BIRS Workshop
Cycles on modular varieties

Diagonal cycles and Euler systems for real quadratic fields

Henri Darmon

A report on joint work with Victor Rotger
(as well as earlier work with Bertolini, Dasgupta, Prasanna...)

October 2011

Summary of Victor Rotger's Lecture

Algebraic cycles in the triple product of modular curves/ Kuga-Sato varieties can be used to construct rational points on elliptic curves ("Zhang points").

These points make it possible to relate:

- Certain extension classes (of mixed motives) arising in the pro-unipotent fundamental groups of modular curves;
- Special values of L-functions of modular forms.

General philosophy (Deligne, Wojtkowiak, ...) relating $\pi_1^{\mathrm{unip}}(X)$ to values of L-functions.

Questions

- Are these points "genuinely new"?
- New cases of the Birch and Swinnerton-Dyer conjecture?
- Relation with Stark-Heegner points?

The fact that "Zhang points" are defined over $\mathbb{Q}$ and controlled by $L'(E/\mathbb{Q}, 1)$ justifies a certain pessimism.

Theme of this talk. Diagonal cycles, *when made to vary in p-adic families*, should yield new applications to the Birch and Swinnerton-Dyer conjecture and to Stark-Heegner points.

Stark-Heegner points: executive summary

Stark-Heegner points arising from $\mathcal{H}_p \times \mathcal{H}$:

• Points in $E(\mathbb{C}_p)$, with $E/\mathbb{Q}$ a (modular) elliptic curve with $p|N_E$.

• They are computed as images of certain real one-dimensional null-homologous cycles on $\Gamma\backslash(\mathcal{H}_p \times \mathcal{H})$, (with $\Gamma \subset \mathbf{SL}_2(\mathbb{Z}[1/p])$) under a kind of Abel-Jacobi map.

• The cycles are indexed by ideals in real quadratic orders.

• The resulting local points on a (modular) elliptic curve $E/\mathbb{Q}$ are *conjecturally* defined over ring class fields of *real* quadratic fields.

Stark-Heegner points and the BSD conjecture

Theorem (Bertolini-Dasgupta-D and Longo-Rotger-Vigni)

Assume the conjectures on Stark-Heegner points attached to the real quadratic field F (in a stronger, more precise form given in Samit Dasgupta's PhD thesis). Then

$$L(E/F, \chi, 1) \neq 0 \implies (E(H) \otimes \mathbb{C})^{\chi} = 0,$$

for all ring class $\chi : \mathrm{Gal}(H/F) \longrightarrow \mathbb{C}^{\times}$.

This result draws a connection between

1. Stark-Heegner points and explicit class field theory for real quadratic fields;
2. certain concrete cases of the BSD conjecture.

BDD-LRV without Stark-Heegner points?

We would like to prove the BDD-LRV result *unconditionally*, without appealing to Stark-Heegner points.

Key Ingredients in our approach:

1. A p-adic Gross-Kudla formula relating certain Garrett Rankin triple product p-adic L-functions to the images of (generalised) diagonal cycles under the p-adic Abel-Jacobi map.

2. A "p-adic deformation" of this formula.

Triples of modular forms

Definition

A triple of eigenforms

$$f \in S_k(\Gamma_0(N_f), \varepsilon_f), \quad g \in S_\ell(\Gamma_0(N_g), \varepsilon_g), \quad h \in S_m(\Gamma_0(N_h), \varepsilon_h)$$

is said to be *self-dual* if

$$\varepsilon_f \varepsilon_g \varepsilon_h = 1.$$

In particular, $k + \ell + m$ is even.
It is said to be *balanced* if each weight is strictly smaller than the sum of the other two.

Generalised Diagonal cycles

Assume for simplicity $N = N_f = N_g = N_h$.

$$k = r_1 + 2, \quad \ell = r_2 + 2, \quad m = r_3 + 2, \quad r = \frac{r_1 + r_2 + r_3}{2}.$$

$\mathcal{E}^r(N) = r$-fold Kuga-Sato variety over $X_1(N)$; $\dim = r + 1$.

$$V = \mathcal{E}^{r_1}(N) \times \mathcal{E}^{r_2}(N) \times \mathcal{E}^{r_3}(N), \quad \dim V = 2r + 3.$$

Victor's lecture: When (k, ℓ, m) is balanced, there is an *essentially unique* interesting way of embedding $\mathcal{E}^r(N)$ as a null-homologous cycle in V. (**Generalised Gross-Kudla Schoen cycle**.)

$$\Delta = \mathcal{E}^r \subset V, \quad \Delta \in CH^{r+2}(V).$$

Diagonal cycles and L-series

The height of the (f, g, h)-isotypic component of the generalised diagonal cycle Δ should be related to the central critical derivative

$$L'(f \otimes g \otimes h, r + 2).$$

Work of Gross-Kudla, vastly extended by Yuan-Zhang-Zhang, represents substantial progress in this direction, when $r_1 = r_2 = r_3 = 0$. (Cf. this afternoon's talks).

Goal of the p-adic Gross-Kudla formula: to describe relationships between Δ and p-adic L-series attached to (f, g, h).

Hida families

$\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]$: Iwasawa algebra.

Weight space: $W = \hom(\Lambda, \mathbb{C}_p) \subset \hom((1 + p\mathbb{Z}_p)^\times, \mathbb{C}_p^\times)$.

The integers form a dense subset of W via $k \leftrightarrow (x \mapsto x^k)$.

Classical weights: $W_{\mathrm{cl}} := \mathbb{Z}^{\geq 2} \subset W$.

If $\tilde{\Lambda}$ is a finite flat extension of Λ, let $\tilde{\mathcal{X}} = \hom(\tilde{\Lambda}, \mathbb{C}_p)$ and let

$$\kappa : \tilde{\mathcal{X}} \longrightarrow W$$

be the natural projection to weight space.

Classical points: $\tilde{\mathcal{X}}_{\mathrm{cl}} := \{x \in \tilde{\mathcal{X}} \text{ such that } \kappa(x) \in W_{\mathrm{cl}}\}$.

Hida families, cont'd

Definition

A *Hida family* of tame level N is a triple $(\Lambda_f, \Omega_f, \underline{f})$, where

1. Λ_f is a finite flat extension of Λ;
2. $\Omega_f \subset \mathcal{X}_f := \hom(\Lambda_f, \mathbb{C}_p)$ is a non-empty open subset (for the p-adic topology);
3. $\underline{f} = \sum_n \mathbf{a}_n q^n \in \Lambda_f[[q]]$ is a formal q-series, such that $\underline{f}(x) := \sum_n x(\mathbf{a}_n) q^n$ is the q series of the *ordinary p-stabilisation* $f_x^{(p)}$ of a normalised eigenform, denoted f_x, of weight $\kappa(x)$ on $\Gamma_1(N)$, for all $x \in \Omega_{f,\mathrm{cl}} := \Omega_f \cap \mathcal{X}_{f,\mathrm{cl}}$.

Hida's theorem

f = normalised eigenform of weight $k \geq 1$ on $\Gamma_1(N)$.

$p \nmid N$ an ordinary prime for f (i.e., $a_p(f)$ is a p-adic unit).

Theorem (Hida)

There exists a Hida family $(\Lambda_f, \Omega_f, \underline{f})$ *and a classical point* $x_0 \in \Omega_{f,\mathrm{cl}}$ *satisfying*

$$\kappa(x_0) = k, \qquad f_{x_0} = f.$$

As x varies over $\Omega_{f,\mathrm{cl}}$, the specialisations f_x give rise to a "p-adically coherent" collection of classical newforms on $\Gamma_1(N)$, and one can hope to construct p-adic L-functions by interpolating classical special values attached to these eigenforms.

A 'Heegner-type" hypothesis

Triple product L-function $L(f \otimes g \otimes h, s)$ has a functional equation

$$\Lambda(f \otimes g \otimes h, s) = \epsilon(f, g, h)\Lambda(f \otimes g \otimes h, k + \ell + m - 2 - s).$$

$$\epsilon(f, g, h) = \pm 1, \qquad \epsilon(f, g, h) = \prod_{q|N\infty} \epsilon_q(f, g, h).$$

Key assumption: $\epsilon_q(f, g, h) = 1$, for all $q|N$.

This assumption is satisfied when, for example:

- $\gcd(N_f, N_g, N_h) = 1$, or,
- $N_f = N_g = N_h = N$ and $a_p(f)a_p(g)a_p(h) = -1$ for all $p|N$.

$\epsilon(f, g, h) = \epsilon_\infty(f, g, h) = -1$, hence $L(f, g, h, c) = 0$.
$(c = \frac{k+\ell+m-2}{2})$

Triple product p-adic Rankin L-functions

They interpolate the *central critical values*

$$\frac{L(\underline{f}_x \otimes \underline{g}_y \otimes \underline{h}_z, c)}{\Omega(f_x, g_y, h_z)} \in \bar{\mathbb{Q}}.$$

Four *distinct* regions of interpolation in $\Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}$:

1. $\Sigma_f = \{(x,y,z) : \kappa(x) \geq \kappa(y) + \kappa(z)\}$. $\Omega = *\langle f_x, f_x \rangle^2$.
2. $\Sigma_g = \{(x,y,z) : \kappa(y) \geq \kappa(x) + \kappa(z)\}$. $\Omega = *\langle g_y, g_y \rangle^2$.
3. $\Sigma_h = \{(x,y,z) : \kappa(z) \geq \kappa(x) + \kappa(y)\}$. $\Omega = *\langle h_z, h_z \rangle^2$.
4. $\Sigma_{\mathrm{bal}} = (\mathbb{Z}^{\geq 2})^3 - \Sigma_f - \Sigma_g - \Sigma_h$.
   $\Omega(f_x, h_y, g_z) = *\langle f_x, f_x \rangle^2 \langle g_y, g_y \rangle^2 \langle h_z, h_z \rangle^2$.

Resulting p-adic L-functions: $L_p^f(\underline{f} \otimes \underline{g} \otimes \underline{h})$, $L_p^g(\underline{f} \otimes \underline{g} \otimes \underline{h})$, and $L_p^h(\underline{f} \otimes \underline{g} \otimes \underline{h})$ respectively.

Garrett's formula

Let (f, g, h) be a triple of eigenforms with unbalanced weights (k, ℓ, m),

$$k = \ell + m + 2n, \qquad n \geq 0.$$

Theorem (Garrett, Harris-Kudla)

The central critical value $L(f, g, h, c)$ is a multiple of

$$\langle f, g\delta_m^n h \rangle^2,$$

where

$$\delta_k = \frac{1}{2\pi i}\left(\frac{d}{d\tau} + \frac{k}{\tau - \bar{\tau}}\right) : S_k(\Gamma_1(N))^! \longrightarrow S_{k+2}(\Gamma_1(N))^!$$

is the Shimura-Maass operator on "nearly holomorphic" modular forms, and

$$\delta_m^n := \delta_{m+2n-2} \cdots \delta_{m+2}\delta_m.$$

The p-adic L-function

Theorem (Hida, Harris-Tilouine)

There exists a (unique) element $\mathscr{L}_p{}^f(\underline{f}, \underline{g}, \underline{h}) \in Frac(\Lambda_f) \otimes \Lambda_g \otimes \Lambda_h$ *such that, for all* $(x, y, z) \in \Sigma_f$, *with* $(k, \ell, m) := (\kappa(x), \kappa(y), \kappa(z))$ *and* $k = \ell + m + 2n$,

$$\mathscr{L}_p{}^f(\underline{f}, \underline{g}, \underline{h})(x, y, z) = \frac{\mathscr{E}(f_x, g_y, h_z)}{\mathscr{E}(f_x)} \frac{\langle f_x, g_y \delta_m^n h_z \rangle}{\langle f_x, f_x \rangle},$$

where, after setting $c = \frac{k+\ell+m-2}{2}$,

$$\begin{aligned}
\mathscr{E}(f_x, g_y, h_z) &:= \left(1 - \beta_{f_x}\alpha_{g_y}\alpha_{h_z}p^{-c}\right) \times \left(1 - \beta_{f_x}\alpha_{g_y}\beta_{h_z}p^{-c}\right) \\
&\quad \times \left(1 - \beta_{f_x}\beta_{g_y}\alpha_{h_z}p^{-c}\right) \times \left(1 - \beta_{f_x}\beta_{g_y}\beta_{h_z}p^{-c}\right), \\
\mathscr{E}(f_x) &:= \left(1 - \beta_{f_x}^2 p^{-k}\right) \times \left(1 - \beta_{f_x}^2 p^{1-k}\right).
\end{aligned}$$

Complex Abel-Jacobi maps

The cycle Δ is null-homologous:

$$\mathrm{cl}(\Delta) = 0 \text{ in } H^{2r+4}(V(\mathbb{C}), \mathbb{Q}).$$

Our formula of "Gross-Kudla-Zhang type" will not involve heights, but rather p-adic analogues of the *complex Abel-Jacobi map* of Griffiths and Weil:

$$\begin{aligned} \mathrm{AJ} : \mathrm{CH}^{r+2}(V)_0 \quad &\longrightarrow \quad \frac{H^{2r+3}_{\mathrm{dR}}(V/\mathbb{C})}{\mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(V/\mathbb{C}) + H^{2r+3}_{B}(V(\mathbb{C}), \mathbb{Z})} \\ &= \frac{\mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(V/\mathbb{C})^{\vee}}{H_{2r+3}(V(\mathbb{C}), \mathbb{Z})}. \end{aligned}$$

$$\mathrm{AJ}(\Delta)(\omega) = \int_{\partial^{-1}\Delta} \omega.$$

p-adic étale Abel-Jacobi maps

$$\begin{array}{ccc}
\mathrm{CH}^{r+2}(V/\mathbb{Q})_0 & \xrightarrow{\mathrm{AJ}_{\mathrm{et}}} & H^1_f(\mathbb{Q}, H^{2r+3}_{\mathrm{et}}(\bar{V}, \mathbb{Q}_p)(r+2)) \\
\downarrow & & \downarrow \\
\mathrm{CH}^{r+2}(V/\mathbb{Q}_p)_0 & \xrightarrow[\mathrm{AJ}_{\mathrm{et}}]{} & H^1_f(\mathbb{Q}_p, H^{2r+3}_{\mathrm{et}}(\bar{V}, \mathbb{Q}_p)(r+2)) \\
 & & \| \\
 & & \mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(V/\mathbb{Q}_p)^{\vee}
\end{array}$$

The dotted arrow is called the *p-adic Abel-Jacobi map* and denoted AJ_p.

p-adic Gross-Kudla: Relate $\mathrm{AJ}_p(\Delta)$ to *certain* Rankin triple product p-adic L-functions, à la Gross-Kudla-Zhang.

More notations

$$\omega_f = (2\pi i)^{r_1+1} f(\tau) dw_1 \cdots dw_{r_1} d\tau \in \mathrm{Fil}^{r_1+1} H_{\mathrm{dR}}^{r_1+1}(\mathcal{E}^{r_1}).$$

$\eta_f \in H_{\mathrm{dR}}^{r_1+1}(\mathcal{E}^{r_1}/\bar{\mathbb{Q}}_p)$ = representative of the f-isotypic part on which Frobenius acts as a p-adic unit, normalised so that

$$\langle \omega_f, \eta_f \rangle = 1.$$

Lemma

If (k, ℓ, m) is balanced, then the (f_k, g_ℓ, h_m)-isotypic part of the $\bar{\mathbb{Q}}_p$ vector space $\mathrm{Fil}^{r+2} H_{\mathrm{dR}}^{2r+2}(V/\bar{\mathbb{Q}}_p)$ is generated by the classes of

$$\omega_{f_k} \otimes \omega_{g_\ell} \otimes \omega_{h_m}, \quad \eta_{f_k} \otimes \omega_{g_\ell} \otimes \omega_{h_m}, \quad \omega_{f_k} \otimes \eta_{g_\ell} \otimes \omega_{h_m}, \quad \omega_{f_k} \otimes \omega_{g_\ell} \otimes \eta_{h_m}.$$

The p-adic Gross-Kudla formula

Given $(x_0, y_0, z_0) \in \Sigma_{\mathrm{bal}}$, write $(f, g, h) = (f_{x_0}, g_{y_0}, h_{z_0})$, and $(k, \ell, m) = (\kappa(x_0), \kappa(y_0), \kappa(z_0))$.

Recall that $\mathrm{sign}(L(f \otimes g \otimes h, s)) = -1$, hence $L(f \otimes g \otimes h, c) = 0$.

Theorem (Rotger-D)

$$\mathscr{L}_p{}^f(\underline{f} \otimes \underline{g} \otimes \underline{h}, x_0, y_0, z_0) = \frac{\mathscr{E}(f)}{\mathscr{E}(f, g, h)} \times \mathrm{AJ}_p(\Delta_{k,\ell,m})(\eta_f \otimes \omega_g \otimes \omega_h),$$

and likewise for $\mathscr{L}_p{}^g$ *and* $\mathscr{L}_p{}^h$.

What next?

Consequences of p-adic Gross-Kudla:

• The Abel-Jacobi images of diagonal cycles encode the special values of the *three distinct* p-adic L-functions attached to $(\underline{f}, \underline{g}, \underline{h})$ at the points in Σ_{bal}.

• The p-adic Gross-Kudla formula supplies evidence for a "p-adic Bloch-Beilinson conjecture" for the rank 8 motive whose ℓ-adic realisation is $V_f \otimes V_g \otimes V_h$, when (f, g, h) is self-dual and balanced.

What about the Birch and Swinnerton-Dyer conjecture?

The Birch Swinnerton-Dyer point

Let $\underline{f}$, $\underline{g}$ and $\underline{h}$ be Hida families such that

1. f_{x_0} is attached to an (ordinary) elliptic curve $E/\mathbb{Q}$, for some $x_0 \in \Omega_f$ with $\kappa(x_0) = 2$;

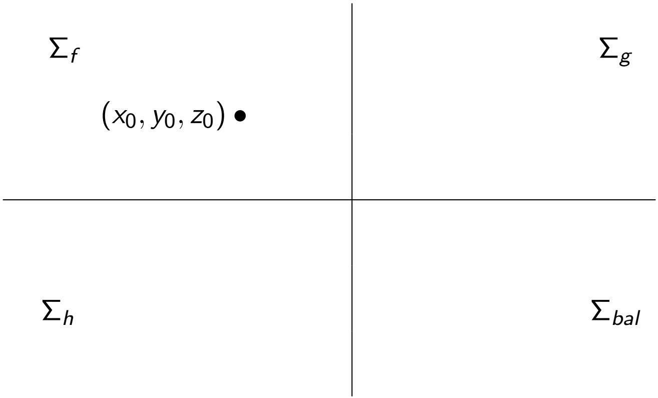
2. g_{y_0} is a *classical* modular form of weight 1 attached to an Artin representation ρ_1, for some $y_0 \in \Omega_g$ with $\kappa(y_0) = 1$;

3. h_{z_0} is a classical modular form of weight 1 attached to an Artin representation ρ_2, for some $z_0 \in \Omega_h$ with $\kappa(z_0) = 1$.

The behaviour of $\mathscr{L}_p{}^f(\underline{f}, \underline{g}, \underline{h})$, $\mathscr{L}_p{}^g(\underline{f}, \underline{g}, \underline{h})$ and $\mathscr{L}_p{}^h(\underline{f}, \underline{g}, \underline{h})$ at the point (x_0, y_0, z_0) should somehow control

$$\hom_{G_{\mathbb{Q}}}(\rho_1 \otimes \rho_2, E(\bar{\mathbb{Q}}) \otimes \mathbb{C}).$$

A picture



$$\mathscr{L}_p{}^f(\underline{f}, \underline{g}, \underline{h})(x_0, y_0, z_0) = *L(E, \rho_1 \otimes \rho_2, 1).$$

What about $\mathscr{L}_p{}^g$, $\mathscr{L}_p{}^h$? p-adic Gross-Kudla?

From cycles to cohomology classes

We can use the cycles $\Delta_{k,\ell,m}$ to construct global classes

$$\mathrm{AJ}_{\mathrm{et}}(\Delta_{k,\ell,m}) \in H^1(\mathbb{Q}, H^{2r+3}_{\mathrm{et}}(V_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(r+2)).$$

Künneth:

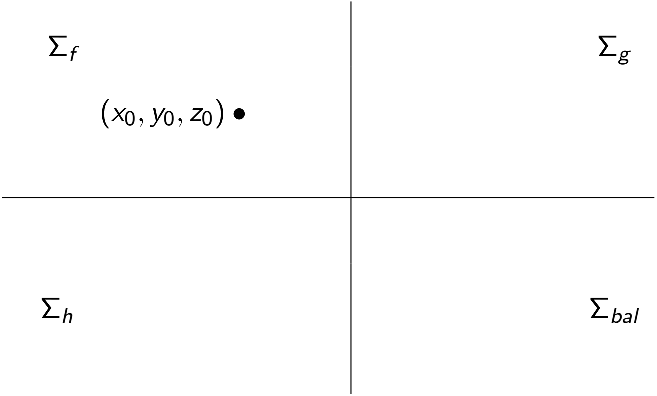
$$\begin{aligned} H^{2r+3}_{\mathrm{et}}(V_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(r+2) &\longrightarrow \bigotimes_{j=1}^{3} H^{r_j+1}_{\mathrm{et}}(\mathcal{E}^{r_j}_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(r+2) \\ &\longrightarrow V_{f_x} \otimes V_{g_y} \otimes V_{h_z}(r+2). \end{aligned}$$

By projecting $\mathrm{AJ}_{\mathrm{et}}(\Delta)$ we obtain a cohomology class

$$\xi(x,y,z) \in H^1(\mathbb{Q}, V_{f_x} \otimes V_{g_y} \otimes V_{h_z}(r+2)),$$

for each $(x,y,z) \in \Sigma_{\mathrm{bal}}$.

p-adic interpolation of $\xi(x, y, z)$



Idea: Extend the assignment $(x, y, z) \mapsto \xi(x, y, z)$ to all of Σ.

p-adic interpolation of diagonal cycle classes

For each $(y, z) \in \Omega_g \times_W \Omega_h$ with $\ell := \kappa(y) = \kappa(z) \geq 2$, the triple (x_0, y, z) is balanced, so we can consider the cohomology classes

$$\kappa(f, g_y, h_z) \in H^1(\mathbb{Q}, V_f \otimes V_{g_y} \otimes V_{h_z}(\ell)).$$

$$\kappa(f, g_y, h_z) \in H^1(\mathbb{Q}, V_p(E) \otimes V_{g_y} \otimes V_{h_z}(\ell - 1)).$$

p-adic interpolation of Galois representations

Theorem (Hida, Wiles,...) There exists a Λ-adic representation $\underline{V}_g$ of $G_{\mathbb{Q}}$ satisfying

$$\underline{V}_g \otimes_{\Lambda_g, y} \bar{\mathbb{Q}}_p = V_{g_y}, \qquad \text{for almost all } y \in \Omega_{g,\mathrm{cl}},$$

and similarly for $\underline{V}_h$.

Corollary There exists a Galois representation $\underline{V}_{gh}$, of rank 4 over $\Lambda_{gh} := \Lambda_g \otimes_\Lambda \Lambda_h$, satisfying

$$\underline{V}_{gh} \otimes_{\Lambda_{gh},(y,z)} \bar{\mathbb{Q}}_p = V_{g_y} \otimes V_{h_z}(\ell - 1).$$

Families of cycles, cont'd

Recall that

$$\xi(f, g_y, h_z) \in H^1(\mathbb{Q}, V_p(E) \otimes V_{g_y} \otimes V_{h_z}(\ell - 1)).$$

Let

$$\mathrm{ev}_{y,z} : H^1(\mathbb{Q}, \underline{V}_{gh}) \longrightarrow H^1(\mathbb{Q}, V_{g_y} \otimes V_{h_z}(\ell - 1)).$$

Theorem (Rotger, D)

There exists a "big" cohomology class

$$\underline{\xi} \in H^1(\mathbb{Q}, V_p(E) \otimes \underline{V}_{gh})$$

such that

$$\underline{\xi}(y, z) := \mathrm{ev}_{y,z}(\underline{\xi}) = \xi(f, g_y, h_z)$$

for almost all $(y, z) \in \Omega_g \times_W \Omega_h$.

p-adic interpolation of cohomology classes

Similar interpolation results have been obtained and exploited in other contexts:

1. Kato: p-adic interpolation of classes arising from Beilinson elements in $H^1(\mathbb{Q}, V_p(f)(2))$. Their weight k specialisations encode higher weight Beilinson elements (A. Scholl, unpublished.)
2. Ben Howard: p-adic interpolation of classes arising from Heegner points. Their higher weight specialisations encode the images of higher weight Heegner cycles under p-adic Abel-Jacobi maps (Francesc Castella, in progress).

The BSD class

Consider the specialisation

$$\begin{aligned}\underline{\xi}(x_0, y_0, z_0) &\in H^1(\mathbb{Q}, V_f \otimes V_{g_{y_0}} \otimes V_{h_{z_0}}(1)) \\ &= H^1(\mathbb{Q}, V_p(E) \otimes \rho_1 \otimes \rho_2).\end{aligned}$$

The BSD point (x_0, y_0, z_0) is not in $\Sigma_{\rm bal}$, and therefore $\underline{\xi}(x_0, y_0, z_0)$ lies *outside* the range of "geometric interpolation" defining the family $\underline{\xi}$.

In particular, the restriction

$$\underline{\xi}(x_0, y_0, z_0)_p \in H^1(\mathbb{Q}_p, V_p(E) \otimes \rho_1 \otimes \rho_2)$$

need not be cristalline.

The dual exponential map

p-adic exponential map:

$$\exp : \Omega^1(E/\mathbb{Q}_p)^\vee \longrightarrow E(\mathbb{Q}_p) \otimes \mathbb{Q}_p.$$

The dual map (exploiting Tate local duality):

$$\exp^* : \frac{H^1(\mathbb{Q}_p, V_p(E))}{H^1_f(\mathbb{Q}_p, V_p(E))} \longrightarrow \Omega^1(E/\mathbb{Q}_p).$$

Analogous map for $V_p(E) \otimes \rho_1 \otimes \rho_2$:

$$\exp^* : \frac{H^1(\mathbb{Q}_p, V_p(E) \otimes \rho_1 \otimes \rho_2)}{H^1_f(\mathbb{Q}_p, V_p(E) \otimes \rho_1 \otimes \rho_2)} \longrightarrow \Omega^1(E/\mathbb{Q}_p) \otimes \rho_1 \otimes \rho_2.$$

A reciprocity law

Question: Relate $\exp^*(\underline{\xi}(x_0, y_0, z_0)) \in \Omega^1(E/\mathbb{Q}_p) \otimes \rho_1 \otimes \rho_2$ to L-functions?

Conjecture (Rotger, D)

The image of the class $\underline{\xi}(x_0, y_0, z_0)$ under $\exp^*$ is non-zero if and only if $L(E \otimes \rho_1 \otimes \rho_2, 1) \neq 0$.

The strategy for proving this, based on ideas of Perrin-Riou, Colmez, Ochiai.... is clear.

The details are not yet fully written up.

One should get a formula relating $\exp^*(\underline{\xi}(x_0, y_0, z_0))$ to $L(E \otimes \rho_1 \otimes \rho_2, 1)$.

The BSD theorem

$$E = \text{ elliptic curve over } \mathbb{Q};$$

$$\rho_1, \rho_2 = \text{ odd 2-dimensional representations of } G_{\mathbb{Q}},$$

$$\det(\rho_1)\det(\rho_2) = 1.$$

The classes $\underline{\xi}(x_0, y_0, z_0)$ and the reciprocity law above should enable us to show:

Theorem? (Rotger, D: still in progress, and far from complete!) *Assume that there exists $\sigma \in G_{\mathbb{Q}}$ for which $\rho_1 \otimes \rho_2(\sigma)$ has distinct eigenvalues. If $L(E \otimes \rho_1 \otimes \rho_2, 1) \neq 0$, then*

$$\hom(\rho_1 \otimes \rho_2, E(K_{\rho_1} K_{\rho_2}) \otimes \mathbb{C}) = 0.$$

Application to elliptic curves and real quadratic fields

Let F be a real quadratic field,

$$\chi_1, \chi_2 : G_F \longrightarrow \mathbb{C}^\times$$

two characters of signature $(+, -)$.

$$\rho_1 = \mathrm{Ind}_F^{\mathbb{Q}} \chi_1, \qquad \rho_2 = \mathrm{Ind}_F^{\mathbb{Q}} \chi_2.$$

$$\rho_1 \otimes \rho_2 = \mathrm{Ind}_F^{\mathbb{Q}}(\chi_1\chi_2) \oplus \mathrm{Ind}_F^{\mathbb{Q}}(\chi_1\chi_2').$$

This set-up would yield BDD-LRV, unconditionally.

The parallel with Kato's method

Rotger-D	Kato
$(f, \underline{g}, \underline{h})$	$(f, E_k(1,\chi), E_k(\chi^{-1},1))$
p-adic Gross-Kudla	p-adic Beilinson (Coleman-de Shalit, Brunault)
Diagonal cycles	Beilinson elements
$L(f \otimes g_\ell \otimes h_\ell, \ell)$	$L(f, j), j \geq 2$
$\Downarrow$	$\Downarrow$
$L(f \otimes \rho_1 \otimes \rho_2, 1)$	$L(f, \chi, 1)$

Cf. the lectures by Brunault and Bertolini this Thursday.

Thank you for your attention.

Time for lunch!