Number Theory and Representation Theory

A conference in honor of the 60th birthday of Benedict Gross

Harvard University, Cambridge

June 2010

Elliptic curves over real quadratic fields, and the Birch and Swinnerton-Dyer conjecture

...

A survey of the mathematical contributions of Dick Gross

which have most influenced and inspired me.

Henri Darmon

McGill University, Montreal

June 3, 2010

The theorem of Gross-Zagier-Kolyvagin

I became Dick's student in 1987, when the following was still new:

Theorem (Gross-Zagier (1985), Kolyvagin (1987))

Let E be a (modular) elliptic curve over $\mathbb{Q}$. If $\operatorname{ord}_{s=1} L(E,s) \leq 1$, then $Ш(E/\mathbb{Q})$ is finite, and

$$\operatorname{rank}(E(\mathbb{Q})) = \operatorname{ord}_{s=1} L(E,s).$$

In 1987, this result was tremendously exciting;

It is still the best theoretical evidence for the BSD conjecture.

Key ingredients in the proof:

1. The Gross-Zagier Theorem;
2. Kolyvagin's descent.

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Modularity

Modularity comes in two flavours:

- (General form) The elliptic curve E is *modular* if

$$L(E,s) = L(f,s),$$

  for some normalised newform $f \in S_2(\Gamma_0(N))$ (with $N =$conductor(E)).

- (Stronger, geometric form): There is a non-constant morphism

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  were $J_0(N)$ is the Jacobian of the modular curve $X_0(N)$.

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Modular curves

Recall: $X_0(N)$ is the modular curve of level N.

- $X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$;
- $X_0(N)(F) =$ the set of pairs (A, C) where
  - A is a (generalised) elliptic curve over F;
  - C is a cyclic subgroup scheme of $A[N]$ over F

  (up to $\bar{F}$-isomorphism.)

Heegner points

K= imaginary quadratic field satisfying the

Heegner hypothesis (HH): There exists an ideal $\mathfrak{N}$ of $\mathcal{O}_K$ of norm N, with $\mathcal{O}_K/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$.

Definition

The Heegner points on $X_0(N)$ of level c attached to K are the points given by pairs $(A, A[\mathfrak{N}])$ with $\mathrm{End}(A) = \mathbb{Z} + c\mathcal{O}_K$.

They are defined over the ring class field of K of conductor c.

$$P_K := \pi_E((A_1, A_1[\mathfrak{N}]) + \cdots + (A_h, A_h[\mathfrak{N}]) - h(\infty)) \in E(K).$$

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The Gross-Zagier Theorem

The Gross-Zagier theorem in its most basic form:

Theorem (Gross-Zagier)

For all K satisfying (HH), the L-series $L(E/K, s)$ vanishes to odd order *at $s = 1$, and*

$$L'(E/K, 1) = \langle P_K, P_K \rangle \langle f, f \rangle \pmod{\mathbb{Q}^\times}.$$

In particular, P_K is of infinite order iff $L'(E/K, 1) \neq 0$.

Kolyvagin's Theorem

Theorem (Kolyvagin)

If P_K is of infinite order, then $\mathrm{rank}(E(K)) = 1$, *and* $Ш(E/K) < \infty$.

- The Heegner point P_K is part of a norm-coherent system of algebraic points on E;
- This collection of points satisfies the axioms of an *Euler system* (a *Kolyvagin system* in the sense of Mazur-Rubin) which can be used to bound the p-Selmer group of E/K.

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Proof of the GZK Theorem

Theorem (Gross-Zagier, Kolyvagin)

If $\mathrm{ord}_{s=1} L(E,s) \leq 1$, *then* $Ш(E/\mathbb{Q})$ *is finite and*

$$\mathrm{rank}(E(\mathbb{Q})) = \mathrm{ord}_{s=1} L(E,s).$$

Proof.

1. Bump-Friedberg-Hoffstein, Murty-Murty $\Rightarrow$ there exists a K satisfying (HH), with $\mathrm{ord}_{s=1} L(E/K,s) = 1$.
2. Gross-Zagier $\Rightarrow$ the Heegner point P_K is of infinite order.
3. Koyvagin $\Rightarrow$ $E(K) \otimes \mathbb{Q} = \mathbb{Q} \cdot P_K$, and $Ш(E/K) < \infty$.
4. Explicit calculation $\Rightarrow$

$$\text{the point } P_K \text{ belongs to } \begin{cases} E(\mathbb{Q}) & \text{if } L(E,1) = 0, \\ E(K)^- & \text{if } L(E,1) \neq 0. \end{cases}$$

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Gross's advice

In 1988, Dick gave me the following advice:

1. Ask Massimo Bertolini to explain Kolyvagin's ideas;
2. Extend Kolyvagin's theorem to ring class characters.

Theorem (Bertolini, D (1989))

Let H be the ring class field of K of conductor c, let $P \in E(H)$ be a Heegner point of conductor c, and let

$$P_\chi := \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi^{-1}(\sigma) P^\sigma \in (E(H) \otimes \mathbb{C})^\chi$$

be its "χ-component". If $P_\chi \neq 0$, then $(E(H) \otimes \mathbb{C})^\chi$ is a one-dimensional complex vector space.

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The proof is an easy extension of Kolyvagin's result. When combined with (less easy) results of Zhang generalising Gross-Zagier to ring class characters, it gives:

Theorem (GZK for characters)

If $L'(E/K, \chi, 1) \neq 0$, then $(E(H) \otimes \mathbb{C})^{\chi}$ is a one-dimensional complex vector space.

Question

What if the imaginary quadratic field K is replaced by a real quadratic field?

The question is still open!

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Are there "Heegner points attached to real quadratic fields"?

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Quadratic cycles

Let $\Psi : K \hookrightarrow M_2(\mathbb{Q})$ be an embedding of a quadratic algebra.

1. If K is imaginary, $\tau_\Psi :=$ fixed point of $\Psi(K^\times) \circlearrowleft \mathcal{H}$;
   $\Delta_\Psi := \{\tau_\Psi\}$.
2. If K is real, $\tau_\Psi, \tau'_\Psi :=$ fixed points of $\Psi(K^\times) \circlearrowleft (\mathcal{H} \cup \mathbb{R})$;
   $\Upsilon_\Psi = \text{geodesic}(\tau_\Psi \to \tau'_\Psi)$.

$$\Delta_\Psi = \Upsilon_\Psi / \langle \Psi(\mathcal{O}_K^\times) \rangle \subset Y(\mathbb{C}).$$

These "real quadratic cycles" have been extensively studied (Shintani, Zagier, Gross-Kohnen-Zagier, Waldspurger, Alex Popa) and related to special values of L-series.

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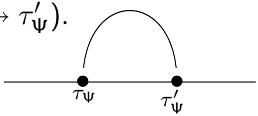

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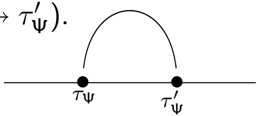

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Another statement of the question

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What objects play the role of real quadratic cycles, when K is real quadratic and the sign in $L(E/K, s)$ is -1?

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1. A thesis, containing a few (not so exciting) theorems;
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Zhang's theorems for totally real fields

The mathematical objects exploited by Gross-Zagier and Kolyvagin continue to be available when $\mathbb{Q}$ is replaced by a *totally real field F* of degree $n > 1$.

Definition

An elliptic curve E/F is *modular* if there is a Hilbert modular form $G \in S_2(N)$ over F such that

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Modularity is often known, and will be assumed from now on.

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Geometric modularity

Geometrically, the Hilbert modular form G corresponds to a (2^n-dimensional) subspace

$$\Omega_G \subset \Omega^n_{\mathrm{har}}(V(\mathbb{C}))^G,$$

where V is a suitable *Hilbert modular variety* of dimension n.

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The elliptic curve E/F is said to satisfy the *Jacquet-Langlands hypothesis* (JL) if either $[F:\mathbb{Q}]$ is odd, or $\mathrm{ord}_p(N)$ is odd for some prime $p|N$ of F.

Theorem (Geometric modularity)

Suppose that E/F is modular and satisfies (JL). There there exists a Shimura curve *X/F and a non-constant morphism*

$$\pi_E : \mathrm{Jac}(X) \longrightarrow E.$$

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Shimura curves, like modular curves, are equipped with a plentiful supply of CM points.

Theorem (Zhang, 2001)

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Zhang, Shouwu. *Heights of Heegner points on Shimura curves.* Ann. of Math. (2) **153** (2001).

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Let E/F be a modular elliptic curve satisfying hypothesis (JL). If $\mathrm{ord}_{s=1} L(E/F, s) \leq 1$, then $Ш(E/F)$ is finite and

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Congruences between modular forms $\Rightarrow$ the Galois representation $E[p^n]$ occurs in $J_n[p^n]$, where $J_n = \mathrm{Jac}(X_n)$ and X_n is a Shimura curve X_n whose level may (and does) depend on n.
Use CM points on X_n to bound the p^n-Selmer group of E. □

Challenge: When $\mathrm{ord}_{s=1} L(E/F, s) = 1$ but (JL) is not satisfied, produce the point in $E(F)$ whose existence is predicted by BSD.

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Elliptic curves with everywhere good reduction

Simplest case where (JL) fails to hold:

$$F = \mathbb{Q}(\sqrt{N}), \text{ a real quadratic field,}$$

$$E/F \text{ has everywhere good reduction.}$$

Fact: $E(F)$ has even analytic rank and hence Longo's theorem applies.

Consider the twist E_K of E by a quadratic extension K/F.

Proposition

1. *If K is totally real or CM, then E_K has even analytic rank.*
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The Conjecture on ATR twists

Conjecture (on ATR twists)

Let E_K be an ATR twist of an elliptic curve E of conductor 1 over F. If $L'(E_K/F, 1) \neq 0$, then $E_K(F)$ has rank one and $Ш(E_K/F) < \infty$.

This is a very special case of the BSD conjecture.

It appears close to existing results, but presents genuine new difficulties.

ATR cycles

Problem: Produce a point $P_K \in E_K(F)$, when (JL) fails and hence no Shimura curve is available.

Let Y be the (open) Hilbert modular surface attached to E/F:

$$Y(\mathbb{C}) = \mathbf{SL}_2(\mathcal{O}_F)\backslash(\mathcal{H}_1 \times \mathcal{H}_2).$$

There are $h := \# \operatorname{Pic}^+(\mathcal{O}_K)/\operatorname{Pic}^+(\mathcal{O}_F)$ distinct $\mathcal{O}_F$-algebra embeddings

$$\Psi_1, \ldots, \Psi_h : \mathcal{O}_K \longrightarrow M_2(\mathcal{O}_F).$$

To each $\Psi = \Psi_j$, one can attach a cycle $\Delta_\Psi \subset Y(\mathbb{C})$ of real dimension one which is analogous to a real quadratic cycle, but "behaves like a Heegner point".

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$\tau_\Psi^{(1)} :=$ fixed point of $\Psi(K^\times) \circlearrowleft \mathcal{H}_1$;

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$\Upsilon_\Psi = \{\tau_\Psi^{(1)}\} \times \mathrm{geodesic}(\tau_\Psi^{(2)} \to \tau_\Psi^{(2)\prime})$.

$\Delta_\Psi = \Upsilon_\Psi / \langle \Psi(\mathcal{O}_K^\times) \rangle \subset Y(\mathbb{C})$.

Key fact: The cycles Δ_Ψ are *null-homologous*.

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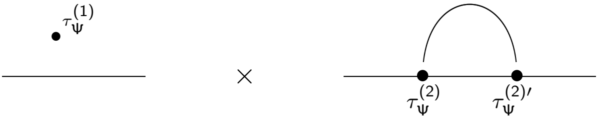

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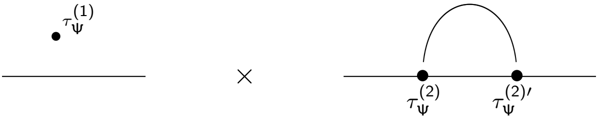

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Points attached to ATR cycles

For any 2-form $\omega_G \in \Omega_G$,

$$P_\Psi^?(G) := \int_{\partial^{-1}\Delta_\Psi} \omega_G \quad \in \quad \mathbb{C}/\Lambda_G.$$

Conjecture (Oda (1982))

For a suitable choice of ω_G, we have $\mathbb{C}/\Lambda_G \sim E(\mathbb{C})$. In particular $P_\Psi^?(G)$ can then be viewed as a point in $E(\mathbb{C})$.

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The points $P_\Psi^?(G)$ belongs to $E(H) \otimes \mathbb{Q}$, where H is the ring class field of K of conductor 1. The points $P_{\Psi_1}^?(G), \ldots, P_{\Psi_h}^?(G)$ are conjugate to each other under $\mathrm{Gal}(H/K)$. Finally, the point $P_K^?(G) := P_{\Psi_1}^?(G) + \cdots + P_{\Psi_h}^?(G)$ is of infinite order iff $L'(E/K, 1) \neq 0$.

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ATR points are defined over abelian extensions of a quadratic ATR extension K of a real quadratic field F.

This setting is "overly complicated", and does not capture the more natural setting of Heegner points over ring class fields of real quadratic fields.

Simplest case: $E/\mathbb{Q}$ is an elliptic curve of prime conductor p, and K is a real quadratic field in which p is inert.

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F real quadratic	$\mathbb{Q}$
$\infty_0,\ \infty_1$	$p,\ \infty$
E/F of conductor 1	$E/\mathbb{Q}$ of conductor p
$\mathbf{SL}_2(\mathcal{O}_F)\backslash(\mathcal{H}\times\mathcal{H})$	$\mathbf{SL}_2(\mathbb{Z}[1/p])\backslash(\mathcal{H}_p\times\mathcal{H})$
K/F ATR	$K/\mathbb{Q}$ real quadratic, with p inert
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One can develop the notions in the right-hand column to the extent of

1. Attaching to $f \in S_2(\Gamma_0(p))$ a "Hilbert modular form" G on $\mathbf{SL}_2(\mathbb{Z}[1/p])\backslash(\mathcal{H}_p \times \mathcal{H})$.
2. Making sense of the expression

$$\int_{\partial^{-1}\Delta_\Psi} \omega_G \quad \in \quad K_p^\times / q^{\mathbb{Z}} = E(K_p)$$

for any "p-adic ATR cycle" Δ_Ψ.

The resulting local points are defined (*conjecturally*) over ring class fields of K. They are called "Stark-Heegner points" ...

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Gross-Stark units are p-adic analogues of Stark-units (in which classical Artin L-functions at $s = 0$ are replaced by the p-adic L-functions attached to totally real fields by Deligne-Ribet.)

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Summary

The Gross-Zagier formula and the p-adic Gross-Stark conjectures are two fundamental contributions of Dick Gross which have been, and continue to be, tremendously influential.

Thank you, Dick,

and

Happy

60th

Birthday!!