathrm{Mp}_2(\mathbb{Z}/2m\mathbb{Z}) \longrightarrow \mathrm{GL}(V_m),$$

by linear operators intertwining the representations U and U^γ, the representation U twisted by an element $\gamma \in \mathrm{Sp}(E[m])$. There are explicit formulas for this representation. For example, if we denote by S, T some lifts to $\mathrm{Mp}_2(\mathbb{Z}/2m\mathbb{Z})$ of the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/2m\mathbb{Z}),$$

respectively, then the action of S, T on a basis $\{\delta_r\}_{r \in \mathbb{Z}/m\mathbb{Z}}$ of delta functions for V_m is given by ([27], Satz 2):

$$\begin{aligned} \rho_m(T)(\delta_r) &= \zeta_{2m}^{-r^2}\,\delta_r \\ \rho_m(S)(\delta_r) &= \frac{\Omega(\zeta_{2m},\sqrt{m})}{\sqrt{m}} \sum_{s\in\mathbb{Z}/m\mathbb{Z}} \zeta_{2m}^{2rs}\,\delta_s, \end{aligned} \tag{2.14}$$

where $\zeta_{2m} \in k^\times$ is a primitive $2m$-th root of unity, $\sqrt{m} \in k^\times$ is a choice of square root of m and

$$\Omega(\zeta_{2m},\sqrt{m}) := \frac{1}{\sqrt{m}} \sum_{r\in\mathbb{Z}/m\mathbb{Z}} \zeta_{2m}^{r^2}$$

is a primitive eight root of unity depending on the choices of ζ_{2m} and $\sqrt{m}$.

Let now $E = E_1$. We can choose a basis for $E_1[2m]$ such that the automorphism σ acts on $E_1[2m]$ via the matrix:

$$\sigma_{E_1[2m]} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Sp}(E_1[2m]),$$

for example by picking a generator P for a cyclic subgroup of order $2m$ and let $\{P, \sigma(P)\}$ be a basis. With respect to this basis, the generator $\tilde{\sigma} \in \mathrm{Aut}(E_1/k, V_m)$ acts on $\mathrm{Aut}(E_1[2m], q)$ via S, hence by (2.14) we can assume that it acts on V_m by:

$$\rho_m(S)(\delta_r) = \frac{\sqrt{i}}{\sqrt{m}} \sum_{s\in\mathbb{Z}/m\mathbb{Z}} \zeta_{2m}^{2rs}\,\delta_s$$

where the choice of $\sqrt{i}$ is the same as in (2.3). Thus:

$$\widetilde{\sigma}_{\mathcal{V}_m} = \rho_m(S) = \sqrt{i}\,\mathrm{DFT}(m)$$

where DFT is the *discrete Fourier transform* on $\mathbb{Z}/m\mathbb{Z}$. The matrix $\mathrm{DFT}(m)$ can be diagonalized: the eigenvalues are $1, -1, i, -i$ and their multiplicities have been computed by Schur (see for example [1], Theorem I.1.2$'$). They are given precisely by Table 2.1. Thus we must have:

$$\sqrt{i}\,\sigma_{\mathcal{J}_m} = \tilde{\sigma}_{\mathcal{V}_m} \tag{2.15}$$

for all m.

Similarly, let $E = E_2$. We can choose a basis for $E_2[2m]$ such that the automorphism τ acts on $E_2[2m]$ via the matrix:

$$\tau_{E_2[2m]} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Sp}(E_2[2m]),$$

for example by picking a generator P for a cyclic subgroup of order $2m$ and let $\{P, \tau(P)\}$ be a basis. Note that:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Sp}(E_2[2m]),$$

and therefore the automorphism $\widetilde{\tau}$ generating $\mathrm{Aut}(E_2/k, V_m)$ acts on $(E_2[2m], V_m)$ as ST and thus

$$\widetilde{\tau}_{\mathcal{V}_m} = \rho_m(ST).$$

Using the formulas (2.14) we can compute:

$$\mathrm{Tr}(\widetilde{\tau}_{\mathcal{V}_m}) = \frac{\Omega(\zeta_{2m}, \sqrt{m})}{\sqrt{m}} \sum_{r \in \mathbb{Z}/m\mathbb{Z}} \zeta_{2m}^{r^2} = \Omega(\zeta_{2m}, \sqrt{m})^2 \tag{2.16}$$

for all m, where we can assume that

$$\frac{\Omega(\zeta_{2m}, \sqrt{m})^2}{-\zeta} = \sqrt{-\zeta},$$

where $\sqrt{-\zeta}$ is chosen as in (2.5).

We are now ready to put everything together. By (2.11) we know that

$$\widetilde{\sigma}_{\mathcal{V}_m} \cdot \widetilde{\sigma}_{\underline{\omega}^{-1/2}} = \sigma_{\mathcal{J}_m} \cdot \sigma_{\mathcal{L}}$$

where $\widetilde{\sigma}_{\mathcal{V}_m}$ and $\sigma_{\mathcal{J}_m}$ are $m \times m$-dimensional linear transformations and $\widetilde{\sigma}_{\underline{\omega}^{-1/2}}$, $\sigma_{\mathcal{L}}$ are scalars. By (2.15), we know that

$$\widetilde{\sigma}_{\underline{\omega}^{1/2}} \cdot \sigma_{\mathcal{L}} = \sqrt{i},$$

but $\widetilde{\sigma}_{\underline{\omega}^{1/2}} = \sqrt{i}$ by (2.3), hence

$$\sigma_{\mathcal{L}} = 1.$$

Similarly, we have:

$$\widetilde{\tau}_{\mathcal{V}_m} \cdot \widetilde{\tau}_{\underline{\omega}^{-1/2}} = \tau_{\mathcal{J}_m} \cdot \tau_{\mathcal{L}},$$

and thus

$$\widetilde{\tau}_{\underline{\omega}^{1/2}} \cdot \tau_{\mathcal{L}} = \frac{\mathrm{Tr}(\widetilde{\tau}_{\mathcal{V}_m})}{\mathrm{Tr}(\tau_{\mathcal{J}_m})}.$$

By (2.13) and (2.16) we can compute the ratio of the traces on the right, so that:

$$\widetilde{\tau}_{\underline{\omega}^{1/2}} \cdot \tau_{\mathcal{L}} = \sqrt{-\zeta}.$$

But $\widetilde{\tau}_{\underline{\omega}^{1/2}} = \sqrt{-\zeta}$ by (2.5), thus

$$\tau_{\mathcal{L}} = 1,$$

and the theorem is proved. □

Corollary 2.2.17. *For any integer $k \in \mathbb{Z}$ there is a canonical isomorphism*

$$\mathcal{V}_m \otimes \underline{\omega}^{k-1/2} \simeq \mathcal{J}_m \otimes \underline{\omega}^k$$

over the modular stack $\mathcal{M}_1 \to Sch[1/3m]$, defined up to an element of $\Gamma(\mathcal{M}_1, \mathcal{O}^{\times}_{\mathcal{M}_1})$.

Remark 2.2.18. The notation $\mathcal{J}_m$ has been chosen because of the relation between this sheaf and the notion of *Jacobi forms* of [14]. This relation is fleshed out in Section 3.2.4 below. In particular, we will show that Corollary 2.2.17, and especially its refinement Corollary 2.2.21 given below, gives an algebraic proof of Theorem 5.1 of [14] relating vector-valued modular forms to Jacobi forms.

2.2.5 Heat equations and algebraic theta functions

To simplify matters, in this section we consider the sheaf $\mathcal{V}_m$ as being defined over the metaplectic stack $\mathcal{M}_{1/2}$. This can be done in light of Remark 2.2.15, or simply by replacing $\mathcal{V}_m$ by $(\mathcal{V}_m \otimes \underline{\omega}^{-1/2}) \otimes \underline{\omega}^{1/2}$, which is indeed defined over $\mathcal{M}_{1/2}$.

Let $\mathcal{M}_1(2m) \to \mathrm{Spec}(\mathbb{Z}[1/2m])$ be the moduli scheme of full level $2m$ arithmetic level structures on elliptic curves, i.e. isomorphisms

$$\lambda : E[2m] \xrightarrow{\simeq} \mathbb{Z}/2m\mathbb{Z} \times \mu_{2m},$$

which are required to be symplectic with respect to the Weil pairing on $E[2m]$ and the natural symplectic pairing

$$e_d((x_1, \zeta_1), (x_2, \zeta_2)) = \zeta_2^{x_1} (\zeta_1^{x_2})^{-1}$$

on $\mathbb{Z}/2m\mathbb{Z} \times \mu_{2m}$.

Consider the fiber product:

$$\mathcal{M}_{1/2}(2m) := \mathcal{M}_1(2m) \times_{\mathcal{M}_1} \mathcal{M}_{1/2},$$

in the category of Deligne-Mumford stacks. The fundamental theorem of this section is that $\mathcal{V}_m$ can be *trivialized* over $\mathcal{M}_{1/2}(2m)$:

Theorem 2.2.19. *There exists an isomorphism*

$$p_{2m}^{*}\mathcal{V}_m \simeq \bigoplus_{r\in\mathbb{Z}/m\mathbb{Z}} \mathcal{O}_{\mathcal{M}_{1/2}(2m)} \cdot \delta_r \tag{2.17}$$

over the stack $p_{2m} : \mathcal{M}_{1/2}(2m) \to \mathcal{M}_{1/2}$, *for some constant everywhere non-zero sections* δ_r. *In particular,* $\mathcal{V}_m$ *trivializes over* $\mathcal{M}_{1/2}(2m)$.

Proof. Let $(E/S, \mathcal{V}_H, \lambda)$ be a triple of an elliptic curve over a scheme $S \in \mathrm{Ob}(Sch[1/m])$, a Schrödinger representation (Definition 1.3.8)

$$\mathcal{V}_H = \mathcal{A}^{(1)}(H\backslash\mathcal{G}_E(\mathcal{L}_m))$$

of the level m Heisenberg group, for $H \subset \mathcal{G}_E(\mathcal{L}_m)$ a lagrangian subgroup, and an arithmetic level $2m$ structure:

$$\lambda : E[2m] \simeq \mathbb{Z}/2m\mathbb{Z} \times \mu_{2m}.$$

We want to show that the level structure λ can be used to trivialize $\mathcal{V}_H$. For ease of notation, let $\mathcal{G} := \mathcal{G}_E(\mathcal{L}_m)$. Let H_1 be the projection of H onto $E[m]$. To trivialize $\mathcal{V}_H$ we want to find an isomorphism

$$\mathcal{A}^{(1)}(H\backslash\mathcal{G}) \simeq \{\text{Functions } f : \hat{H}_1 \to \mathcal{O}_S\}, \tag{2.18}$$

for then the lagrangian subgroup $\hat{H}_1 \subset E[m]$ can be trivialized as a constant group scheme using λ and we get

$$\{\text{Functions } f : \hat{H}_1 \to \mathcal{O}_S\} \simeq \bigoplus_{r\in\mathbb{Z}/m\mathbb{Z}} \mathcal{O}_S\, \delta_r$$

for the basis of delta functions on $\hat{H}_1$. To construct the isomorphism 2.18, we can proceed as in [20], V.3.3.3.2. Namely, suppose we can find a section

$$\sigma : H_1\backslash E[m] \longrightarrow H\backslash\mathcal{G}.$$

Then we can construct an isomorphism:

$$\{\text{Functions } f : H_1\backslash E[m] \to \mathcal{O}_S\} \simeq \mathcal{A}^{(1)}(H\backslash\mathcal{G})$$

by sending f to the function $\varphi_\sigma(f)$ defined by:

$$\varphi_\sigma(f)(t\sigma(k)) = tf(k),$$

where we decomposed an arbitrary element of $\mathcal{G}$ as $g = t\sigma(k)$. Since $H_1\backslash E[m] \simeq \hat{H}_1$ canonically, we would then be done.

In order to construct σ, note that such a section exists affine-locally on S ([20], V.3.3.3).

This is because if $U = \mathrm{Spec}(R) \subset S$ is the spectrum of a local ring, then $H\backslash\mathcal{G} \times U$ is a $\mathbb{G}_m$-torsor over $\hat{H}_1$, but $\mathrm{Pic}(\hat{H}_1 \times U) = 0$ since $\hat{H}_1$ is semi-local. Thus we can cover S by affine opens U_i and find sections σ_i over them. Over the intersections $U_i \cap U_j$, σ_i and σ_j are both lifts of $H_1\backslash E[m]$ and they must differ by a morphism

$$\alpha_{ij} : H_1\backslash E[m] \to \mathbb{G}_m.$$

If we let $\sigma_j = \sigma_i \alpha_{ij}$, we then have:

$$(\varphi_{\sigma_j}(f) - \varphi_{\sigma_i}(f))(t\sigma(k)) = t(\alpha_{ij}(k)^{-1} - 1)f(k)$$

over the intersections $U_i \cap U_j$. Now we want the isomorphism (2.18) to preserve the involution ι on $\mathrm{End}(\mathcal{V}_H)$, thus we must require

$$\alpha_{ij}(-k) = \alpha_{ij}(k)$$

for all $k \in H_1\backslash E[m]$, i.e. $\alpha_{ij}(k) \subset \mu_2$ for all i, j.

Consider now the map

$$E[2m] \to E[m]$$

given by $P \to [2]P$. As in [22], §2, this map extends to a map of Heisenberg groups:

$$\begin{aligned} \eta_2 : \mathcal{G}_E(\mathcal{L}_{2m}) &\longrightarrow \mathcal{G} \\ (t, P) &\longmapsto (t^2, [2]P). \end{aligned}$$

Let now $\tilde{H}_1$ be a maximal isotropic subgroup in $E[2m]$ such that $[2]\tilde{H}_1 = H_1$ (which can be found using the level structure), and let $\tilde{H} \subset \mathcal{G}_E(\mathcal{L}_{2m})$ be a lagrangian subgroup such that $\eta_2(\tilde{H}) = H$. Proceeding as above, on affine charts U_i we can find sections

$$\tilde{\sigma}_i : \tilde{H}_1\backslash E[2m] \to \mathcal{G}_E(\mathcal{L}_{2m})$$

which differ on the intersections U_{ij} by morphisms:

$$\tilde{\alpha}_{ij} : \tilde{H}_1\backslash E[2m] \to \mu_2.$$

Set now

$$\sigma_i(k) := \eta_2(\tilde{\sigma}(\tilde{k}))$$

for $k \in H_1\backslash E[m]$ and $\tilde{k}$ such that $[2]\tilde{k} = k$. These are well-defined local sections $\sigma_i : H_1\backslash E[m] \to H\backslash\mathcal{G}$, since η_2 has for kernel the 2-torsion of $\mathcal{G}_E(\mathcal{L}_{2m})$. Over the intersections

$U_i \cap U_i$, we have:

$$(\varphi_{\sigma_j}(f) - \varphi_{\sigma_i}(f))(t\sigma(k)) = t(\tilde{\alpha}_{ij}^2(k)^{-1} - 1)f(k) = 0$$

therefore the σ_i define the desired *global* isomorphism

$$\{\text{Functions } f : H_1\backslash E[m] \to \mathcal{O}_S\} \simeq \mathcal{A}^{(1)}(H\backslash\mathcal{G})$$

which trivializes $\mathcal{V}_H$.

Now the triple $(E, \mathcal{V}_H, \lambda)$ is classified by a morphism:

$$s : S \to \mathcal{M}_{1/2}(2m),$$

where again we have implicitly identified the stacks $\mathcal{M}_{1/2}$ and $\mathcal{K}(\mathcal{A}_m, \iota)$. In particular, $s^*\mathcal{V}_m = \mathcal{V}_H$ can be trivialized over S as above. If we have a morphism $\phi : (E, \mathcal{V}_H, \lambda) \to (E', \mathcal{V}_{H'}, \lambda')$ of elliptic curves with a Schrödinger representation and a full level $2m$ arithmetic level structure, the trivializations of $\mathcal{V}_H$ and $\mathcal{V}_{H'}$ constructed above are functorial with respect to ϕ, therefore we obtain the desired trivialization of $\mathcal{V}_m$ over the stack $\mathcal{M}_{1/2}(2m)$. □

By the Theorems 2.2.19 and 2.2.16 the sheaves

$$\mathcal{V}_m, \quad \mathcal{J}_m \otimes \underline{\omega}^{1/2},$$

defined over the metaplectic stack $\mathcal{M}_{1/2}$, are trivial over $\mathcal{M}_{1/2}(2m)$. In particular, we can normalize the isomorphism between $\mathcal{V}_m$ and $\mathcal{J}_m \otimes \underline{\omega}^{1/2}$ by requiring that a basis of trivializing sections for $\mathcal{J}_m \otimes \underline{\omega}^{1/2}$ maps to the basis $\{\delta_r\}$ of (2.17). This uniquely determines the isomorphism up to a constant $\mathbb{Z}[1/3m]^\times$, as opposed to an element of $\Gamma(\mathcal{M}_1, \mathcal{O}_{\mathcal{M}_1}^\times)$. In more abstract languange, we are choosing the isomorphism not only in the category of $\mathcal{G}(\mathcal{L}_m)$-representations, but in the category of *locally constant* representations of $\mathcal{G}(\mathcal{L}_m)$. We thus obtain:

THEOREM 2.2.20 (Main Theorem, Second Version). *Let $m \in 2\mathbb{Z}_{>0}$ and let $\mathcal{M}_1 \to Sch[1/3m]$ be the modular stack of elliptic curves over schemes were 3 and m are invertible. Then there is a canonical isomorphism*

$$\mathcal{V}_m \otimes \underline{\omega}^{-1/2} \simeq \mathcal{J}_m$$

of locally free $\mathcal{O}_{\mathcal{M}_1}$-modules of rank m over $\mathcal{M}_1$, defined up to multiplication by an element in $\mathbb{Z}^\times[1/3m]$, and compatible with the isomorphism

$$\mathcal{V}_m \simeq \mathcal{J}_m \otimes \underline{\omega}^{1/2},$$

as étale locally constant sheaves over $\mathcal{M}_{1/2}$.

COROLLARY 2.2.21. *For any integer $k \in \mathbb{Z}$ there is a canonical isomorphism*

$$\mathcal{V}_m \otimes \underline{\omega}^{k-1/2} \simeq \mathcal{J}_m \otimes \underline{\omega}^k$$

over the modular stack $\mathcal{M}_1 \to Sch[1/3m]$, defined up to an element of $\mathbb{Z}[1/3m]^\times$.

The structure of $\mathcal{J}_m \otimes \underline{\omega}^{1/2}$ as a locally constant sheaf over $\mathcal{M}_{1/2}$ directly leads to a notion of algebraic theta functions of level m. In fact, another way of stating that $\mathcal{J}_m \otimes \underline{\omega}^{1/2}$ is locally constant is to say that $\mathcal{J}_m$ is endowed with a canonical integrable connection ∇, defined up to μ_2, whose monodromy is the Weil representation of the metaplectic group

$$0 \to \mu_2 \to \mathrm{Mp}_2(\mathbb{Z}/2m\mathbb{Z}) \to \mathrm{Sp}_2(\mathbb{Z}/2m\mathbb{Z}) \to 0,$$

the unique nontrivial central extension of $\mathrm{Sp}(\mathbb{Z}/2m\mathbb{Z})$ by μ_2, viewed as the automorphism group of the cover

$$\mathcal{M}_{1/2}(2m) \to \mathcal{M}_1.$$

By the theory of Welters ([37], [18]) the connection ∇ is an algebraic analog of the heat equations satisfied by theta functions. In particular, we can find sections

$$\{\vartheta_{m,r}\}_{r\in\mathbb{Z}/m\mathbb{Z}} \in \Gamma(\mathcal{M}_1(2m), \mathcal{J}_m), \tag{2.19}$$

which correspond to horizontal sections of ∇ trivializing $\mathcal{J}_m \otimes \underline{\omega}^{1/2}$ over $\mathcal{M}_{1/2}(2m)$, hence mapping to the delta functions δ_r of (2.17) up to a constant. These sections $\vartheta_{m,r}$ can be viewed as algebraic analogs of the classical level m theta functions.

2.2.6 Tate curves

We would now like to illustrate some aspects of the theory so far developed by computing explicitly with q-expansions over Tate curves. As an application of these computations, we will be able to define q-expansions for vector-valued modular forms in Section 2.2.7.

To begin, we recall the description of the Tate curve $\mathrm{Tate}(q)$, as found for example in [13]. In particular, let $\mathrm{Spec}(\mathbb{Z}[\![q]\!][x_i, y_i])$ be an infinite collection of affine planes over $\mathbb{Z}[\![q]\!]$ indexed by $i \in \mathbb{Z}$ and let $\overline{\mathbb{G}}_m^q$ be the scheme obtained as the union of all the affine schemes:

$$U_j = \mathrm{Spec}(\mathbb{Z}[\![q]\!][x_{j-1/2}, y_{j+1/2}]/(x_{j-1/2}, y_{j+1/2} - q)), \quad j \in 1/2 + \mathbb{Z},$$

glued along the subschemes

$$T_i = U_{i-1/2} \cap U_{i+1/2}$$

by

$$x_i = y_i^{-1}.$$

This is a scheme, locally of finite type over $\mathrm{Spec}(\mathbb{Z}[\![q]\!])$, whose fiber above $q = 0$ is an infinite chain of projective lines, given by

$$D_i := \overline{T_i \cap V(q)}, \quad i \in \mathbb{Z},$$

and linked so that the i-th copy is attached to the $i+1$-th copy by gluing 0 on the i-th copy to ∞ on the $i+1$-th copy. Over $\mathrm{Spec}(\mathbb{Z}(\!(q)\!))$ all the charts U_j are glued together by:

$$x_i = t^{-i} x_0, \quad y_i = t^i y_0,$$

so that the scheme $\overline{\mathbb{G}}_m^q$ over $\mathrm{Spec}(\mathbb{Z}(\!(q)\!))$ is given by $T_0 = \mathrm{Spec}(\mathbb{Z}(\!(q)\!)[x_0, x_0^{-1}])$. In particular, the function

$$x := x_0$$

gives a rational function to all of $\overline{\mathbb{G}}_m^q$, with divisor

$$\mathrm{div}(x) = \sum_{i \in \mathbb{Z}} D_i.$$

Now to form the Tate curve $\mathrm{Tate}(q)$ we restrict $\overline{\mathbb{G}}_m^q$ to infinitesimal neighborhoods $\mathrm{Spec}(\mathbb{Z}[\![q]\!]/(q^n))$ of the special fiber $V(q)$, so that non-consecutive U_j's are now disjoint. There is an action of $\mathbb{Z}$ on the U_i's given by

$$(i, x) = q^{-i} x$$

which sends U_i to U_{i-1}. The quotients

$$\mathfrak{Tate}_n(q) := \overline{\mathbb{G}}_m^q \times \mathrm{Spec}(\mathbb{Z}[\![q]\!]/(q^n))/q^{\mathbb{Z}}$$

thus exist and the limit

$$\mathfrak{Tate}(q) := \lim_{\rightarrow} \mathfrak{Tate}_n(q)$$

is a formal scheme over $\mathrm{Spf}(\mathbb{Z}[\![q]\!])$. This formal scheme can be algebrized by a scheme $\overline{\mathrm{Tate}}(q)$ over $\mathrm{Spec}(\mathbb{Z}[\![q]\!])$, whose restriction over $\mathrm{Spec}(\mathbb{Z}(\!(q)\!))$ is the elliptic curve $\mathrm{Tate}(q)$. The Tate curves $\mathrm{Tate}(q^{1/k})$, for $k \in \mathbb{Z}$, are similarly obtained, by taking the quotients

$$\mathfrak{Tate}_n(q^{1/k}) := \overline{\mathbb{G}}_m^{q^{1/k}} \times \mathrm{Spec}(\mathbb{Z}[\![q^{1/k}]\!]/(q^{n/k}))/q^{\mathbb{Z}}$$

and letting $\overline{\mathrm{Tate}}(q^{1/k})$ be the algebraization of the formal scheme $\mathfrak{Tate}(q^{1/k}) := \lim_{\rightarrow} \mathfrak{Tate}_n(q^{1/k})$. The restriction of $\overline{\mathrm{Tate}}(q^{1/k})$ to $\mathrm{Spec}(\mathbb{Z}(\!(q^{1/k})\!))$ is the elliptic curve $\mathrm{Tate}(q^{1/k})$.

Let now $m \in 2\mathbb{Z}_{>0}$ be a positive even integer. There is a canonical inclusion of group schemes ([13], VII.1.12.3)

$$\mu_m \subset \mathrm{Tate}(q)[m],$$

which provides the m-th torsion $\mathrm{Tate}(q)[m]$ with a canonical maximal isotropic subgroup. We now have:

Lemma 2.2.22. *The subgroup $\mu_m \subset \mathrm{Tate}(q)[m]$ lifts canonically to a lagrangian subgroup $H_{\mathrm{can}} \subset \mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m)$, where $\mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m)$ is the central extension*

$$0 \to \mathbb{G}_m \to \mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m) \xrightarrow{\pi} \mathrm{Tate}(q)[m] \to 0,$$

the level m Heisenberg group of $\mathrm{Tate}(q)$.

Proof. The group $\mu_m \subset \mathrm{Tate}(q)[m]$ is isotropic, thus $\pi^{-1}(\mu_m)$ is a *commutative* extension of $\mathbb{G}_m$. But any commutative extension of a commutative finite group scheme by $\mathbb{G}_m$ over an affine scheme is trivial ([35], Theorem 1) thus $\pi^{-1}(\mu_m) \simeq \mathbb{G}_m \times \mu_m$ as a group scheme, and we can take $H_{\mathrm{can}} := \{1\} \times \mu_m$. □

Therefore, $\mathrm{Tate}(q)$ is endowed with a canonical Schrödinger representation

$$V_m^{\mathrm{can}} := V_{H_{\mathrm{can}}}. \tag{2.20}$$

This representation can be canonically trivialized over $\mathrm{Tate}(q^{1/2m})$, as the following proposition shows.

Proposition 2.2.23. *Over* $\mathrm{Tate}(q^{1/2m}) \to \mathrm{Spec}(\mathbb{Z}[1/m]((q^{1/2m}))$*, there is a canonical trivialization*

$$V_m^{\mathrm{can}} \simeq \{\textit{Functions } f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}[1/m]((q))\} \simeq \bigoplus_{r \in \mathbb{Z}/m\mathbb{Z}} \mathbb{Z}[1/m]((q^{1/2m})) \cdot \delta_r, \tag{2.21}$$

by a basis of delta functions $\{\delta_r\}_{r \in \mathbb{Z}/m\mathbb{Z}}$.

Proof. By [13], VII.1.16.4, over $\mathrm{Tate}(q^{1/2m})$ there is a canonical splitting of:

$$0 \to \mu_m \to \mathrm{Tate}(q^{1/2m})[m] \to \mathbb{Z}/m\mathbb{Z} \to 0$$

given by sending the constant function 1 to the constant function $q^{1/m}$. Thus we have a canonical decomposition

$$\mathrm{Tate}(q^{1/2m})[m] \simeq H_1 \times \hat{H}_1$$

into maximal isotropic subgroups, where $H_1 = \mu_m$. Now by Lemma 2.2.22, H_1 lifts canonically to a lagrangian subgroup H_{can} of $\mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m)$, and by definition we have:

$$V_m^{\mathrm{can}} = \mathcal{A}^{(1)}(H_{\mathrm{can}} \backslash \mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m)).$$

By the proof of Lemma V.3.3.3 of [20], in particular V.3.3.3.2, if we choose a lifting

$$\sigma : \hat{H}_1 \to \mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m)$$

of $\hat{H}_1$ to a lagrangian subgroup of $\mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m)$ then there is an isomorphism

$$\mathcal{A}^{(1)}(H_{\mathrm{can}}\backslash\mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m)) = \{\text{Functions } f : \hat{H}_1 \to \mathbb{Z}[1/m]((q))\}.$$

Thus, to prove the proposition, we need to provide a canonical such lift σ.

To do so, we need to compute the action of $\mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m)$ on sections of $\mathcal{L}_m$, given by (2.10). In particular, we need to compute the action on $\mathcal{L}_m$ of translation by the element $q^{1/m}$, and find an explicit isomorphism:

$$\varphi : \mathcal{L}_m \to q^{1/m,*}\mathcal{L}_m.$$

The map $\sigma(q^{1/m}) = (\varphi, q^{1/m})$ is then the required lift.

To compute the action of $q^{1/m}$ on $\mathcal{L}_m$, note first that over $\mathrm{Tate}(q^{1/2m})$ the sheaf of relative differentials is trivial, thus

$$\mathcal{L}_m \simeq \mathcal{O}_{\mathrm{Tate}(q^{1/2m})}(m\, e),$$

where e is the identity section. Now over $\overline{\mathbb{G}}_m^q$, consider the divisor given by

$$D := \sum_{i\in\mathbb{Z}} \frac{i^2}{2} D_i^{(q)}.$$

The q-invariant sections of the invertible sheaf corresponding to this divisor over the quotients $\mathfrak{Tate}_n(q)$ descend to the sections of $\mathcal{O}_{\mathrm{Tate}(q)}(e)$ over $\mathrm{Tate}(q)$. Under the map

$$\pi : \overline{\mathbb{G}}_m^{q^{1/2m}} \to \overline{\mathbb{G}}_m^q,$$

given by raising $q^{1/2m}$ to the $2m$-th power, we have

$$\pi^*(D) = \sum_{i\in\mathbb{Z}} \frac{1}{2}\, i^2 D_i^{(q^{1/2m})}.$$

At the level of the underlying elliptic curve, this map corresponds to a cyclic isogeny of degree $2m$

$$\pi : \mathrm{Tate}(q^{1/2m}) \to \mathrm{Tate}(q),$$

for which $\pi^*\mathcal{O}_{\mathrm{Tate}(q)}(e) = \mathcal{O}_{\mathrm{Tate}(q^{1/2m})}(2m\, e)$. Now we are interested in the invertible sheaf

$\mathcal{O}_{\mathrm{Tate}(q^{1/2m})}(m\,e)$, whose sections are the q-invariant sections over $\overline{\mathbb{G}}_m^{q^{1/2m}}$ of divisor

$$D(m) := \sum_{i\in\mathbb{Z}} \frac{1}{4}\, i^2 D_i^{(q^{1/2m})}.$$

Now the action of $q^{1/m}$ on $\overline{\mathbb{G}}_m^{q^{1/2m}}$ sends D_i to D_{i-2}, thus:

$$\begin{aligned}
q^{1/m,*}D(m) &= \sum_{i\in\mathbb{Z}} \frac{1}{4}\, i^2 D_{i-2} \\
&= \sum_{i\in\mathbb{Z}} \frac{1}{4}\,(i-2)^2 D_{i-2} + \sum_{i\in\mathbb{Z}} i\, D_{i-2} - \sum_{i\in\mathbb{Z}} D_{i-2} \\
&= D(m) + \sum_{i\in\mathbb{Z}} (i-2)\, D_{i-2} + \sum_{i\in\mathbb{Z}} D_{i-2} \\
&= D(m) + \sum_{i\in\mathbb{Z}} i\, D_i + \sum_{i\in\mathbb{Z}} D_i \\
&= D(m) + \mathrm{div}(x) + \mathrm{div}(q^{1/2m}).
\end{aligned}$$

In other words, we can choose a canonical isomorphism:

$$\mathcal{L}_m \simeq q^{1/m,*}\mathcal{L}_m$$

simply by multiplying sections by the rational function $xq^{1/2m}$. This provides the required map

$$\sigma : \hat{H}_1 \to \mathcal{G}_{\mathrm{Tate}(q)}(\mathcal{L}_m)$$

by sending $q^{1/m}$ to $(xq^{1/2m}, q^{1/m})$. □

2.2.7 q-expansions

We would now like to define q-expansions of vector-valued modular forms, in a way analogous to the q-expansions of modular forms of integral weight. In particular, consider the Tate elliptic curve $\mathrm{Tate}(q) \to \mathrm{Spec}(\mathbb{Z}((q)))$. The Hodge bundle $\underline{\omega}_{\mathrm{Tate}(q)}$ has a canonical everywhere non-vanishing section ω_{can} ([13], VII.1.12.2), which gives a canonical trivialization:

$$\underline{\omega}_{\mathrm{Tate}(q)} \simeq \mathbb{Z}((q)) \cdot \omega_{\mathrm{can}}.$$

Therefore $\underline{\omega}_{\mathrm{Tate}(q)}$-valued quadratic forms are in bijection with $\mathbb{Z}((q))$-valued quadratic forms, and in particular we can find a $\underline{\omega}_{\mathrm{Tate}(q)}$-valued rank 1 non-degenerate quadratic form

$$q : \mathcal{Q} \to \underline{\omega}_{\mathrm{Tate}(q)},$$

corresponding to $f \mapsto f^2$, together with a canonical trivialization:

$$\mathcal{Q} \simeq \mathbb{Z}(\!(q)\!) \cdot \omega_{\mathrm{can}}^{1/2}. \tag{2.22}$$

Next, for $m \in 2\mathbb{Z}_{>0}$, consider the Tate curve $\mathrm{Tate}(q^{1/2m})$ together with its canonical Schrödinger representation (2.20). The triple $(\mathrm{Tate}(q^{1/2m}), \mathcal{Q}, V_m^{\mathrm{can}})$ taken over $\mathbb{Z}[1/m](\!(q^{1/2m})\!)$ corresponds to a morphism

$$\psi : \mathrm{Spec}(\mathbb{Z}[1/m](\!(q^{1/2m})\!)) \to \mathcal{K}(\mathcal{A}_m, \iota) \times_{\mathcal{M}_1}^{\mu_2} \mathcal{M}_{1/2},$$

with the property that:

$$\psi^* \left(\mathcal{V}_m \otimes \underline{\omega}^{k/2}\right) \simeq V_m^{\mathrm{can}} \otimes \mathcal{Q}^k$$

for any integer $k \in \mathbb{Z}$. In particular, using the trivializations (2.22) and (2.21) we see that a vector-valued modular form $f \in \Gamma(\mathcal{K}(\mathcal{A}_m, \iota) \times_{\mathcal{M}_1}^{\mu_2} \mathcal{M}_{1/2}, \mathcal{V}_m \otimes \underline{\omega}^{k/2})$ defines a unique m-dimensional vector:

$$\psi^*(f) = \{f_r\}_{r \in \mathbb{Z}/m\mathbb{Z}} \in \mathbb{Z}[1/m](\!(q^{1/2m})\!)^m.$$

Definition 2.2.24. The vector $\psi^*(f)$ is called the *q-expansion* of f.

Remark 2.2.25. The q-expansions of $\mathcal{V}_m^\vee$-valued modular forms (Remark 2.2.12) can similarly be obtained by considering trivializations dual to (2.22), and (2.21).

Using q-expansions, we can also define the notion of a *holomorphic* vector-valued modular form, although this notion more properly belongs to compactifications of the moduli stacks, which we do not discuss in this work.

Definition 2.2.26. Let $m \in 2\mathbb{Z}_{>0}$ and let R be a ring containing $1/m$. A weight $k/2$ $\mathcal{V}_m$-valued modular form f defined over R is *holomorphic* if its q-expansion lies in $(\mathbb{Z}[\![q^{1/2m}]\!] \otimes R)^m$. More precisely, we require:

$$\psi_R^*(f) \in (\mathbb{Z}[\![q^{1/2m}]\!] \otimes R)^m,$$

where

$$\psi_R : \mathrm{Spec}(\mathbb{Z}(\!(q^{1/2m})\!) \otimes R) \to \mathcal{K}(\mathcal{A}_m, \iota) \times_{\mathcal{M}_1}^{\mu_2} \mathcal{M}_{1/2}$$

is the morphism obtained by extending the triple $(\mathrm{Tate}(q^{1/2m}), \mathcal{Q}_{\mathrm{Tate}(q^{1/2m})}, V_m^{\mathrm{can}})$ to R.

Remark 2.2.27. A similar definition can be given for *holomorphic* $\mathcal{V}_m^\vee$-valued modular forms.

Remark 2.2.28. Note in particular that contrary to the integral weight case, vector-valued modular forms always have q-expansions containing fractional powers of q. This is indeed a feature of the complex analytic theory as well.

2.3 Theta constants and modular forms of half-integral weight

2.3.1 Theta constants

We now introduce the prototypical example of vector-valued modular forms, the vector of level m *theta constants*. In particular, let $m \in 2\mathbb{Z}_{>0}$ be a positive even integer and let $\pi : E \to S$ be an elliptic curve over a scheme $S \in \mathrm{Ob}(Sch[1/m])$ with identity section $e : S \to E$. The invertible sheaf

$$\mathcal{L}_m = \mathcal{O}_E(m\, e) \otimes (\Omega^1_{E/S})^{\otimes m},$$

is normalized along e, by Proposition 2.2.3. Therefore the pull-back e^* along the identity section gives a well-defined 'evaluation' homomorphism:

$$\mathrm{ev}_e : \Gamma(E, \mathcal{L}_m) \to H^0(S, \mathcal{O}_S),$$

which can be viewed as an element

$$\mathrm{ev}_e \in \Gamma(E, \mathcal{L}^\vee).$$

The assignment

$$\{\pi : E \to S\} \longmapsto \mathrm{ev}_e \in \Gamma(E, \mathcal{L}^\vee) = \Gamma(S, \pi_* \mathcal{L}^\vee)$$

is functorial with respect to morphisms of elliptic curves, since e and $\mathcal{L}_m$ are, and it gives a well-defined global section

$$\theta_{\mathrm{null},m} \in \Gamma(\mathcal{M}_1, \mathcal{J}_m^\vee).$$

By the main Theorem 2.2.20, second version, we obtain a vector-valued modular form

$$\theta_{\mathrm{null},m} \in \Gamma(\mathcal{M}_1, \mathcal{V}_m^\vee \otimes \underline{\omega}^{1/2}),$$

defined up to a constant in $\mathbb{Z}^\times[1/3m]$. In particular, $\theta_{\mathrm{null},m}$ is a $\mathcal{V}_m^\vee$-valued modular form of weight 1/2, in the sense of Definition 2.2.11 and Remark 2.2.12.

Definition 2.3.1. The weight 1/2, $\mathcal{V}_m^\vee$-valued modular form $\theta_{\mathrm{null},m}$ is called the *vector of theta constants of level m.*

2.3.2 q-expansions of theta constants

We now want to compute the q-expansions of the vector-valued modular forms $\theta_{\mathrm{null,m}}$, and show that they indeed agree with the classical q-expansions of theta constants, obtained by complex-analytic methods.

In order to compute the q-expansion of $\theta_{\mathrm{null},m}$, we first compute the canonical basis of

$\Gamma(\mathrm{Tate}(q^{1/2m}), \mathcal{L}_m)$ given by the theta functions $\vartheta_{m,r}$ defined by (2.19). To do that, note that over the Tate curve $\mathrm{Tate}(q^{1/2m})$ we have a canonical isomorphism

$$\Gamma(\mathrm{Tate}(q^{1/2m}), \mathcal{L}_m) \simeq \bigoplus_{r \in \mathbb{Z}/m\mathbb{Z}} \mathbb{Z}[1/m]((q)) \cdot \delta_r, \tag{2.23}$$

obtained by pulling back the isomorphism of Theorem 2.2.20 and applying the canonical trivializations (2.22) and (2.17). Thus a canonical basis of $\Gamma(\mathrm{Tate}(q^{1/2m}), \mathcal{L}_m)$ can be specified by computing the images of the delta functions δ_r under this isomorphism which are horizontal for the integrable connection ∇ given by the heat equations, defined at the end of Section 2.2.5. Note that over the Tate curve the connection ∇, in principle only defined on $\mathcal{J}_m \otimes \underline{\omega}^{1/2}$, descends to a canonical integrable connection on $\mathcal{J}_m$, since we have trivialized $\underline{\omega}^{1/2}$.

Proposition 2.3.2. *Let $m \in 2\mathbb{Z}_{>0}$ be a positive even integer and let $\mathrm{Tate}(q^{1/2m})$ be the Tate elliptic curve over $\mathrm{Spec}(\mathbb{Z}((q^{1/2m})))$. The invertible sheaf $\mathcal{L}_m$ over $\mathrm{Tate}(q^{1/2m})$ has a canonical basis of global sections:*

$$\theta_{m,r}(x) = \left(\sum_{n \equiv r \mod m} q^{n^2/2m} x^n \right) \omega_{\mathrm{can}}^m, \quad r \in \mathbb{Z}/m\mathbb{Z},$$

defined up to a constant in $\mathbb{Z}[1/m]^\times$, such that $\theta_{m,r}$ corresponds to the delta function δ_r in (2.23).

Proof. The canonical differential ω_{can} sets up an isomorphism

$$\mathcal{L}_m \simeq \mathcal{O}_{\mathrm{Tate}(q)}(me),$$

so we have to show that the theta functions correspond to the canonical trivialization of this sheaf.

Recall from the proof of (2.21) that the invertible sheaf $\mathcal{O}_{\mathrm{Tate}(q^{1/2m})}(m\,e)$ is given by the q-invariant sections over $\overline{\mathbb{G}}_m^{q^{1/2m}}$ of divisor

$$D(m) := \sum_{i \in \mathbb{Z}} \frac{1}{4}\, i^2 D_i^{(q^{1/2m})}.$$

We want to construct these q-invariant sections explicitly. To lighten notation, set $D_i =$

$D_i^{(q^{1/2m})}$. Note that the action of q on $\overline{\mathbb{G}}_m^{q^{1/2m}}$ sends D_i to D_{i-2m}, thus:

$$\begin{aligned}
q^* D(m) &= \sum_{i\in\mathbb{Z}} \frac{1}{4}\, i^2 D_{i-2m} \\
&= \sum_{i\in\mathbb{Z}} \frac{1}{4}\,(i-2m)^2 D_{i-2m} + \sum_{i\in\mathbb{Z}} im\, D_{i-2m} - \sum_{i\in\mathbb{Z}} m^2\, D_{i-2m} \\
&= D(m) + \sum_{i\in\mathbb{Z}} (i-2m)m\, D_{i-2m} + \sum_{i\in\mathbb{Z}} m^2\, D_{i-2m} \\
&= D(m) + \sum_{i\in\mathbb{Z}} im\, D_i + \sum_{i\in\mathbb{Z}} m^2\, D_i \\
&= D(m) + \operatorname{div}(x^m) + \operatorname{div}(q^{m/2}).
\end{aligned}$$

In particular, q acts on sections of divisor $D(m)$ by multiplication by $x^m q^{m/2}$. To construct q-invariant sections, define the 'slash operator'

$$f(x)|_q = x^m q^{m/2}\, f(qx),$$

and apply it infinitely many times to $1 \in \Gamma(\overline{\mathbb{G}}_m^{q^{1/2m}}, \mathcal{L}(D(m)))$. We thus obtain the q-invariant section:

$$\vartheta_{m,0} := \sum_{n\in\mathbb{Z}} q^{mn^2/2} x^{mn}.$$

This section descends to a section over the formal scheme $\mathfrak{Tate}(q^{1/2m})$ and then to a section of $\mathcal{O}_{\mathrm{Tate}(q^{1/2m})}(m\, e)$ over $\mathrm{Tate}(q^{1/2m})$. We would like to compute the action of the level m Heisenberg group

$$0 \to \mathbb{G}_m \to \mathcal{G}_{\mathrm{Tate}(q^{1/2m})}(m) \to \mathrm{Tate}(q^{1/2m})[m] \to 0$$

on $\vartheta_{m,0}$, and on sections of $\mathcal{L}_m$ in general. In order to do so, it suffices to compute the actions of the lagrangian subgroups H_{can} and its dual $\hat{H}_{\mathrm{can}}$. Now by Lemma 2.2.22, the lagrangian subgroup

$$H_{\mathrm{can}} \simeq \{1\} \times \mu_m \subset \mathcal{G}_{\mathrm{Tate}(q^{1/2m})}(m)$$

acts simply by translation by a root of unity $\zeta \in \mu_m$, i.e.

$$(1, \mu_m)\theta(x) = \theta(\zeta^{-1} x)$$

for any section $\vartheta(x)$ of $\mathcal{L}_m$. On the other hand, by the proof of (2.21), we have:

$$\hat{H}_{\mathrm{can}} \simeq \{(x^r\, q^{r/2m}, q^{r/m})\}_{r\in\mathbb{Z}/m\mathbb{Z}},$$

so that

$$(x^r\, q^{r/2m}, q^{r/m})\vartheta(x) = x^r\, q^{r/2m}\theta(q^{-r/m}x).$$

We thus *define* the theta functions

$$\vartheta_{m,r} := (x^r\, q^{r/2m}, q^{r/m})\vartheta_{m,0}(x), \quad r \in \mathbb{Z}/m\mathbb{Z},$$

by translating $\vartheta_{m,0}$ by all the elements in $\hat{H}_{\text{can}}$. That these form a basis for $\Gamma(\text{Tate}(q^{1/2m}), \mathcal{L}_m)$ can be seen from the fact that their span is $\mathcal{G}_{\text{Tate}(q^{1/2m})}(m)$-invariant and there are no non-trivial $\mathcal{G}_{\text{Tate}(q^{1/2m})}(m)$-invariant subspaces. Moreover, we claim that under the canonical trivialization (2.23), we have:

$$\vartheta_{m,r} = \delta_r, \quad r \in \mathbb{Z}/m\mathbb{Z}$$

up to a constant in $\mathbb{Z}[1/m]^\times$. To see this, note that δ_0 and $\vartheta_{0,r}$ are both fixed by the action of H_{can}, thus

$$\vartheta_{0,r} = f \cdot \delta_0$$

where $f \in \mathbb{Z}[1/m](\!(q^{1/2m})\!)$. But $\vartheta_{0,r}$ is also horizontal for the connection ∇, since over the Tate curve this connection is given by the heat operator

$$\nabla = (\partial_q - \frac{1}{2m}\partial_x^2)\frac{dq}{q},$$

where

$$\partial_q := q\frac{\partial}{\partial q}, \quad \partial_x := x\frac{\partial}{\partial x},$$

(as can be deduced from the computations of [18], §3). Thus $\vartheta_{0,r}$ is the unique (up to a constant in $\mathbb{Z}[1/m]^\times$) horizontal image of δ_0 in $\Gamma(\text{Tate}(q^{1/2m}), \mathcal{L}_m)$. Now the result follows by noting that δ_r is the translate of δ_0 by elements of $\hat{H}_{\text{can}}$, and that all the theta functions $\vartheta_{m,r}$ satisfy the heat equation. □

We can use the Proposition to find the q-expansions of theta constants, as follows. Let

$$\psi : \text{Spec}(\mathbb{Z}[1/m](\!(q)\!)) \longrightarrow \mathcal{M}_{1/2}$$

be the point classifying the triple $(\text{Tate}(q^{1/2m}), V_m^{\text{can}}, \mathcal{Q})$. Then $V_m^{\text{can}} \simeq \psi^*\mathcal{V}_m$ and the isomorphism of Theorem 2.2.20 gives an isomorphism:

$$V_m^{\text{can}} \otimes \mathcal{Q}^{-1} \simeq \pi_*\mathcal{L}_m$$

where $\pi : \text{Tate}(q^{1/2m}) \to \text{Spec}(\mathbb{Z}[1/3m](\!(q)\!))$ is the structure morphism. For ease of notation, let $R = \mathbb{Z}[1/3m](\!(q)\!)$. There is a commutative diagram of isomorphisms:

$$\begin{array}{ccc}
V_m^{\mathrm{can}} \otimes \mathcal{Q}^{-1} & \longrightarrow & \pi_* \mathcal{L}_m \\
\downarrow & & \downarrow \\
\bigoplus_{r \in \mathbb{Z}/m\mathbb{Z}} R\,(\delta_r \otimes \omega_{\mathrm{can}}^{-1/2}) & \longrightarrow & \bigoplus_{r \in \mathbb{Z}/m\mathbb{Z}} R\,\vartheta_{m,r}
\end{array}$$

where the vertical arrows are given by Proposition 2.3.2 and the trivializations (2.20), (2.22). Moreover, the isomorphisms in the diagram are isomorphisms of locally constant $\mathcal{G}_{\mathrm{Tate}(q^{1/2m})}(\mathcal{L}_m)$-modules, so that in particular δ_r maps to $\vartheta_{m,r}$, up to multiplication by a constant.

There is a corresponding commutative diagram of duals

$$\begin{array}{ccc}
V_m^{\mathrm{can},\vee} \otimes \mathcal{Q} & \longrightarrow & \pi_* \mathcal{L}_m^{\vee} \\
\downarrow & & \downarrow \\
\bigoplus_{r \in \mathbb{Z}/m\mathbb{Z}} R\,(\delta_r^* \otimes \omega_{\mathrm{can}}^{1/2}) & \longrightarrow & \bigoplus_{r \in \mathbb{Z}/m\mathbb{Z}} R\,\vartheta_{m,r}^*,
\end{array}$$

where the $*$ superscript indicates that we are taking dual bases. In particular, the element $\mathrm{ev}_e \in \Gamma(\mathcal{M}_1, \mathcal{J}_m^{\vee})$, defined in Section 2.3.1, pulls back over the Tate curve as

$$\psi^*(\mathrm{ev}_e) = \sum_{r \in \mathbb{Z}/m\mathbb{Z}} \vartheta_{m,r}(q,1)\,\vartheta_{m,r}^*$$

and thus

$$\psi^*(\theta_{\mathrm{null},m}) = \sum_{r \in \mathbb{Z}/m\mathbb{Z}} \vartheta_{m,r}(q,1)\,\delta_r^* \otimes \omega_{\mathrm{can}}^{1/2}.$$

These are precisely the q-expansions of the classical level m theta constants, up to multipplication by the ever-present element of $\mathbb{Z}[1/3m]^{\times}$.

2.3.3 Theta structures

We would now like to explain the relationship between our theory of the Schrödinger representations $\mathcal{V}_m$ and that of Mumford's *theta structures* ([22], [23]).

We begin by recalling the basic facts from the theory of theta structures, as first laid out in much greater generality by Mumford in [23], §6. Let S be a scheme in $\mathrm{Ob}(Sch[1/m])$ and set:

$$\begin{aligned}
H(m) &:= \mathbb{Z}/m\mathbb{Z} \\
\widehat{H}(m) &:= \mu_m \\
K(m) &:= H(m) \oplus \widehat{H}(m)
\end{aligned}$$

as group schemes over S. Consider the central extension $\mathcal{G}(m)$ given as a S-scheme by:

$$\mathcal{G}(m) := \mathbb{G}_m \times H(m) \times \widehat{H}(m)$$

and with multiplication defined by:

$$(a_1, x_1, \zeta_1) \cdot (a_2, x_2, \zeta_2) = (a_1\, a_2\, \zeta_2^{x_1}, x_1 + x_2, \zeta_1\, \zeta_2).$$

The group scheme $\mathcal{G}(m)$ is a Heisenberg group. A model for its weight 1, rank m representation can be constructed as follows: let

$$V(m) := \{\text{Functions } f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}[1/m]\}.$$

This is a free module over $\mathbb{Z}[1/m]$, hence it gives rise to a free sheaf $\mathcal{V}(m)$ of rank m over $\mathrm{Spec}(\mathbb{Z}[1/m])$. This sheaf is endowed with a representation ([22] §1, [23] §6):

$$\begin{aligned} \mathcal{G}(m) &\longrightarrow \mathrm{GL}(\mathcal{V}(m)) \\ (a, x, \zeta) &\longmapsto U_{(a,x,\zeta)} f(y) := a\, \zeta^y\, f(x + y) \end{aligned}$$

which is clearly of weight 1 (and rank m).

In [23], §6 Mumford shows how these 'abstract' Heisenberg groups can be used to trivialize Heisenberg groups on elliptic curves. In particular, suppose that an elliptic curve $E \to S$ is endowed with an arithmetic level m structure (see Section 2.2.5 for definition)

$$\lambda : E[m] \xrightarrow{\simeq} \mathbb{Z}/m\mathbb{Z} \times \mu_m.$$

Then starting from λ, we can also trivialize the Heisenberg group of level m of $E \to S$, as follows:

Definition 2.3.3. Let $m \in 2\mathbb{Z}_{>0}$. A *level m theta structure* is an isomorphism $(\mathrm{id}, \Theta, \lambda)$, simply denoted by Θ, of exact sequences of group schemes over S:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{G}_E(\mathcal{L}_m) & \longrightarrow & E[m] & \longrightarrow & 0 \\ & & \updownarrow{\scriptstyle \mathrm{id}} & & \updownarrow{\scriptstyle \Theta} & & \updownarrow{\scriptstyle \lambda} & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{G}(m) & \longrightarrow & H(m) & \longrightarrow & 0. \end{array}$$

Remark 2.3.4. The existence of theta structures is discussed in [23].

The group $\mathcal{G}(m)$ is a symmetric Heisenberg group, and it has an automorphism $D_{-1} \in$

$\mathrm{Aut}(\mathcal{G}(m))$ given by the formula:

$$D_{-1}(a,x,\zeta) = (a,-x,\zeta^{-1}). \tag{2.24}$$

We want this automorphism to be compatible with the automorphism δ_{-1} of $\mathcal{G}_E(\mathcal{L}_m)$ (see proof of Proposition 2.2.5 for definition).

DEFINITION 2.3.5. A theta structure Θ for $\mathcal{L}_m$ is called *symmetric* if

$$\Theta \circ \delta_{-1} = D_{-1} \circ \Theta.$$

Consider now the functor:

$$\begin{aligned} Sch[1/m] &\longrightarrow Sets \\ S &\longmapsto \left\{ \begin{array}{l} \text{isomorphism classes } (E,\Theta) \text{ of elliptic curves } E \to S \\ \text{and a symmetric level } m \text{ theta structure } \Theta \end{array} \right\}. \end{aligned} \tag{2.25}$$

By [23], §6 this functor is representable by a scheme $\mathcal{M}_{m,2m}$ over $\mathbb{Z}[1/m]$. The 'forget the theta structure' functor gives a 1-morphism of stacks:

$$\mathcal{M}_{m,2m} \longrightarrow \mathcal{M}_1.$$

The following theorem, essentially contained in [23], shows that over the fiber product

$$\widetilde{\mathcal{M}}_{m,2m} := \mathcal{M}_{m,2m} \times_{\mathcal{M}_1} \mathcal{M}_{1/2},$$

taken in the category of Deligne-Mumford stacks, the sheaf $\mathcal{V}_m$ of Schrödinger representations can be decomposed into m copies of an invertible sheaf.

THEOREM 2.3.6. *Let $m \in 2\mathbb{Z}_{>0}$. Over $p_{m,2m} : \widetilde{\mathcal{M}}_{m,2m} \to \mathcal{M}_{1/2}$, there is an invertible sheaf $\mathcal{L}_{\theta,m}$ and a canonical isomorphism*

$$p^*_{m,2m}(\mathcal{V}_m) \simeq \mathcal{L}_{\theta,m}^{\oplus m},$$

defined up to multiplication by an element in $\Gamma(\widetilde{\mathcal{M}}_{m,2m}, \mathcal{O}^\times_{\widetilde{\mathcal{M}}_{m,2m}})$.

Proof. Since the functor (2.25) is representable, there exists a universal elliptic curve

$$\mathcal{E} \longrightarrow \mathcal{M}_{m,2m}$$

canonically endowed with a symmetric level m theta structure Θ. Via this theta structure, the $\mathcal{O}_{\mathcal{M}_{m,2m}}$-modules $\mathcal{V}(m) \otimes \mathcal{O}_{\mathcal{M}_{m,2m}}$ and $\mathcal{V}_m \otimes \underline{\omega}^{-1/2}$, considered over $\mathcal{M}_{m,2m}$, are both weight

1, rank m representations of the level m Heisenberg group $\mathcal{G}_{\mathcal{E}}(\mathcal{L}_m)$. Hence by Theorem 1.3.3 we must have:

$$\mathcal{V}_m \otimes \underline{\omega}^{-1/2} \simeq \mathcal{V}(m) \otimes \mathcal{L}$$

for some invertible sheaf $\mathcal{L}$ over $\mathcal{M}_{m,2m}$. But $\mathcal{V}(m)$ is free of rank m, hence:

$$\mathcal{V}_m \otimes \underline{\omega}^{-1/2} \simeq \mathcal{L}^{\oplus m},$$

over $\mathcal{M}_{m,2m}$. The theorem is now obtained by tensoring both sides by $\underline{\omega}^{1/2}$ and setting $\mathcal{L}_{\theta,m} := \mathcal{L} \otimes \underline{\omega}^{1/2}$. □

Thus the theory of theta structures produces invertible sheaves $\mathcal{L}_{\theta,m}$ which decompose the Schrödinger representation over $\widetilde{M}_{m,2m}$.

Remark 2.3.7. By [22], §2, there is a morphism

$$\phi_m : \mathcal{M}_{1/2}(2m) \to \widetilde{\mathcal{M}}_{m,2m},$$

which is finite étale of degree 4 on geometric fibers. Therefore $\mathcal{L}_{\theta,m}$ must be of order 4 in $\mathrm{Pic}(\widetilde{\mathcal{M}}_{m,2m})$, i.e. $\mathcal{L}_{\theta,m}^{\otimes 4}$ is trivial. But we can say even more. By Theorem 2.2.19, we know that the invertible sheaves $\mathcal{L}_{\theta,m}$ trivialize over $\mathcal{M}_{1/2}(2m)$. Moreover, for each copy of $\mathcal{L}_{\theta,m}$ in the decomposition of Theorem 2.3.6, we must have

$$\phi_m^*(\mathcal{L}_{\theta,m}) \simeq \mathcal{O}_{\mathcal{M}_{1/2}(2m)}\, \delta_r$$

for some $r \in \mathbb{Z}/m\mathbb{Z}$. This is because both the decomposition of Theorem 2.3.6 and the trivialization of Theorem 2.2.19 are equivariant with respect to the Heisenberg action. If we combine this observation with Theorem 2.2.20, we deduce that the invertible sheaves $\mathcal{L}_{\theta,m} \otimes \underline{\omega}^{-1/2}$ each correspond to one of the level m theta functions $\vartheta_{m,r}$ of (2.19), i.e. the theta function $\vartheta_{m,r}$ trivializes the r-th copy of $\mathcal{L}_{\theta,m}$ over $\mathcal{M}_{1/2}(2m)$.

2.3.4 Shimura's modular forms of half-integral weight

As a final application of our algebraic theory of vector-valued modular forms, we would like to construct algebro-geometric analogs of Shimura's modular forms of half-integral weight ([31]), usually just called *modular forms of half-integral weight.*

To this end, let $m \in 2\mathbb{Z}_{>0}$ be a positive even integer an consider pairs $(E/S, C)$ of an elliptic curve E over a scheme $S \in \mathrm{Ob}(Sch[1/m])$, together with a cyclic subgroup of order $2m$:

$$C \subset E[2m].$$

Let $\mathcal{M}_0(2m)$ be the moduli stack of all such pairs and let

$$\widetilde{\mathcal{M}}_0(2m) := \mathcal{M}_0(2m) \times_{\mathcal{M}_1} \mathcal{M}_{1/2}$$

be the fiber product in the category of Deligne-Mumford stacks, classifying triples $(E/S, C, \mathcal{Q})$ of an elliptic curve over a scheme $S \in \mathrm{Ob}(Sch[1/m])$ together with a cyclic subgroup C of order $2m$ and a rank 1 non-degenerate $\underline{\omega}_{E/S}$-valued quadratic form $\mathcal{Q}$. The sheaf $\mathcal{V}_m$ of Schrödinger representations pulls back to $\widetilde{\mathcal{M}}_0(2m)$ under the natural forgetful map $\widetilde{\mathcal{M}}_0(2m) \to \mathcal{M}_{1/2}$. We then have:

THEOREM 2.3.8. *There is an invertible sheaf $\mathcal{D}_m$ over $\widetilde{\mathcal{M}}_0(2m)$ equipped with a canonical injection*

$$\mathcal{D}_{2m} \hookrightarrow \mathcal{V}_m.$$

Proof. Let $(E/S, C, \mathcal{Q})$ be an elliptic curve over a scheme $S \in \mathrm{Ob}(Sch[1/m])$ together with a cyclic subgroup C of order $2m$ and a rank 1 non-degenerate $\underline{\omega}_{E/S}$-valued quadratic form $\mathcal{Q}$. The subgroup

$$H_1 := [2]C \subset E[m]$$

is cyclic of order m, hence it must be maximal isotropic in $E[m]$. By the same argument as in the proof of Theorem 2.2.19, we also have a canonical lift

$$H_1 \to \mathcal{G}_E(\mathcal{L}_m),$$

to a lagrangian subgroup H of $\mathcal{G}_E(\mathcal{L}_m)$ given by taking local lifts of C to lagrangian subgroups of $\mathcal{G}_E(\mathcal{L}_{2m})$ and projecting them down to $\mathcal{G}_E(\mathcal{L}_m)$ using the map

$$\eta_2 : \mathcal{G}_E(\mathcal{L}_{2m}) \to \mathcal{G}_E(\mathcal{L}_m).$$

We can therefore define a canonical Schrödinger representation $\mathcal{V}_H$ by using the subgroup C. In particular there is an invertible sheaf over S,

$$\mathcal{D}_H := \mathcal{V}_H^H \subset \mathcal{V}_H,$$

given by the H-invariant sections of $\mathcal{V}_H$. Now the assignment:

$$(E, C, \mathcal{Q}) \longmapsto \Gamma(S, \mathcal{V}_H)$$

defines a locally free sheaf $\mathcal{V}$ of rank m over $\widetilde{\mathcal{M}}_0(2m)$, since our choice of Schrödinger representation is functorial in C. By universality of $\mathcal{V}_m$, we must have

$$p_0(2m)^*\mathcal{V}_m \simeq \mathcal{V},$$

under the map $p_0(2m) : \widetilde{\mathcal{M}}_0(2m) \to \mathcal{M}_{1/2}$. On the other hand, the assignment

$$(E, C, \mathcal{Q}) \longmapsto \Gamma(S, \mathcal{D}_H)$$

defines an invertible sheaf $\mathcal{D}_{2m}$ over $\widetilde{\mathcal{M}}_0(2m)$, equipped with the required canonical injection

$$\mathcal{D}_{2m} \hookrightarrow p_0(2m)^*\mathcal{V}_m.$$

□

Remark 2.3.9. Note that in the proof we have implicitly defined a morphism

$$\mathcal{M}_0(2m) \longrightarrow \widetilde{\mathcal{M}}_0(2m),$$

given by sending (E, C) to $(E, \mathcal{V}_H)$.

Remark 2.3.10. The injection in Theorem 2.3.8 is Heisenberg-equivariant, thus it is compatible with the trivialization of Theorem 2.2.19. In particular, over $\mathcal{M}_{1/2}(2m)$ we must have:

$$\phi^*\mathcal{D}_{2m} \simeq \mathcal{O}_{\mathcal{M}_{1/2}(2m)}\, \delta_0,$$

under the forgetful map $\phi : \mathcal{M}_{1/2}(2m) \to \widetilde{\mathcal{M}}_0(2m)$.

Let now $m = 2$, and consider the sheaf:

$$\mathcal{L}_{\mathrm{Shi}} := \mathcal{D}_2^\vee \otimes \underline{\omega}^{1/2}.$$

By Theorem 2.2.14, this invertible sheaf is defined over the stack $\mathcal{M}_0(4)$.

Definition 2.3.11. Let $k \in \mathbb{Z}$ be an integer. A *(Shimura) modular form of half-integral weight* $k/2$ is a global section of the invertible sheaf $\mathcal{L}_{\mathrm{Shi}}^k$ over $\mathcal{M}_0(4)$.

Remark 2.3.12. It would be interesting to draw a comparison between our algebraic theory of that of Nick Ramsey ([30]), who defines algebraic modular forms of half-integral weight over $\mathcal{M}_1(4)$. In particular, Ramsey analyzes what happens at the cusps and defines a theory of Hecke operators for modular forms of half-integral weight, which we do not do in this work.

To define q-expansions of modular forms of half-integral weight, consider the Tate curve

$\mathrm{Tate}(q) \to \mathrm{Spec}(\mathbb{Z}[1/2]((q)))$. This curve is endowed with a canonical cyclic subgroup of order 4

$$\mu_4 \hookrightarrow \mathrm{Tate}(q)[4],$$

hence we can form the sheaf $\mathcal{D}_H$ as in the proof of Theorem 2.3.8, where H is the lagrangian subgroup obtained from μ_4. In particular, we must have an injection

$$\mathcal{D}_H \hookrightarrow V_2^{\mathrm{can}},$$

since $[2]\mu_4 = \mu_2$ (notation as in Section 2.2.7). A simple modification of the argument used in Theorem 2.2.19 now shows:

Proposition 2.3.13. *The invertible sheaf* $\mathcal{D}_H \hookrightarrow V_2^{\mathrm{can}}$ *associated to the pair* $(\mathrm{Tate}(q), \mu_4)$ *has a canonical trivialization:*

$$\mathcal{D}_H \simeq \mathbb{Z}[1/2]((q))\, \delta_0, \tag{2.26}$$

compatible with the trivialization (2.21) *of* V_2^{can} *over* $\mathrm{Tate}(q^{1/4}) \to \mathrm{Tate}(q)$.

Consider now the triple $(\mathrm{Tate}(q), \mu_4, \mathcal{Q})$, where $\mathcal{Q}$ is chosen as in Section 2.2.7. This triple defines a map

$$\psi : \mathrm{Spec}(\mathbb{Z}[1/2]((q))) \longrightarrow \widetilde{\mathcal{M}}_0(4).$$

In particular, for any integer $k \in \mathbb{Z}$ we have a canonical trivialization:

$$\psi^*(\mathcal{L}_{\mathrm{Shi}}^{\otimes k}) \simeq \mathbb{Z}[1/2]((q))\, (\delta_0^* \otimes \omega_{\mathrm{can}}^{1/2})^{\otimes k},$$

obtained by combining the duals of trivializations (2.22) and (2.26). If f is a modular form of half-integral weight, then we can associate to it a well-defined element $f(q)$ of $\mathbb{Z}[1/2]((q))$ by

$$\psi(f) = f(q)\, (\delta_0^* \otimes \omega_{\mathrm{can}}^{1/2})^{\otimes k}.$$

Definition 2.3.14. The element $f(q) \in \mathbb{Z}[1/2]((q))$ is the q-expansion of the modular form of half-integral weight f. The modular form f is *holomorphic* if in in fact we have $f \in \mathbb{Z}[1/2][\![q]\!]$.

The prototypical example of a holomorphic modular form of half-integral weight can be constructed in a way analogous to our construction of the theta constants $\theta_{\mathrm{null},m}$. In particular, let $(E/S, C)$ be a pair of an elliptic curve $\pi : E \to S$ over a scheme $S \in \mathrm{Ob}(Sch[1/m])$, together with a cyclic subgroup C of order $2m$. As we have seen in the proof of Theorem 2.3.8, this curve is endowed with a canonical lagrangian subgroup $H \subset \mathcal{G}_E(\mathcal{L}_m)$. The rule

$$(E/S, C) \longmapsto \Gamma(S, \pi_* \mathcal{L}_m)^H$$

defines an invertible sheaf $\mathcal{J}_{m,0}$ over $\mathcal{M}_0(2m)$, equipped with a canonical injection

$$0 \to \mathcal{J}_{m,0} \to \mathcal{J}_m.$$

Dually, there is a canonical surjection

$$\mathcal{J}_m^\vee \xrightarrow{\beta} \mathcal{J}_{m,0}^\vee \to 0.$$

Define now

$$\mathrm{ev}_{e,0} := \beta(\mathrm{ev}_e).$$

This is a global section of $\mathcal{J}_{m,0}^\vee$. For example, for $m = 2$ we get a modular form θ_0 of half-integral weight, in the sense of Definition 2.3.11, defined by:

$$\begin{aligned} \mathcal{J}_{2,0}^\vee &\longleftrightarrow \mathcal{L}_{\mathrm{Shi}} \\ \mathrm{ev}_{e,0} &\longmapsto \theta_0. \end{aligned}$$

By Proposition 2.3.13, to compute the q-expansion of θ_0 we can pass to $\mathrm{Tate}(q^{1/4})$ where we have:

$$\theta_{\mathrm{null},2}(q) = (\vartheta_{2,0}(q,1)\delta_0^* + \vartheta_{2,1}(q,1)\delta_1^*) \otimes \omega_{\mathrm{can}}^{1/2},$$

by the computations of Section 2.3.2. In particular

$$\theta_0(q) = \vartheta_{2,0}(q,1) = \sum_{n\in\mathbb{Z}} q^{n^2},$$

up to a constant in $\mathbb{Z}[1/3m]^\times$.

Thus we have proven, by purely algebraic/moduli-theoretic means, that the q-expansion

$$\sum_{n\in\mathbb{Z}} q^{n^2}$$

is the q-expansion of a holomorphic modular form of half-integral weight 1/2, in the algebro-geometric sense of Definition 2.3.11. Similarly, the q-expansion

$$\theta_0^3(q) = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + \dots,$$

essentially the generating series of quadratic imaginary class numbers, is the q-expansion of the section θ_0^3 of $\mathcal{L}_{\mathrm{Shi}}^{\otimes 3}$ over $\mathcal{M}_0(4)$, hence a holomorphic modular form of weight 3/2.

Chapter 3

Analytic Theory

The same constructions of Chapter 2 apply when the base scheme S is replaced by an analytic space. In this chapter we do precisely that and recover the usual analytic notions of vector-valued modular forms attached to rank 1 lattices (Section 3.2.3), and of Shimura's modular forms of half-integral weight (3.2.5). We also explain in Section 3.2.4 the relation between vector-valued modular forms and the Jacobi forms of [14] using the Main Theorem 2.2.20.

The key notion in this chapter is that of a *basic orbifold* ([17]), which we review in Section 3.1.1. This is a generalization of the concept of a quotient of a topological space by a group action. It turns out (Theorem 3.1.5) that the stack $\mathcal{M}_{1/2}^{\mathrm{an}}$ of elliptic curves over an analytic space equipped with a quadratic form is equivalent to the orbifold $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$, where $\mathrm{Mp}_2(\mathbb{Z})$ is the metaplectic group given by acting on $\mathfrak{h}$ by linear fractional transformations. Thus the existence of the vector bundles $\mathcal{V}_m$ of Chapter 2 can be translated into transformation laws for holomorphic functions on $\mathfrak{h}$, and similarly for the bundles $\mathcal{L}_{\mathrm{Shi}}^k$ of modular forms of half-integral weight.

3.1 The metaplectic orbifold

3.1.1 Basic Orbifolds

We follow the definition of *basic orbifold* given in [17].

Definition 3.1.1 ([17], §3.1). A basic orbifold is a triple (X, Γ, ρ), often simply denoted by $\Gamma\backslash\backslash X$, where:

(i) X is a connected, simply connected topological space.

(ii) Γ is a discrete group.

(iii) $\rho : \Gamma \to \mathrm{Aut}(X)$ is a homomorphism.

A morphism $(X, \Gamma, \rho) \to (X', \Gamma', \rho')$ between orbifolds is a pair (f, ϕ) of:

(i) A continuous map $f : X \to X'$

(ii) A group homomorphism $\phi : \Gamma \to \Gamma'$,

such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \Big\downarrow{\scriptstyle \rho(\gamma)} & & \Big\downarrow{\scriptstyle \rho'(\phi(\gamma))} \\ X & \xrightarrow{f} & X' \end{array}$$

.

When X is a complex manifold, the orbifold (X, Γ, ρ) can be given a 'complex structure' by letting the holomorphic functions be the holomorphic functions on X which are Γ-invariant ([17], §3.1). A *holomorphic vector bundle* on the orbifold (X, Γ, ρ) is defined by giving the Γ-invariant holomorphic sections of a vector bundle on X ([17], §3.2). The following Proposition then follows from standard cocycle computations:

Proposition 3.1.2.

Let X be a simply connected manifold where every vector bundle is trivial. Let $\Gamma\backslash\!\backslash X$ be a basic orbifold with the inherited complex structure. Then

(i) Line bundles on the complex orbifold $\Gamma\backslash\!\backslash X$ are in 1-1 correspondence with 1-cocycles in $H^1(\Gamma, \mathcal{O}_X^*)$.

(ii) Rank r vector bundles on the complex orbifold $\Gamma\backslash\!\backslash X$ are in 1-1 correspondence with 1-cocycles in $H^1(\Gamma, \mathrm{GL}_r(\mathcal{O}_X))$.

The Proposition highlights the fact that Γ can be thought of as the 'fundamental group' of $\Gamma\backslash\!\backslash X$. This notion can be made rigorous, as in [17], §3.3. In particular, we can define local systems on $\Gamma\backslash\!\backslash X$ as follows:

Definition 3.1.3. Let V be a finite-dimensional complex vector space. A *local system* $\mathbb{V}$ of fiber V on the complex orbifold $\Gamma\backslash\!\backslash X$ is a finite dimensional complex representations $\Gamma \to \mathrm{GL}(V)$.

A complex orbifold $\Gamma\backslash\!\backslash X$ gives rise to a category whose set of objects is the set of points of X and whose set of morphisms is $\Gamma \times X$. This category is a groupoid in the category of analytic spaces, hence a stack. In the following, when we speak of the 'stack' $\Gamma\backslash\!\backslash X$ we mean the above category determined by the orbifold $\Gamma\backslash\!\backslash X$.

3.1.2 The metaplectic orbifold and metaplectic forms

Let $\mathcal{M}_1^{\mathrm{an}} \to AnSp$ be the stack of elliptic curves over a (complex) analytic space. It is well-known (e.g. [21], §7) that this stack is equivalent to the stack determined by the orbifold $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$, where $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathfrak{h}$ by linear fractional transformations. The orbifold $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ is called the *modular orbifold*. Note that $\mathfrak{h}$ is simply connected, and every holomorphic vector bundle over it is trivial, so Proposition 3.1.2 applies.

For each elliptic curve $\pi : E \to S$ over an analytic space, we can define the Hodge bundle $\underline{\omega}_{E/S} := \pi_*\Omega^1_{E/S}$, an invertible sheaf over S. Precisely as in the algebraic setting, this assignment defines the Hodge bundle $\underline{\omega}$ over the stack $\mathcal{M}_1^{\mathrm{an}}$. This invertible sheaf $\underline{\omega}$ corresponds to a holomorphic line bundle over the orbifold $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$, hence by Proposition 3.1.2 its isomorphism class corresponds to the class of a 1-cocycle j_1 in $H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{O}_{\mathfrak{h}}^*)$ which is given by ([17], Lemma 5.13):

$$j_1 : \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathcal{O}_{\mathfrak{h}}^*$$
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto c\tau + d.$$

For $k \in \mathbb{Z}$, the holomorphic sections of $\underline{\omega}^k$ are given by holomorphic functions $f : \mathfrak{h} \to \mathbb{C}$ such that:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

These are the classical modular forms of integral weight k and level 1 (with no growth conditions imposed at the cusps).

Next, consider the stack $\mathcal{M}_{1/2}^{\mathrm{an}} \to AnSp$ of pairs $(E, \mathcal{Q})$ of an elliptic curve over an analytic space together with a non-degenerate rank one $\underline{\omega}_{E/S}$-valued quadratic form. As in in section 2.1.1 this is a μ_2-gerbe over $\mathcal{M}_1^{\mathrm{an}}$, canonically equipped with a square root $\underline{\omega}^{1/2}$ of $\underline{\omega}$. Passing to the category of orbifolds, we know that μ_2-gerbes over $\mathcal{M}_1^{\mathrm{an}} = \mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ must correspond to classes in

$$H^2(\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}, \mu_2) = H^2(\mathrm{SL}_2(\mathbb{Z}), \mu_2) \simeq \mathbb{Z}/2\mathbb{Z}.$$

It is well-known that the H^2 in group cohomology corresponds to central extensions. In particular, the gerbe $\mathcal{M}_{1/2}^{\mathrm{an}}$ corresponds to the unique nontrivial central extension

$$0 \to \mu_2 \to \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}) \to 0$$

of $\mathrm{SL}_2(\mathbb{Z})$ by μ_2.

Definition 3.1.4. The group $\mathrm{Mp}_2(\mathbb{Z})$ is called the *metaplectic group*. It is the set of pairs

$(\gamma, \epsilon) \in \mathrm{SL}_2(\mathbb{Z}) \times \{\pm 1\}$ with group structure given by the multiplication rule:

$$(\gamma_1, \epsilon_1) \cdot (\gamma_2, \epsilon_2) = (\gamma_1\gamma_2, c(\gamma_1, \gamma_2)\epsilon_1\epsilon_2)$$

where c is any 2-cocycle $c : \mathrm{SL}_2(\mathbb{Z})^2 \to \{\pm 1\}$ representing the nontrivial class in $H^2(\mathrm{SL}_2(\mathbb{Z}), \mu_2)$.

An explicit model of the metaplectic group is the set of pairs

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi(\tau)\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),\ \phi^2(\tau) = c\tau + d,\ \phi(\tau) \in \mathcal{O}_{\mathfrak{h}}^*,$$

with multiplication given by:

$$(A_1, \phi_1(\tau)) \cdot (A_2, \phi_2(\tau)) = (A_1A_2, \phi_1(A_2\tau)\phi_2(\tau)).$$

We now have:

THEOREM 3.1.5. *The stack* $\mathcal{M}_{1/2}^{\mathrm{an}}$ *is equivalent to the stack determined by the orbifold* $\mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$*, where* $\mathrm{Mp}_2(\mathbb{Z})$ *acts on* $\mathfrak{h}$ *via the map* $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$*, i.e. via linear fractional transformations of the underlying matrix in* $\mathrm{SL}_2(\mathbb{Z})$*.*

Proof. Consider the the elliptic curve $\pi^{\mathfrak{h}} : \mathcal{E}_{\mathfrak{h}} \to \mathfrak{h}$ over the complex upper half-plane, an elliptic curve over the analytic space $\mathfrak{h}$ given by:

$$(\mathfrak{h} \times \mathbb{C})\,/\mathbb{Z}^2 = \mathcal{E}_{\mathfrak{h}} \xrightarrow{\pi^{\mathfrak{h}}} \mathfrak{h},$$

where $\mathbb{Z}^2$ acts by $(\tau, z) \mapsto (\tau, z + m + n\tau)$. This is the universal framed elliptic curve ([17], Prop. 2.4), i.e. the universal object over $\mathfrak{h}$, viewed as the moduli space of elliptic curves over analytic spaces with 'framings' ([17], Def. 2.1). Now the Hodge bundle $\underline{\omega}_{\mathfrak{h}}$ of $\mathcal{E}_{\mathfrak{h}}$ can be trivialized over $\mathfrak{h}$ by

$$\underline{\omega}_{\mathfrak{h}} \simeq \mathcal{O}_{\mathfrak{h}}\, dz,$$

where dz is the everywhere nonzero section obtained by choosing the invariant differential dz over the elliptic curve $E_\tau = \mathbb{C}/\langle\tau, 1\rangle$. We define a rank 1 non-degenerate $\underline{\omega}_{\mathfrak{h}}$-valued quadratic form as follows. Over the elliptic curve E_τ, the sheaf of differential $\Omega^1_{E_\tau}$ is trivial, thus the square roots of $\Omega^1_{E_\tau}$ are in bijection with $\mathrm{Pic}(E_\tau)[2] \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus there are 4 square roots, the *theta characteristics* of E_τ. It follows from a simple Riemann-Roch argument that only one of them, the 'odd one' $\Omega^{1/2,\mathrm{odd}}_{E_\tau}$ has global sections. In particular $\Omega^{1/2,\mathrm{odd}}_{E_\tau}$ is trivial, and can be trivialized by a generator $\sqrt{dz}$ compatible with our choice of generator for $\Omega^1_{E_\tau}$. These odd theta characteristics on the fibers give rise to a line bundle

$\Omega^{1/2,\mathrm{odd}}_{\mathcal{E}_\mathfrak{h}}$ over the universal elliptic curve. Now set:

$$\mathcal{Q} := \pi^\mathfrak{h}_*(\Omega^{1/2,\mathrm{odd}}_{\mathcal{E}_\mathfrak{h}}).$$

This is a trivial line bundle over $\mathfrak{h}$, trivialized by

$$\mathcal{Q} \simeq \mathcal{O}_\mathfrak{h} \sqrt{dz},$$

and is a non-degenerate rank 1 $\underline{\omega}_\mathfrak{h}$-valued quadratic form:

$$q : \mathcal{Q} \to \underline{\omega}_\mathfrak{h}.$$

The pair $(\mathcal{E}_\mathfrak{h}, \mathcal{Q})$ defines a map:

$$\mathfrak{h} \to \mathcal{M}^{\mathrm{an}}_1.$$

Now there is an isomorphism $\mathrm{SL}_2(\mathbb{Z}) \simeq \mathrm{Aut}(\mathcal{E}_\mathfrak{h})$ given by

$$\gamma \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

therefore the group $\mathrm{Aut}(\mathcal{E}_\mathfrak{h}, \mathcal{Q})$ must be a central extension:

$$0 \to \mu_2 \to \mathrm{Aut}(\mathcal{E}_\mathfrak{h}, \mathcal{Q}) \to \mathrm{SL}_2(\mathbb{Z}) \to 0.$$

We claim this extension is nontrivial. This is because the action of the automorphism $[-1]$ of $\mathcal{E}_\mathfrak{h}$ on $\mathcal{Q}$ generates a cyclic subgroup of order 4. Therefore, the matrix $-I \in \mathrm{SL}_2(\mathbb{Z})$ must lift to an element of order 4 in $\mathrm{Aut}(\mathcal{E}_\mathfrak{h}, \mathcal{Q})$, which cannot happen if the extension is trivial. But the central extensions of $\mathrm{SL}_2(\mathbb{Z})$ by μ_2 are classified by $H^2(\mathrm{SL}_2(\mathbb{Z}), \mu_2) \simeq \mathbb{Z}/2\mathbb{Z}$, generated by the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$, thus

$$\mathrm{Aut}(\mathcal{E}_\mathfrak{h}, \mathcal{Q}) \simeq \mathrm{Mp}_2(\mathbb{Z}).$$

Now the pair $(\mathcal{E}_\mathfrak{h}, \mathcal{Q})$ defines a map of orbifolds

$$\mathfrak{h} \to \mathcal{M}^{\mathrm{an}}_{1/2},$$

by the universal property of $\mathcal{M}^{\mathrm{an}}_{1/2}$. But $\mathfrak{h}$ is simply connected, hence it must be the universal cover of $\mathcal{M}^{\mathrm{an}}_{1/2}$ and we must have:

$$\mathcal{M}^{\mathrm{an}}_{1/2} \simeq \mathrm{Aut}(\mathcal{E}_\mathfrak{h}, \mathcal{Q})\backslash\!\!\backslash \mathfrak{h} \simeq \mathrm{Mp}_2(\mathbb{Z})\backslash\!\!\backslash \mathfrak{h}.$$

□

Definition 3.1.6. The orbifold $\mathrm{Mp}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{h}$ is called the *metaplectic orbifold.*

By construction, the stack $\mathcal{M}^{\mathrm{an}}_{1/2}$ is canonically equipped with an invertible sheaf $\underline{\omega}^{1/2}$, which corresponds to a line bundle over the orbifold $\mathrm{Mp}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{h}$. By Proposition 3.1.2, this line bundle is determined by a corresponding 1-cocylce class in $H^1(\mathrm{Mp}_2(\mathbb{Z}), \mathcal{O}^*_{\mathfrak{h}})$, whose computation is a simple corollary of Theorem 3.1.5.

Proposition 3.1.7. *The line bundle $\underline{\omega}^{1/2}$ over $\mathrm{Mp}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{h}$ corresponds to the 1-cocycle:*

$$j_{1/2} : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathcal{O}^*_{\mathfrak{h}}$$
$$\left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) \longmapsto \phi.$$

For $k \in \mathbb{Z}$, the holomorphic sections of $\underline{\omega}^{k/2}$ are given by holomorphic functions $f : \mathfrak{h} \to \mathbb{C}$ such that:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \phi^k f(\tau), \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) \in \mathrm{Mp}_2(\mathbb{Z}),$$

by Proposition 3.1.7. These are the *metaplectic forms* of weight $k/2$, first considered in [31], §1.

We can also compute the Picard group of $\mathcal{M}^{\mathrm{an}}_{1/2}$ (compare with Proposition 2.1.4):

Proposition 3.1.8. *Let $\mathrm{Pic}(\mathcal{M}^{\mathrm{an}}_{1/2})$ be the group of line bundles on the metaplectic orbifold. Then there is a canonical isomorphism*

$$\mathrm{Pic}(\mathcal{M}^{\mathrm{an}}_{1/2}) \simeq \mathbb{Z}/24\mathbb{Z}$$

given by sending $\underline{\omega}^{1/2} \mapsto 1 \mod 24$.

Proof. Of course we can prove this by the same methods of Proposition 2.1.4. However, we want to give a different, analytic proof of this fact. By Proposition 3.1.2 and Theorem 3.1.5, we have

$$\mathrm{Pic}(\mathcal{M}^{\mathrm{an}}_{1/2}) = \mathrm{Pic}(\mathrm{Mp}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{h}) = H^1(\mathrm{Mp}_2(\mathbb{Z}), \mathcal{O}^\times_{\mathfrak{h}}).$$

To compute $H^1(\mathrm{Mp}_2(\mathbb{Z}), \mathcal{O}^\times_{\mathfrak{h}})$, consider the exponential sequence over $\mathfrak{h}$

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathfrak{h}} \overset{f\mapsto e^{2\pi i f}}{\longrightarrow} \mathcal{O}^\times_{\mathfrak{h}} \to 0,$$

which induces a long exact sequence:

$$\ldots \to H^1(\mathrm{Mp}_2(\mathbb{Z}), \mathcal{O}_{\mathfrak{h}}) \to H^1(\mathrm{Mp}_2(\mathbb{Z}), \mathcal{O}^\times_{\mathfrak{h}}) \to H^2(\mathrm{Mp}_2(\mathbb{Z}), \mathbb{Z}) \to H^2(\mathrm{Mp}_2(\mathbb{Z}), \mathcal{O}_{\mathfrak{h}}) \to \ldots.$$

By considering the Leray spectral sequence of the map $\mathfrak{h} \to \mathrm{Mp}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$, we know that

$$H^1(\mathrm{Mp}_2(\mathbb{Z}), \mathcal{O}_{\mathfrak{h}}) = H^2(\mathrm{Mp}_2(\mathbb{Z}), \mathcal{O}_{\mathfrak{h}}) = 0,$$

since the cohomology over $\mathfrak{h}$ vanishes, and therefore

$$H^1(\mathrm{Mp}_2(\mathbb{Z}), \mathcal{O}_{\mathfrak{h}}^{\times}) = H^2(\mathrm{Mp}_2(\mathbb{Z}), \mathbb{Z}).$$

Now $\mu_2 \subset \mathrm{Mp}_2(\mathbb{Z})$ is normal, so the fact that

$$H^2(\mathrm{Mp}_2(\mathbb{Z}), \mathbb{Z}) \simeq \mathbb{Z}/24\mathbb{Z}$$

follows by applying the Hochschild-Serre spectral sequence and by the classical fact that $H^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}) \simeq \mathbb{Z}/12\mathbb{Z}$.

To show that $\underline{\omega}^{1/2}$ is a generator, note that $(\underline{\omega}^{1/2})^{\otimes 2} \simeq \underline{\omega}$ and $\underline{\omega}$ is a generator for $\mathrm{Pic}(\mathcal{M}_1^{\mathrm{an}})$. □

3.2 Vector-valued modular forms

3.2.1 Heisenberg groups over the complex torus E_τ

Consider the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$, where Λ_τ is the rank 2 lattice $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$, $\tau \in \mathfrak{h}$. For any positive even integer $m \in 2\mathbb{Z}_{>0}$, define the line bundle:

$$\mathcal{L}_m := \mathcal{O}_{E_\tau}(m\,0) \otimes (\Omega^1_{E_\tau})^{\otimes m},$$

the analytic analog of the invertible sheaf defined in (2.9). This line bundle is isomorphic over E_τ to $\mathcal{O}_{E_\tau}(m\,0)$, which by the Appel-Humbert Theorem ([24], p.20) is given by the 1-cocycle:

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z} &\longrightarrow \mathcal{O}^*_{\mathbb{C}} \\ \lambda = (n_1, n_2) &\longmapsto e^{\pi\, H_m(\lambda,\lambda)/2 + \pi\, H_m(z,\lambda)}, \end{aligned} \tag{3.1}$$

where H_m is the Hermitian form on $\mathbb{C}$:

$$H_m(z_1, z_2) := m\,\frac{z_1\,\bar{z}_2}{v}, \quad \tau = u + iv.$$

The level m Heisenberg group $\mathcal{G}_{E_\tau}(\mathcal{L}_m)$ can also be described explicitly. Write $\mathbb{C}$ as a two-dimensional real vector space by writing $z = r_1 + r_2\tau \in \mathbb{C}$ with $r_1, r_2 \in \mathbb{R}$. Let:

$$\mathrm{im}\, H_m(r_1 + r_2\tau, s_1 + s_2\tau) = m(r_1 s_2 - r_2 s_1)$$

be the alternating form on $\mathbb{C}$ determined by H_m. Note that:

$$\operatorname{im} H_m(\Lambda_\tau, \Lambda_\tau) \subset \mathbb{Z}.$$

The form $\operatorname{im} H_m$ determines a Heisenberg group:

$$0 \to S^1 \to \mathrm{Heis}(2, m) \to \mathbb{C} \to 0,$$

where the group law is given by:

$$(t_1, z_1 = r_1 + r_2\tau) \cdot (t_2, z_2 = s_1 + s_2\tau) = (t_1 t_2 e^{\pi i m(r_1 s_2 - r_2 s_1)}, z_1 + z_2).$$

The lattice $\Lambda \subset \mathbb{C}$ can be lifted to a Lagrangian subgroup of $\mathrm{Heis}(2, m)$ via:

$$\begin{aligned} \sigma : \Lambda &\longrightarrow \mathrm{Heis}(2, m) \\ \lambda = n_1 + n_2\tau &\longmapsto (1, \lambda). \end{aligned}$$

We then have an isomorphism of central extensions ([26], Prop. 3.1):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{G}_{E_\tau}(\mathcal{L}_m) & \longrightarrow & E_\tau[m] & \longrightarrow & 0 \\ & & \updownarrow \simeq & & \updownarrow \simeq & & \updownarrow \simeq & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{Norm}_{\mathrm{Heis}(2,m)}(\sigma(\Lambda_\tau))/\sigma(\Lambda_\tau) & \longrightarrow & \Lambda_\tau^\perp/\Lambda_\tau & \longrightarrow & 0 \end{array}$$

where

$$\Lambda^\perp = \{z \in \mathbb{C} : \operatorname{im} H_m(z, \lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda\}.$$

3.2.2 The Weil representation

Write $m = 2m'$ and let $L = (\mathbb{Z}, q_{m'})$ be the rank 1 lattice with quadratic form $q_{m'}(x) = m'x^2$. Let $B_{m'}$ be the associated bilinear form $(x, y) = mxy$ on $\mathbb{Z}^2$ and extend it to $\mathbb{R}^2$. Let:

$$L^\perp = \{r \in \mathbb{R} : B_{m'}(r, x) \in \mathbb{Z}\ \forall x \in L\}.$$

The group $L^\perp/L$ is cyclic of order m and the embedding:

$$L^\perp/L \subset \Lambda_\tau^\perp/\Lambda_\tau$$

can be lifted to a Lagrangian subgroup of $\mathcal{G}_{E_\tau}(\mathcal{L}_m)$. The corresponding representation of $\mathcal{G}_{E_\tau}(\mathcal{L}_m)$ is given by $V_m = \mathbb{C}[L^\perp/L]$ or, equivalently, by the $\mathbb{C}$-vector space of dimension m:

$$V_m = \{f : L^\perp/L \to \mathbb{C}\}.$$

We can now define a local system of rank m on the metaplectic orbifold $\mathrm{Mp}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{h}$ by considering the *Weil representation*:

$$\rho_m : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}(V_m)$$

attached to $L = (\mathbb{Z}, q_{m'})$. This is given by the formulas ([4], §2):

$$\begin{aligned} \rho_m(T)(e_\gamma) &= e^{-\pi i \gamma^2/m}\, e_\gamma \\ \rho_m(S)(e_\gamma) &= \frac{1}{\sqrt{im}} \sum_{\delta \in \mathbb{Z}/m\mathbb{Z}} e^{2\pi i \gamma\delta/m}\, e_\delta \end{aligned}$$

where:

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right)$$

are the standard generators for $\mathrm{Mp}_2(\mathbb{Z})$ and by $e_\gamma, e_\delta, \ldots$ we denote the delta functions in V_m which take the value 1 at γ, δ respectively and 0 everywhere else.

These formulas are the the same formulas (2.14) defining the vector bundle $\mathcal{V}_m$ of Schrödinger representations over $\mathcal{M}^{\mathrm{an}}_{1/2} \to AnSp$ constructed as in Definition 2.2.10. Thus:

Theorem 3.2.1. *The local system $\mathcal{V}_m$ over $\mathrm{Mp}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{h}$ given by the Weil representation ρ_m is canonically isomorphic to the bundle $\mathcal{V}_m$ of Schrödinger representations over $\mathcal{M}^{\mathrm{an}}_{1/2} \to AnSp$ constructed as in Definition 2.2.10.*

Let now $\Gamma(2m)$ be the principal congruence subgroup of level $2m$ and let

$$\widetilde{\Gamma}(2m) \subset \mathrm{Mp}_2(\mathbb{Z})$$

be its preimage under $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$. Then using the classical fact that $\mathcal{M}^{\mathrm{an}}_{2m} \simeq \Gamma(2m)\backslash\mathfrak{h}$, where $\mathcal{M}^{\mathrm{an}}_{2m}$ denotes the moduli space of elliptic curves over an analytic space together with a full level $2m$ structure, we have (notations as in Section 2.2.5):

$$\widetilde{\mathcal{M}}^{\mathrm{an}}_{2m} \simeq \widetilde{\Gamma}(2m)\backslash\!\backslash\mathfrak{h}.$$

In particular, we can prove as in Theorem 2.2.19 that $\mathcal{V}_m$ trivializes over $\widetilde{\mathcal{M}}^{\mathrm{an}}_{2m}$. We thus obtain a geometric proof of the well-known fact (e.g. [7]):

Theorem 3.2.2. *The Weil representation ρ_m factors through the nontrivial extension:*

$$0 \to \mu_2 \to \mathrm{Mp}_2(\mathbb{Z}/2m\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2m\mathbb{Z}) \to 0,$$

viewed as the automorphism group of the map $\widetilde{\mathcal{M}}^{\mathrm{an}}_{2m} \to \mathcal{M}_1$.

3.2.3 Vector-valued modular forms

Analogously to Definition 2.2.11, we define a weight $k/2$, $\mathcal{V}_m$-valued modular form as a section of $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ over $\mathcal{M}^{\mathrm{an}}_{1/2}$. This definition translates thanks to Proposition 3.1.7 and Theorem 3.2.1 into the following.

Definition 3.2.3. A weight $k/2$, $\mathcal{V}_m$-valued modular form is a holomorphic function $f : \mathfrak{h} \to V_m$ such that:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \phi^k \rho_m\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) f(\tau),$$

for any $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) \in \mathrm{Mp}_2(\mathbb{Z})$.

Note that this definition is precisely the definition of vector-valued modular forms associated to the rank 1 lattice $L = (\mathbb{Z}, q_{m'})$ ([14], §5). Moreover, by the analog of Theorem 2.2.14 over the category of analytic spaces, the vector bundle $\mathcal{V}_m \otimes \underline{\omega}^{k/2}$ descends to $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ whenever k is odd.

Remark 3.2.4. The definition of $\mathcal{V}_m^\vee$-valued modular forms (Remark 2.2.12) is similarly obtained by conjugating the action of ρ_m. This is for example the definition of vector-valued modular forms of [4].

3.2.4 Jacobi forms and the Eichler-Zagier Theorem

We would now like to explain how Theorem 2.2.20 is related to Theorem 5.1 of [14], linking Jacobi forms of weight k and index $m/2$ to $\mathcal{V}_m$-valued modular forms of weight $k-1/2$.

To this end, let $\pi^{\mathfrak{h}} : \mathcal{E}_{\mathfrak{h}} \to \mathfrak{h}$ be the universal elliptic curve over the complex upper half-plane, defined as in the proof of Theorem 3.1.5. For m a positive even integer, consider the invertible sheaf over $\mathcal{E}_{\mathfrak{h}}$:

$$\mathcal{L}_m := \mathcal{O}_{\mathcal{E}_{\mathfrak{h}}}(m\, 0_{\mathcal{E}_{\mathfrak{h}}}) \otimes (\Omega^1_{\mathcal{E}_{\mathfrak{h}}/\mathfrak{h}})^{\otimes m},$$

analogous to the invertible sheaf (2.9). Sections of this sheaf are holomorphic functions $f : \mathfrak{h} \times \mathbb{C} \to \mathbb{C}$ such that:

$$f(\tau, z+\lambda) = e^{\pi H_m(\lambda,\lambda)/2 + \pi H_m(z,\lambda)} f(\tau, z), \quad \lambda = m + n\tau,$$

as can be deduced from (3.1).

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{E}_{\mathfrak{h}}$ via the automorphism:

$$\gamma(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{3.2}$$

and thus acts on $\Omega^1_{\mathcal{E}_{\mathfrak{h}}/\mathfrak{h}}$ and $\mathcal{L}_m$ via multiplication by elements $\gamma_{\Omega^1_{\mathcal{E}_{\mathfrak{h}}/\mathfrak{h}}}(\tau), \gamma_{\mathcal{L}_m}(\tau) \in \mathcal{O}^*_{\mathfrak{h}}$. In particular, we have:

Lemma 3.2.5. *The element* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *acts on* $\Omega^1_{\mathcal{E}_{\mathfrak{h}}/\mathfrak{h}}$ *and* $\mathcal{L}_m$ *by:*

(i) $\gamma_{\Omega^1_{\mathcal{E}_{\mathfrak{h}}/\mathfrak{h}}}(\tau) = (c\tau + d)^{-1}$

(ii) $\gamma_{\mathcal{L}_m}(\tau) = 1$

respectively.

Proof.

(i) The invertible sheaf $\Omega^1_{\mathcal{E}_{\mathfrak{h}}/\mathfrak{h}}$ can be trivialized over $\mathcal{E}_{\mathfrak{h}}$ by the everywhere-non-vanishing differential $\omega = dz$, thus the computation of $\gamma_{\Omega^1_{\mathcal{E}_{\mathfrak{h}}/\mathfrak{h}}}(\tau)$ follows directly from (3.2).

(ii) We have that:

$$\mathcal{O}_{\mathcal{E}_{\mathfrak{h}}}(m\, 0_{\mathcal{E}_{\mathfrak{h}}}) = \mathcal{O}_{\mathcal{E}_{\mathfrak{h}}}(0_{\mathcal{E}_{\mathfrak{h}}})^{\otimes m}$$

and

$$\gamma_{\mathcal{O}_{\mathcal{E}_{\mathfrak{h}}}(0_{\mathcal{E}_{\mathfrak{h}}})}(\tau) = c\tau + d$$

as can be seen by noting that the invertible sheaf $\mathcal{O}_{\mathcal{E}_{\mathfrak{h}}}(0_{\mathcal{E}_{\mathfrak{h}}})$ can be trivialized by the section $1 \in \Gamma(E_\tau, \mathcal{O}_{E_\tau}(0))$, which gets sent to $c\tau + d$ by γ. Thus

$$\gamma_{\mathcal{O}_{\mathcal{E}_{\mathfrak{h}}}(m\, 0_{\mathcal{E}_{\mathfrak{h}}})}(\tau) = (c\tau + d)^m$$

which, combined with (i) above, proves the lemma.

□

By the Lemma, we deduce that the invertible sheaf $\Omega^1_{\mathcal{E}_{\mathfrak{h}}/\mathfrak{h}}$ descends to an invertible sheaf $\Omega^1_{\mathcal{E}/\mathcal{M}_1}$ over the orbifold:

$$\mathcal{E} := \mathrm{SL}_2(\mathbb{Z}) \backslash\backslash \mathcal{E}_{\mathfrak{h}}$$

which can be thought of as the universal 'elliptic curve' over $\mathcal{M}_1$, and similarly for $\mathcal{L}_m$. More generally, for any integer $k \in \mathbb{Z}$ the sheaf $\mathcal{L}_m \otimes (\Omega^1_{\mathcal{E}/\mathcal{M}_1})^{\otimes k}$ gives an invertible sheaf over $\mathcal{E}$ whose sections are holomorphic functions $\phi : \mathfrak{h} \times \mathbb{C} \to \mathbb{C}$ obeying the two transformation laws:

(i) $\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k\, \phi(z,\tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$

(ii) $\phi(\tau, z+\lambda) = e^{\pi\, H(\lambda,\lambda)/2 + \pi\, H(z,\lambda)}\, \phi(z,\tau), \quad \forall \lambda \in \Lambda_\tau.$

From this computation, we deduce that the space of global sections

$$J_{k,m} := \Gamma(\mathcal{E}, \mathcal{L}_m \otimes (\Omega^1_{\mathcal{E}/\mathcal{M}_1})^{\otimes k})$$

is essentially the space of Jacobi forms of weight k and index $m/2$, in the sense of Eichler and Zagier ([14]).

Remark 3.2.6. This is not exactly the definition of Jacobi forms of [14]. However, if we correct our Jacobi form ϕ by the simple factor:

$$\phi(z,\tau) \longmapsto \phi'(z,\tau) := e^{\pi m\, z^2/v}\, \phi(z,\tau), \tag{3.3}$$

then the function on the right is a Jacobi form in the sense of [14]. The reason behind this re-normalization is that if we consider the Fourier series

$$\vartheta_{m,0} = \sum_{n\in\mathbb{Z}} e^{2\pi i(mn^2\tau/2 + mz)},$$

obtained by 'plugging in' $q = e^{2\pi i\tau}$ and $x = e^{2\pi iz}$ in our expression (2.3.2) for $\vartheta_{m,0}$, we *do not* get a section of $\mathcal{L}_m$ over E_τ, but rather of the re-normalized line bundle of [14]. Note that this re-normalization is purely transcendental (it depends on $v = \mathrm{im}(\tau)$) and there is no way to make sense of it algebraically. These types of inconsistencies are inevitable when comparing the algebraic and analytic theories of elliptic curves: even when working with the simpler line bundle $\Omega^1_{E_\tau}$, transcendental factors of $2\pi i$ appear when comparing algebraically-defined and analytically-defined differentials.

The algebraic isomorphism of Corollary 2.2.21 carries over to the analytic setting. In particular, by taking global sections of both sides of Corollary 2.2.21 we obtain a canonical isomorphism:

$$\Gamma(\mathcal{M}_1^{\mathrm{an}}, \mathcal{V}_m \otimes \underline{\omega}^{k-1/2}) \simeq \Gamma(\mathcal{M}_1^{\mathrm{an}}, \mathcal{J}_{k,m}).$$

But in this case

$$\mathcal{J}_{k,m} = \pi_*(\mathcal{L}_m \otimes (\Omega^1_{\mathcal{E}/\mathcal{M}_1})^{\otimes k})$$

where

$$\pi : \mathcal{E} \to \mathcal{M}_1^{\mathrm{an}}$$

is the universal elliptic orbifold over $\mathcal{M}_1^{\mathrm{an}}$, thus we have a canonical isomorphism:

$$\Gamma(\mathcal{E}, \mathcal{L}_m \otimes (\Omega^1_{\mathcal{E}/\mathcal{M}_1})^{\otimes k}) = \Gamma(\mathcal{M}_1^{\mathrm{an}}, \mathcal{J}_{k,m}).$$

Combining the two observations, we obtain a canonical isomorphism:

$$\Gamma(\mathcal{M}_1^{\mathrm{an}}, \mathcal{V}_m \otimes \underline{\omega}^{k-1/2}) \simeq J_{k,m}$$

between weight $k-1/2$, $\mathcal{V}_m$-valued modular forms and Jacobi forms of index $m/2$ and weight k. Factoring in the re-normalization of Remark 3.2.6, we have thus given a geometric proof of the following well-known theorem of Eichler and Zagier:

THEOREM 3.2.7 ([14], Theorem 5.1).

Let k be an integer and let m be a positive even integer. Let $h(\tau) = \sum_{\mu\in\mathbb{Z}/m\mathbb{Z}} h_\mu(\tau)$ be a weight $k-1/2$, $\mathcal{V}_m$-valued modular form. For each $\mu \in \mathbb{Z}/m\mathbb{Z}$, let

$$\vartheta_{m,\mu}(\tau, z) := e^{-\pi m\, z^2/v} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\mu \mod m}} e^{2\pi i\,(\tau\, r^2/2m + r\, z)}.$$

Then the function:

$$\phi_h(\tau, z) := \sum_{\mu\in\mathbb{Z}/m\mathbb{Z}} h_\mu(\tau)\, \vartheta_{m,\mu}(\tau, z)$$

is a Jacobi form (in our sense) of weight k and index $m/2$. The map $h(\tau) \to \phi_h(\tau, z)$ induces a canonical isomorphism between weight $k-1/2$, $\mathcal{V}_m$-valued modular forms and Jacobi forms of weight k and index $m/2$.

3.2.5 Theta structures, theta constants and modular forms of half-integral weight

Consider now the analytification $\mathcal{M}^{\mathrm{an}}_{m,2m}$ of the moduli space of level m symmetric theta structures on elliptic curves. According to Mumford ([25]):

$$\mathcal{M}^{\mathrm{an}}_{m,2m} \simeq \Gamma(m, 2m)\backslash\mathfrak{h}$$

as analytic spaces, where:

$$\Gamma(m, 2m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \mod m, \quad b, c \equiv 0 \mod 2m \right\}.$$

Let $\widetilde{\Gamma}(m, 2m)$ be the preimage of $\Gamma(m, 2m)$ under the map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$, so that (notations as in 2.3.3):

$$\widetilde{\mathcal{M}}^{\mathrm{an}}_{m,2m} \simeq \widetilde{\Gamma}(m, 2m)\backslash\backslash\mathfrak{h}.$$

Now by the analytic analog of Theorem 2.3.6 we have a decomposition:

$$p_{m,2m}^*(\mathcal{V}_m^\vee) \simeq \mathcal{L}_{\theta,m}^{\vee,\oplus m}$$

over $p_{m,2m} : \widetilde{\mathcal{M}}_{m,2m}^{\text{an}} \to \mathcal{M}_{1/2}^{\text{an}}$, for some line bundle $\mathcal{L}_{\theta,m}$. Thus $\mathcal{L}_{\theta,m}^\vee$ determines a 1-cocycle $\widetilde{\Gamma}(m,2m) \longrightarrow \mathcal{O}_{\mathfrak{h}}^*$, which we want to determine explicitly. This can be done by specifying an extension of the Legendre symbol $\left(\frac{c}{d}\right)$ as follows ([5], §5). The symbol is multiplicative in both c and d. When d is an odd prime then it is the usual Legendre symbol. If $d = 2$ then it is 1 if $c \equiv \pm 1 \mod 8$ and -1 otherwise. When $d = -1$ it is 1 if $c > 0$ and -1 if $c < 0$. Moreover, $\left(\frac{0}{\pm 1}\right) = \left(\frac{\pm 1}{0}\right) = 1$. We then have ([5], Theorem 5.4):

Proposition 3.2.8. *The line bundle $\mathcal{L}_{\theta,m}^\vee$ over $\widetilde{\mathcal{M}}_{m,2m}^{\text{an}}$ corresponds to the 1-cocycle:*

$$\widetilde{\Gamma}(m,2m) \longrightarrow \mathcal{O}_{\mathfrak{h}}^*$$
$$\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right) \longmapsto \chi_\theta(\gamma)^{\left(\frac{-1}{m}\right)} \left(\frac{d}{2m}\right)$$

where χ_θ is the character:

$$\chi_\theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right) = \begin{cases} \pm\left(\frac{c}{d}\right) & \text{if } d \equiv 1 \mod 4 \\ \pm(-i)\left(\frac{c}{d}\right) & \text{if } d \equiv 3 \mod 4 \end{cases}$$

and $\sqrt{\ }$ is the principal value of the square root, with $-\pi/2 < \arg(\sqrt{\ }) \leq \pi/2$.

Remark 3.2.9. Note that in [5], vector-valued modular forms are $\mathcal{V}_m^\vee$-valued modular forms.

Remark 3.2.10. As in Section 2.3.1, we can given a purely geometric definition of the section

$$\text{ev}_e \in \Gamma(\mathcal{M}_1^{\text{an}}, \mathcal{J}_m^\vee)$$

and by the analytic analog of Theorem 2.2.20, we can give a geometric construction of the $\mathcal{V}_m^\vee$-valued modular form of weight 1/2

$$\theta_{\text{null},m} \in \Gamma(\mathcal{M}_1^{\text{an}}, \mathcal{V}_m^\vee \otimes \underline{\omega}^{1/2}).$$

As functions of τ, the components of this vector must transform according to Proposition 3.2.8. We have thus given a geometric proof of the transformations laws of single-variable, level m theta constants.

For $m = 2$, we can define the line bundle

$$\mathcal{L}_{\text{Shi}} = \mathcal{D}_2^\vee \otimes \underline{\omega}^{1/2},$$

defined as in Section 2.3.4, is a line bundle over the orbifold $\mathcal{M}_0^{\text{an}}(4) = \Gamma_0(4)\backslash\backslash\mathfrak{h}$. By definition, the formation of the line bundle $\mathcal{D}_2^\vee$ is compatible with the decomposition of Proposition 3.2.8. In particular, $\mathcal{D}_2^\vee$ must be given by the cocycle χ_θ. Therefore $\mathcal{L}_{\text{Shi}}$ is given by the 1-cocycle:

$$\Gamma_0(4) \longrightarrow \mathcal{O}_{\mathfrak{h}}^*$$
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \chi_\theta(\gamma)\sqrt{c\tau + d}.$$

For $k \in \mathbb{Z}$, sections of $\mathcal{L}_{\text{Shi}}^{\otimes k}$ over $\mathcal{M}_0^{\text{an}}(4)$ are holomorphic functions $f : \mathfrak{h} \to \mathbb{C}$ such that:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\chi_\theta(\gamma)\sqrt{c\tau + d}\right)^k f(\tau), \tag{3.4}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$. This is precisely the transformation law of Shimura's modular forms of half-integral weight ([31]).

Chapter 4

Conclusions and future directions

In this final chapter we would like to highlight some of the key themes that have emerged in this work, and indicate some of the most natural directions towards which the geometric theory of vector-valued modular forms could be developed.

The first theme, and the most general, is that geometric vector-valued modular forms should be considered as stack-theoretic objects. There seems to be no way around this aspect of the theory. If on the one hand this might seem discouraging, due to the many nuances and technicalities of stack theory, on the other it must be noted that the modular stack $\mathcal{M}_1$ (hence the metaplectic stack $\mathcal{M}_{1/2}$) is a very well-understood object, and many computations on it reduce to computations on elliptic curves over fields and very basic cohomological arguments. Moreover, modular forms of integral weight are also essentially stack-theoretic objects: the coarse moduli space of $\mathcal{M}_1$ is the affine j-line, which has no nontrivial line bundles. The existence of modular forms of level one such as E_4 and E_6 is a purely stack-theoretic phenomenon, as is the fact that the Δ function is of weight 12. We seem to be perfectly comfortable working with these aspects of stack theory. Part of the effort in this thesis has been devoted to convince the reader that the metaplectic stack is a perfectly workable environment for the study of modular forms.

The second theme, a bit more specialized, is that when working with theta functions and modular forms of half-integral weight we are better off working with vector-valued modular forms. This philosophy has been pioneered by authors such as Borcherds, Bruinier and Ono (e.g. [4], [5], [6], [7]) and has been successfully employed by many others. From our point of view, a closer look at Section 2.3.4 reveals just how cumbersome it is to work with modular forms of half-integral weight alone: their q-expansions are essentially defined ad-hoc to recover $\theta_0(q)$. Natural operators such as the U and V operators (e.g. [28], §3.2), which are not defined in this thesis but whose geometric definition is not hard to guess, do not preserve the space of modular forms of half-integral weight, whereas they have very natural geometric interpretations when extended to vector-valued modular forms. More generally, from a geometric point view it seems very hard to justify the way we cut out the sheaf of modular forms of half-integral weight, whereas the whole sheaf of Schrödinger representations $\mathcal{V}_m$ is a much more natural object to work with.

In the following paragraphs I would like to highlight some future directions that deserve to be explored in future work.

Higher rank lattices. In the analytic case, the theory of vector-valued modular forms has been developed as to include modular forms which transform according to the Weil representation of lattices of any rank and signature, and not just rank 1 lattices like in this thesis. It seems that geometrically the way to proceed is to consider Schrödinger representations over the r-fold product of an elliptic curve and its dual. The relationship with theta functions coming from higher rank lattices could be worked out along the lines of Theorem 2.2.20.

Hecke operators and Shimura lifts. Missing from our geometric theory of vector-valued modular forms is a geometric theory of Hecke operators on them. In the complex-analytic setting, these have been defined by Bruinier and Stein ([7]) by extending the Weil representation to matrices of square determinant. Thus it seems very possible to give a geometric interpretation to their construction in terms of isogenies of elliptic curves together with a quadratic form. These isogenies of course will need to be of square degree, if we want them to act functorially on the quadratic form. Moreover, in connection with the previous paragraph, it would be interesting to flesh out the combinatorial relation between eigenforms of different weights. It seems that this relation must come from comparing the action of Hecke operators on vector-valued modular forms of lattices of different rank, for example of rank 3 and 4. Similar relations exist for modular forms of weight 3/2 and 2, a phenomenon known as Shimura lifting ([31]). Our geometric approach to modular forms of half-integral weight seems perfectly suitable to sudy such relations.

Compactifications. A natural direction to pursue in expanding the results of this thesis is to extend the notion of $\mathcal{V}_m$-valued modular forms to the compactified moduli stack $\overline{\mathcal{M}_1}$ of generalized elliptic curves ([13]). This would lead directly to the construction of finite-rank R-modules of *vector-valued cusp forms*, for any ring R, and possibly to dimension formulas for these spaces. Given our very general approach to the metaplectic stack and Schrödinger representations, and given the extensive literature on Heisenberg groups (esp. [20], V) it does not seem out of reach to study Schrödinger representations and quadratic forms over generalized elliptic curves, and their relation to geometric representations. In particular, the fact that the q-expansions of theta functions are holomorphic at ∞ suggests that the sheaves $\mathcal{J}_m$ and their integrable connection should extend to $\overline{\mathcal{M}_1}$, the one complication being of course that the connection is only defined up to μ_2. The analog of Theorem 2.2.20 should then hold for these 'extended' sheaves over $\overline{\mathcal{M}_1}$.

p-adic vector-valued modular forms The work of Katz on p-adic modular forms of integral weight ([19]) has been a major influence of this thesis. It is thus very natural to expand our geometric theory in a p-adic direction. In fact, perhaps the main advantage of working with modular forms geometrically is that it allows to define spaces of mod p and p-adic modular forms in a very natural way. For example, the theory of mod p, $\mathcal{V}_m$-valued modular forms of weight $k/2$ is already contained in this thesis, at least for $p \nmid m$: all that we have to do is specialize our vector bundles to the moduli stack of elliptic curves over a $\mathbb{F}_p$-scheme. For example, we can prove as in Section 2.3.1 that the q-series $\theta_0(q)$ reduced modulo p *is* a mod p modular form of weight 1/2, in our algebraic sense. But the really interesting aspect of the mod p theory is to study the action of Frobenius on our spaces of modular forms. In particular, over the ordinary locus of the modular stack there should be canonical lifts of Frobenius acting on our spaces of Schrödinger representations, which would bring a p-adic theory of vector-valued modular forms to life.

Serre duality and harmonic weak Maass forms According to Serre duality, over $\overline{\mathcal{M}}_1$ there should be an isomorphism between spaces of $\mathcal{V}_m$-valued (weakly holomorphic) modular forms of weight k and $\mathcal{V}_m^\vee$-valued cusp forms of weight $2 - k$. In the analytic setting this observation was, for example, the starting point for Borcherds' generalization of the Gross-Kohnen-Zagier formula ([4]). Serre duality also explains the existence of $\mathcal{V}_m$-valued harmonic weak Maass forms (e.g. [6]). In particular, given a weakly holomorphic $\mathcal{V}_m$-valued form f of weight k, we would like to find a $\mathcal{V}_m^\vee$-valued cusp form of weight $2-k$ in some way canonically associated to f by Serre duality. The problem usually is that there are *no* holomorphic sections of $\mathcal{V}_m^\vee \otimes \underline{\omega}^{-k} \otimes \Omega^1_{\overline{\mathcal{M}}_1}$. The theory of harmonic weak Maass forms essentially picks such canonical 'duals' by looking at *anti-holomorphic* sections of $\mathcal{V}_m^\vee \otimes \underline{\omega}^{-k} \otimes \Omega^1_{\overline{\mathcal{M}}_1}$. The results of this process can be surprising. For example, if g is a cusp form of weight 3/2, then we can associate to (what is essentially) $\overline{g}$ a 'dual' f which is a weakly holomorphic modular form of weight 1/2. If g is the Shimura lift of a cusp form of weight 2, then Bruinier and Ono ([6]) show that the coefficients of f encode information about derivatives of L-series of quadratic twists of f. Now as I have shown in [9], the construction of harmonic weak Maass forms, hence the production of such canonical 'duals' of modular forms, can be given a geometric interpretation in the integral weight case. This geometric interpretation leads to p-adic analogs of (scalar-valued, integral weight) harmonic weak Mass forms, as in my M.Sc. thesis ([8]).

With the geometric interpretation of vector-valued modular forms and modular forms of half-integral weight presented in this thesis, we could attempt to construct harmonic weak Maass forms geometrically and p-adically, mimicking [9] and [8]. Among other things, this construction might explain the relation between the above-mentioned results of Bruinier-Ono and the p-adic analogs of Darmon-Tornaria ([10]).

Bibliography

[1] L. Auslander and R. Tolimieri. Is computing with the finite Fourier transform pure or applied mathematics? *Bull. Amer. Math. Soc. (N.S.)*, 1(6):847–897, 1979.

[2] F. R. Beyl. The Schur multiplicator of $\mathrm{SL}(2, \mathbf{Z}/m\mathbf{Z})$ and the congruence subgroup property. *Math. Z.*, 191(1):23–42, 1986.

[3] R. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998.

[4] R. Borcherds. The Gross-Kohnen-Zagier theorem in higher dimensions. *Duke Math. J.*, 97:219–233, 1999.

[5] R. Borcherds. Reflection groups of Lorentzian lattices. *Duke. Math. J.*, 104:319–366, 2000.

[6] J. Bruinier and K. Ono. Heegner divisors, L-functions and harmonic weak Maass forms. *Ann. Math.*, 172:2135–2181, 2010.

[7] J. Bruinier and O. Stein. The Weil representation and Hecke operators for vector valued modular forms. *Math. Z.*, 264(2):249–270, 2010.

[8] L. Candelori. Towards a p-adic theory of harmonic weak Maass forms. Master's thesis, McGill University, 2010.

[9] L. Candelori. Harmonic weak Maass forms: a geometric approach. *Math. Ann.*, to appear.

[10] H. Darmon and G. Tornaría. Stark-Heegner points and the Shimura correspondence. *Compos. Math.*, 144(5):1155–1175, 2008.

[11] P. Deligne. Formes modulaires et représentations ℓ-adiques. *Séminaire Bourbaki 1968/69*, 355:139–172, 1969.

[12] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[13] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 143–316. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973.

[14] M. Eichler and D. Zagier. *The Theory of Jacobi Forms.* Birkhäuser, 1985.

[15] J. Giraud. *Cohomologie non abélienne.* Springer-Verlag, Berlin, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 179.

[16] A. Grothendieck. Le groupe de Brauer: I. Algèbre d'Azumaya et interprétations diverses. *Séminaire N. Bourbaki*, 290:199–219, 1964-66.

[17] R. Hain. Lectures on moduli spaces of elliptic curves. In S. Yau L. Ji, editor, *Transformation Groups and Moduli Spaces of Curves*, number 16 in Advanced Lectures in Mathematics, pages 95–166. Higher Education Press, Beijing, 2010.

[18] N. J. Hitchin. Flat connections and geometric quantization. *Comm. Math. Phys.*, 131(2):347–380, 1990.

[19] N. Katz. p-adic properties of modular schemes and modular forms. *Modular Functions of One Variable*, 3:69–190, 1973.

[20] L. Moret-Bailly. Pinceaux de varieties abeliennes. *Asterisque*, 129, 1985.

[21] D. Mumford. Picard groups of moduli problems. pages 33–81, 1965.

[22] D. Mumford. On the equations defining abelian varieties I. *Invent. Math.*, 1:287–354, 1966.

[23] D. Mumford. On the equations defining abelian varieties II. *Invent. Math.*, 3:75–135, 1967.

[24] D. Mumford. *Abelian Varieties.* Oxford University Press, 1970.

[25] D. Mumford. The structure of the moduli spaces of curves and Abelian varieties. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 457–465. Gauthier-Villars, Paris, 1971.

[26] D. Mumford. *Tata lectures on theta. III.* Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2007.

[27] A. Nobs. Die irreduziblen Darstellungen der Gruppen $SL_2(Z_p)$, insbesondere $SL_2(Z_2)$. I. *Comment. Math. Helv.*, 51(4):465–489, 1976.

[28] K. Ono. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, volume 102. CBMS regional conference series in mathematics, 2004.

[29] A. Polishchuk. Analogue of weil representation for abelian schemes. *J. reine angew. Math*, pages 1–37.

[30] N. Ramsey. Geometric and p-adic modular forms of half-integral weight. *Ann. Inst. Fourier (Grenoble)*, 56(3):599–624, 2006.

[31] G. Shimura. On Modular forms of half-integral weight. *Ann. Math.*, 97:440–481, 1973.

[32] S. W. Shin. Abelian varieties and Weil representations. *Algebra Number Theory*, 6(8):1719–1772, 2012.

[33] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu, 2013.

[34] J. Stevens. Degenerations of elliptic curves and equations for cusp singularities. *Math. Ann.*, 311(2):199–222, 1998.

[35] W. C. Waterhouse. Principal homogeneous spaces and group scheme extensions. *Trans. Amer. Math. Soc.*, 153:181–189, 1971.

[36] A. Weil. Sur certains groupes d'opérateurs unitaires. *Acta Math.*, 111:145–211, 1964.

[37] G. E. Welters. Polarized abelian varieties and the heat equations. *Compositio Math.*, 49(2):173–194, 1983.