Heegner Points, Stark-Heegner points, and values of $L$-series

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Elliptic Curves

\( E = \text{elliptic curve over a number field } F \)

\( L(E/F, s) = \text{its Hasse-Weil } L\text{-function}. \)

**Birch and Swinnerton-Dyer Conjecture.**

\( \operatorname{ord}_{s=1} L(E/F, s) = \operatorname{rank}(E(F)). \)

**Theorem** (Gross-Zagier, Kolyvagin)

Suppose \( \operatorname{ord}_{s=1} L(E/Q, s) \leq 1 \). Then the Birch and Swinnerton-Dyer conjecture is true.

Key special case: if \( L(E/Q, 1) = 0 \) and \( L'(E/Q, 1) \neq 0 \), then \( E(Q) \) is infinite.

Essential ingredient: Heegner points
Modularity

Write \( L(E/Q, s) = \sum_{n \geq 1} a_n n^{-s} \).

Consider
\[
    f(\tau) = \sum_n a_n e^{2\pi i n \tau}, \quad \text{quad} \tau \in cH.
\]

**Theorem** The function \( f \) is a modular form of weight two on \( \Gamma_0(N) \), where \( N \) is the conductor of \( E \).

*Modular parametrisation* attached to \( E \):

\[
    \Phi : \mathcal{H}/\Gamma_0(N) \longrightarrow E(\mathbb{C}).
\]

\[
    \Phi^*(\omega) = 2\pi i f(\tau) d\tau
\]

\[
    \log_E(\Phi(\tau)) = \int_{i\infty}^{\tau} 2\pi i f(z) dz = \sum_{n=1}^{\infty} \frac{a_n}{n} e^{2\pi i n \tau}.
\]
CM points

$K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field.

**Theorem.** If $\tau$ belongs to $\mathcal{H} \cap K$, then $\Phi(\tau)$ belongs to $E(K^{ab})$.

This theorem produces a systematic and well-behaved collection of algebraic points on $E$ defined over class fields of $K$. 
Heegner points

Let $D$ be a negative discriminant.

**Heegner hypothesis:** $D \equiv s^2 \pmod{N}$.

\[ \mathcal{F}_D^{(N)} = \{ Ax^2 + Bxy + Cy^2 \text{ such that } B^2 - 4AC = D, N|A, B \equiv s \pmod{N} \} \]

Gaussian Composition:

\[ \Gamma_0(N) \backslash \mathcal{F}_D^{(N)} = \text{SL}_2(\mathbb{Z}) \backslash \mathcal{F}_D = G_D \]

is an abelian group under composition, and is identified with the class group of the order of discriminant $D$.

Given $F \in \mathcal{F}_D^{(N)}$, the point

\[ P_F := \Phi(tau), \text{ where } F(\tau, 1) = 0, \]

is called the Heegner point (of discriminant $D$) attached to $F$. 
Heegner points

Class field theory:

\[ \text{rec} : G_D \longrightarrow \text{Gal}(H_D/K), \]

where \( H_D \) is the ring class field attached to \( D \).

Write

\[ \Gamma_0(N)\mathcal{F}_D^{(N)} = \{F_1, \ldots, F_h\}. \]

**Theorem** The Heegner points \( P_{F_j} \) belong to \( E(H_D) \) and

\[ P_{\sigma F} = \text{rec}(\sigma^{-1})P_F. \]

In particular, letting \( D = \text{disc}(K) \),

\[ P_K := P_{F_1} + \cdots + P_{F_h} \]
belongs to $E(K)$.

**Theorem** (Gross-Zagier)

$$L'(E/K, \mathcal{O}_K, 1) = \hat{h}(P_K) \cdot \text{(period)}$$
Kolyvagin’s theorem

**Theorem** (Kolyvagin)

If $P_K$ is of infinite order, then $E(K)$ has rank one and $\text{III}(E/K)$ is finite. (Hence, BSD holds for $E/K$.)

Main ingredient: $P_K$ does not come alone, but is part of a norm-compatible collection of points in $E(K^{ab})$.

**Corollary.** If $\text{ord}_{s=1} L(E, s) \leq 1$, then the Birch and Swinnerton-Dyer conjecture holds for $E$.

**Sketch of Proof.** Choose a quadratic field $K$ satisfying the Heegner hypothesis, for which $\text{ord}_{s=1} L(E/K, s) = 1$.

By Gross-Zagier, $P_K$ is of infinite order.

By Kolyvagin, the BSD conjecture holds for $E/K$.

BSD for $E/Q$ follows.
Totally real fields

**Question:** Does the above scheme generalise to other number fields?

Suppose $E$ is defined over a totally real field $F$.

**Definition:** $E$ is *arithmetically uniformisable* if $[F : \mathbb{Q}]$ is odd or if $N$ is not a square.

If $E$ is modular, and arithmetically uniformisable, there is a *Shimura curve parametrisation*

$$
\Phi : \text{Jac}(X) \longrightarrow E
$$
defined over $F$.

Also, $X$ is equipped with a collection of CM points attached to orders in CM extensions of $F$.

**Theorem** (Zhang, Kolyvagin). Suppose that $E$ is modular and arithmetically uniformisable. If $\text{ord}_{s=1} L(E/F, s) \leq 1$, then BSD holds for $E/F$. 
Non arithmetically uniformisable curves

**Theorem** (Longo, Tian). Suppose that $E$ is modular. If $\text{ord}_{s=1} L(E/F, s) = 0$, then BSD holds for $E/F$.

**Sketch of proof:** Let $f$ be the modular form on $\text{GL}_2(F)$ attached to $E$. One can produce modular forms that are congruent to $f$, and correspond to quotients of Shimura curves. For each $n \geq 1$, there is a Shimura curve $X_n$ for which $J_n[p^n]$ has $E[p^n]$ as a constituent.

**Key formula:** Relate Heegner points attached to $K$, on $X_n$, to $L(EK, 1)$ modulo $p^n$.

**Question.** If $E$ is not arithmetically uniformisable, and $\text{ord}_{s=1} L(E/F, s) = 1$, show that $\text{rank}(E(F)) = 1$?

E.g. If $E$ has everywhere good reduction over a real quadratic field.
Stark-Heegner points

**Wish:** There should be generalisations of Heegner points making it possible to

a) prove BSD for elliptic curves in analytic rank $\leq 1$, for more general $E/F$;

b) Construct class fields of $K$;

**Paradox:** Sometimes we can write down precise formulae for points whose existence is not proved.

**General setting:** $E$ defined over a number field $F$;

$K = \text{auxiliary quadratic extension of } F$;

I will present three contexts.
1. $F = \mathbb{Q}$, $K$ = real quadratic field;

2. $F$ = totally real field, $K$ = ATR extension ("Almost Totally Real"). (Logan)

3. $F$ = imaginary quadratic field. (Trifkovic)
Real quadratic fields

Set-up: $E$ has conductor $N = pM$, with $p \nmid M$.

$H_p := C_p - Q_p$ (A $p$-adic analogue of $H$)

$K = \text{real quadratic field, embedded both in } \mathbb{R} \text{ and } C_p$.

Naive motivation for $H_p$: $H \cap K = \emptyset$, but $H_p \cap K$ need not be empty!

Goal: Define a $p$-adic “modular parametrisation”

$$\Phi : H_p^D / \Gamma_0(M) \xrightarrow{?} E(H_D),$$

for positive discriminants $D$. 
Modular symbols

Set $\omega_f := \text{Re}(2\pi i f(z)dz)$.

Fact: There exists a real period $\Omega$ such that

$$I_f\{r \to s\} := \frac{1}{\Omega} \int_r^s \omega_f \text{mod} \mathbb{Z},$$

for all $r, s \in \mathbb{P}_1(\mathbb{Q})$.

Mazur-Swinnerton-Dyer measure:

There is a measure on $\mathbb{Z}_p$ defined by

$$\mu_f(a + p^n\mathbb{Z}_p) = I_f\{a/p^n \to \infty\}.$$
Systems of measures

Let

\[ \Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \text{ such that } M | c \right\}. \]

**Proposition** There exists a unique collection of measures \( \mu \{ r \to s \} \) on \( \mathbb{P}_1(\mathbb{Q}_p) \) satisfying

1. \( \mu \{ r \to s \} |_{\mathbb{Z}_p} = \mu_f. \)

2. \( \gamma^* \mu \{ \gamma r \to \gamma s \} = \mu \{ r \to s \}, \) for all \( \gamma \in \Gamma. \)

3. \( \mu \{ r \to s \} + \mu \{ s \to t \} = \mu \{ r \to t \}. \)
Rigid analytic functions

\[ f\{r \to s\}(z) := \int_{\mathbb{Q}_p} d\mu_{r\to s} z - t. \]

Properties:

1. \( f\{\gamma r \to \gamma s\}(\gamma z) = (cz + d)^2 f\{r \to s\}(z), \) for all \( \gamma \in \Gamma. \)

2. \( f\{r \to s\} + f\{s \to t\} = f\{r \to t\}. \)
Stark’s conjecture

$K = \text{number field}.$

$v_1, v_2, \ldots, v_n = \text{Archimedean place of } K.$

Assume: $v_2, \ldots, v_n$ real.

$s(x) = \text{sign}(v_2(x)) \cdots \text{sign}(v_n(x)).$

$\zeta(K, \mathcal{A}, s) = N(\mathcal{A})^s \sum_{x \in \mathcal{A}/(\mathcal{O}_K^+)^\times} s(x)N(x)^{-s}.$

$H = \text{Narrow Hilbert class field of } K.$

$\tilde{v}_1 : H \to \mathbb{C}$ extending $v_1 : K \to \mathbb{C}.$

**Conjecture** (Stark) There exists $u(\mathcal{A}) \in \mathcal{O}_H^\times$ such that

$\zeta'(K, \mathcal{A}, 0) \doteq \log |\tilde{v}_1(u(\mathcal{A}))|.$

$u(\mathcal{A})$ is called a **Stark unit** attached to $H/K.$
**Stark Question:** Is there an explicit analytic formula for $\tilde{v}_1(u(A))$, and not just its absolute value?

Some evidence that the answer is “Yes”: Sczech-Ren. (Also, ongoing work of Charollois-D.)

If $\tilde{v}_1$ is real,

$$\tilde{v}_1(u(A)) \equiv \pm \exp(\zeta'(K, A, 0)).$$

If $\tilde{v}_1$ is complex, it is harder to recover $\tilde{v}_1(u(A))$ from its absolute value.

$$\log(\tilde{v}_1(u(A))) = \log |\tilde{v}_1(u(A))| + i\theta(A) \in \mathbb{C}/2\pi i\mathbb{Z}.$$

Applications to *Hilbert’s Twelfth problem* $\Rightarrow$ *Explicit class field theory for $K$.*

The **Stark Question** has an analogue for elliptic curves.
Elliptic Curves

\( E \) = elliptic curve over \( K \)

\( L(E/K, s) = \) its Hasse-Weil \( L \)-function.

**Birch and Swinnerton-Dyer Conjecture.** If \( L(E/K, 1) = 0 \), then there exists \( P \in E(K) \) such that

\[ L'(E/K, 1) = \hat{h}(P) \cdot (\text{explicit period}). \]

**Stark-Heegner Question:** Fix \( v : K \to \mathbb{C} \).

\( \Omega = \) Period lattice attached to \( v(E) \).

Is there an *explicit analytic formula* for \( P \), or rather, for

\[ \log_E(v(P)) \in \mathbb{C}/\Omega? \]

A point \( P \) for which such an explicit analytic recipe exists is called a **Stark-Heegner point**.
The prototype: Heegner Points

*Modular parametrisation* attached to $E$:

$$
\Phi : \mathcal{H}/\Gamma_0(N) \longrightarrow E(\mathbb{C}).
$$

$K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field.

$$
\log_E(\Phi(\tau)) = \int_{i\infty}^{\tau} 2\pi i f(z)dz = \sum_{n=1}^{\infty} \frac{a_n}{n} e^{2\pi i n \tau}.
$$

**Theorem.** If $\tau$ belongs to $\mathcal{H} \cap K$, then $\Phi(\tau)$ belongs to $E(K^{ab})$.

This theorem produces a *systematic* and *well-behaved* collection of algebraic points on $E$ defined over class fields of $K$. 
Heegner points

Given $\tau \in \mathcal{H} \cap K$, let

$$F_\tau(x, y) = Ax^2 + Bxy + Cy^2$$

be the primitive binary quadratic form with

$$F_\tau(\tau, 1) = 0, \quad N|A.$$ Define $\text{Disc}(\tau) := \text{Disc}(F_\tau)$.

$$\mathcal{H}^D := \{ \tau \text{ s.t. } \text{Disc}(\tau) = D. \}.$$ 

$H_D = \text{ring class field of } K \text{ attached to } D.$

**Theorem 1.** If $\tau$ belongs to $\mathcal{H}^D$, then $P_D := \Phi(\tau)$ belongs to $E(H_D)$.

2. (Gross–Zagier)

$$L'(E/K, \mathcal{O}_K, 1) = \hat{h}(P_D) \cdot (\text{period})$$
The Stark-Heegner conjecture

**General setting:** $E$ defined over $F$;

$K$ = auxiliary quadratic extension of $F$;

The Stark-Heegner points belong (*conjecturally*) to ring class fields of $K$.

So far, three contexts have been explored:

1. $F = $ totally real field, $K = $ ATR extension ("Almost Totally Real").

2. $F = \mathbb{Q}$, $K = $ real quadratic field

3. $F = $ imaginary quadratic field.

(Trifkovic, Balasubramaniam, in progress).
ATR extensions

$E$ of conductor 1 over a totally real field $F$,

$\omega_E =$ associated Hilbert modular form on $(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)/\text{SL}_2(\mathcal{O}_F)$.

$K =$ quadratic ATR extension of $F$; (“Almost Totally Real”): $v_1$ complex, $v_2, \ldots, v_n$ real.

D-Logan: A “modular parametrisation”

$$\Phi : \mathcal{H}/\text{SL}_2(\mathcal{O}_F) \rightarrow E(\mathbb{C})$$

is constructed, and $\Phi(\mathcal{H} \cap K) \subset E(K^{\text{ab}})$.

$\Phi$ defined analytically from periods of $\omega_E$.

- Experimental evidence (Logan);

- Replacing $\omega_E$ with a weight two Eisenstein series yields a conjectural affirmative answer to the Stark Question for $K$ (work in progress with Charollois).
Real quadratic fields

Set-up: \( E \) has conductor \( N = pM \), with \( p \nmid M \).

\( \mathcal{H}_p := \mathbb{C}_p - \mathbb{Q}_p \) \((A p\text{-adic analogue of } \mathcal{H})\)

\( K \) = real quadratic field, embedded both in \( \mathbb{R} \) and \( \mathbb{C}_p \).

Motivation for \( \mathcal{H}_p \): \( \mathcal{H} \cap K = \emptyset \), but \( \mathcal{H}_p \cap K \) need not be empty!

Goal: Define a \( p\text{-adic “modular parametrisation”} \)

\[ \Phi : \mathcal{H}_p^D / \Gamma_0(M) \to E(H_D), \]

for \textit{positive} discriminants \( D \).

In defining \( \Phi \), I follow an approach suggested by \textit{Dasgupta’s thesis}. 
**Hida Theory**

$U = p$-adic disc in $\mathbb{Q}_p$ with $2 \in U$;

$\mathcal{A}(U) =$ ring of $p$-adic analytic functions on $U$.

**Hida.** There exists a unique $q$-expansion

$$f_\infty = \sum_{n=1}^{\infty} a_n q^n, \quad a_n \in \mathcal{A}(U),$$

such that $\forall k \geq 2, k \in \mathbb{Z}, k \equiv 2 \pmod{p-1}$,

$$f_k := \sum_{n=1}^{\infty} a_n(k) q^n$$

is an eigenform of weight $k$ on $\Gamma_0(N)$, and

$$f_2 = f_E.$$

For $k > 2$, $f_k$ arises from a newform of level $M$, which we denote by $f_k^\dagger$. 
Heegner points for real quadratic fields

**Definition.** If $\tau \in \mathcal{H}_p/\Gamma_0(M)$, let $\gamma_\tau \in \Gamma_0(M)$ be a generator for $\text{Stab}_{\Gamma_0(M)}(\tau)$.

Choose $r \in \mathbb{P}_1(\mathbb{Q})$, and consider the “Shimura period” attached to $\tau$ and $f_k^\dagger$:

$$J_\tau^\dagger(k) := \Omega_E^{-1} \int_r^{\gamma\tau r} (z - \tau)^{k-2} f_k^\dagger(z) dz.$$ 

This does not depend on $r$.

**Proposition.** There exist $\lambda_k \in \mathbb{C}^\times$ such that $\lambda_2 = 1$ and

$$J_\tau(k) := \lambda_k^{-1} (a_p(k)^2 - 1) J_\tau^\dagger(k)$$

takes values in $\bar{\mathbb{Q}} \subset \mathbb{C}_p$ and extends to a $p$-adic analytic function of $k \in U$. 

23
The definition of \( \Phi \)

Note: \( J_\tau(2) = 0 \). We define:

\[
\log_E \Phi(\tau) := \frac{d}{dk} J_\tau(k)|_{k=2}.
\]

There are more precise formulae giving \( \Phi(\tau) \) itself, and not just its formal group logarithm.

**Conjecture** 1. If \( \tau \) belongs to \( \mathcal{H}_D^p \), then \( P_D := \Phi(\tau) \) belongs to \( E(H_D) \).

2. (“Gross-Zagier”)

\[
L'(E/K, \mathcal{O}_K, 1) = \hat{h}(P_D) \cdot \text{(period)}
\]
Computational Issues

The definition of $\Phi$ is well-suited to numerical calculations. (Green (2000), Pollack (2004)).

Magma package $\texttt{shp}$: software for calculating Stark-Heegner points on elliptic curves of prime conductor.


The key new idea in this efficient algorithm is the theory of overconvergent modular symbols developed by Stevens and Pollack.
Numerical examples

\[ E = X_0(11) : y^2 + y = x^3 - x^2 - 10x - 20. \]

> HP, P, hD := stark_heegner_points(E, 8, Qp);

The discriminant \( D = 8 \) has class number 1

Computing point attached to quadratic form \((1,2,-1)\)

Stark-Heegner point (over \( \mathbb{C}p \)) =

\[ (-2088624084707821, 1566468063530870w + 2088624084707825) + O(11^{15}) \]

This point is close to \([9/2, 1/8(7s - 4), 1]\)

\((9/2 : 1/8(7s - 4) : 1)\) is a global point on \( E(K) \).
A second example

\[ E = 37A : y^2 + y = x^3 - x, \quad D = 1297. \]

The discriminant \( D = 1297 \) has class number 11.

1. Computing point for quadratic form \((1,35,-18)\)
2. Computing point for quadratic form \((-4,33,13)\)
3. Computing point for quadratic form \((16,9,-19)\)
4. Computing point for quadratic form \((-6,25,28)\)
5. Computing point for quadratic form \((-8,23,24)\)
6. Computing point for quadratic form \((2,35,-9)\)
7. Computing point for quadratic form \((9,35,-2)\)
8. Computing point for quadratic form \((12,31,-7)\)
9. Computing point for quadratic form \((-3,31,28)\)
10. Computing point for quadratic form \((12,25,-14)\)
11. Computing point for quadratic form \((14,17,-18)\)

Sum of the Stark-Heegner points (over \( \mathbb{Q}_p \)) = 
\((0 : -1 : 1)) + (37^{100})\)

This \( p \)-adic point is close to \([0,-1,1]\)

\((0 : -1 : 1)\) is indeed a global point on \( E(K) \).
Polynomial $h_D$ satisfied by the $x$-coordinates:

$$961x^{11} - 4035x^{10} - 3868x^9 + 19376x^8 + 13229x^7$$
$$- 27966x^6 - 21675x^5 + 11403x^4 + 11859x^3$$
$$+ 1391x^2 - 369x - 37$$

> G := GaloisGroup(hD);

Permutation group $G$ acting on a set of cardinality 11
(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)

> #G;

22
A theoretical result

\[ \chi : G_D := \text{Gal}(H_D/K) \rightarrow \pm 1 \]

\[ \zeta(K, \chi, s) = L(s, \chi_1)L(s, \chi_2). \]

\[ P(\chi) := \sum_{\sigma \in G_D} \chi(\sigma)\Phi(\tau^\sigma), \quad \tau \in \mathcal{H}_p^D. \]

\[ H(\chi) := \text{extension of } K \text{ cut out by } \chi. \]

**Theorem** (Bertolini, D).

If \( a_p(E)\chi_1(p) = -\text{sign}(L(E, \chi_1, s)) \), then

1. \( \log_E P(\chi) = \log_E \tilde{P}(\chi), \) with \( \tilde{P}(\chi) \in E(H(\chi)). \)

2. The point \( \tilde{P}(\chi) \) is of infinite order, if and only if \( L'(E/K, \chi, 1) \neq 0. \)

The proof rests on an idea of Kronecker ("Kronecker’s solution of Pell’s equation in terms of the Dedekind eta-function").
Kronecker’s Solution of Pell’s Equation

\( D = \text{negative} \) discriminant.

Replace \( \mathcal{H}_p^D / \Gamma_0(N) \) by \( \mathcal{H}^D / \text{SL}_2(\mathbb{Z}) \).

Replace \( \Phi \) by

\[ \eta^*(\tau) := |D|^{-1/4} \sqrt{\text{Im}(\tau)} |\eta(\tau)|^2. \]

\( \chi = \) genus character of \( \mathbb{Q}(\sqrt{D}) \), associated to \( D = D_1 D_2, \quad D_1 > 0, \quad D_2 < 0. \)

**Theorem** (Kronecker, 1865).

\[ \prod_{\sigma \in G_D} \eta^*(\tau^\sigma) \chi(\sigma) = \epsilon^{2h_1 h_2 / w_2}, \]

where

\( h_j = \) class number of \( \mathbb{Q}(\sqrt{D_j}) \).

\( \epsilon = \) Fundamental unit of \( \mathcal{O}^\times_{D_1} \).
Kronecker’s Proof

Three key ingredients:

1. Kronecker limit formula:
\[ \zeta'(K, \chi, 0) = \sum_{\sigma \in G_D} \chi(\sigma) \log \eta^*(\tau^\sigma). \]

2. Factorisation Formula:
\[ \zeta(K, \chi, s) = L(s, \chi_{D_1})L(s, \chi_{D_2}). \]
In particular
\[ \zeta'(K, \chi, 0) = L'(0, \chi_{D_1})L(0, \chi_{D_2}). \]

3. Dirichlet’s Formula.
\[ L'(0, \chi_{D_1}) = h_1 \log(\epsilon), \quad L(0, \chi_{D_2}) = 2h_2/w_2. \]

Note: Complex multiplication is not used!
The Stark-Heegner setting

Assume $\chi = \text{trivial character}$.

$P_K = \text{“trace” to } K \text{ of } P_D$.

1. A “Kronecker limit formula”

$$\frac{d^2}{dk^2} L_p(f_k/K, k/2) = \frac{1}{4} \log_p (P_K + a_p(E) \bar{P}_K)^2.$$ 

If $a_p(E) = -\text{sign}(L(E/Q, s))$, then

$$\frac{d^2}{dk^2} L_p(f_k/K, k/2) = \log_p (P_K)^2.$$

2. Factorisation formula:

$$L_p(f_k/K, k/2) = L_p(f_k, k/2) L_p(f_k, \chi_D, k/2).$$

$L_p(f_k, k/2) = \text{specialisation to the critical line } s = k/2 \text{ of } L_p(f_k, k, s)$ (Mazur's two-variable $p$-adic $L$-function.)
An analogue of Dirichlet’s Formula

Suppose $a_p = -\text{sign}(L(E/Q, s)) = 1$.

**Theorem over $Q$ (Bertolini, D)**

The function $L_p(f_k, k/2)$ vanishes to order $\geq 2$ at $k = 2$, and there exists $P_Q \in E(Q) \otimes Q$ such that

1. $\frac{d^2}{dk^2} L_p(f_k, k/2) = -\log^2(P_Q)$.

2. $P_Q$ is of infinite order iff $L'(E/Q, 1) \neq 0$. 
Proof of theorem over \( \mathbb{Q} \)

Introduce a suitable auxiliary imaginary quadratic field \( K \).

A “Kronecker limit formula”

\[
\frac{d^2}{dk^2} L_p(f_k/K, k/2) = \log_p(P_K)^2,
\]

where \( P_K \) is a Heegner point arising from a Shimura curve parametrisation.

Key Ingredients: Cerednik-Drinfeld Theorem.


End of Proof

We now use the factorisation formula

\[ L''_p(f_k/K, k/2) = L''_p(f_k, k/2)L_p(f_k, \chi_D, 1) \]

to conclude.

The structure of the argument

Heegner points + Cerednik-Drinfeld

⇒ Theorem for \( K \) imaginary quadratic

⇒ Theorem for \( \mathbb{Q} \)

⇒ Theorem for \( K \) real quadratic.

This argument seems to shed no light on the rationality of the Stark-Heegner point \( P_D \) (unless the class group has exponent two).