Automorphic forms and Number Theory

International Center, Goa

August 2010
Apology

I will not talk about

$p$-adic weak harmonic Maass forms,

as I had advertised...
The Birch and Swinnerton-Dyer conjecture for $\mathbb{Q}$-curves
and Oda’s period relations

... 

Joint work in progress with
Victor Rotger (Barcelona),
Yu Zhao (Montreal)

Henri Darmon
The Birch and Swinnerton-Dyer conjecture

\[ E = \text{an elliptic curve over a number field } F. \]

\[ L(E/F, s) = \text{its Hasse-Weil } L\text{-series.} \]

Conjecture (Birch and Swinnerton-Dyer)

\[ L(E/F, s) \text{ has analytic continuation to all } s \in \mathbb{C} \text{ and} \]

\[ \text{ord}_{s=1} L(E/F, s) = \text{rank}(E(F)) \]
The BSD conjecture for analytic rank $\leq 1$

Assume $F = \mathbb{Q}$.

Then $L(E, s)$ is known to have analytic continuation thanks to modularity.

Theorem (Gross-Zagier, Kolyvagin)

If $\text{ord}_{s=1} L(E, s) \leq 1$, then $\mathcal{H}(E/\mathbb{Q})$ is finite, and

$$\text{rank}(E(\mathbb{Q})) = \text{ord}_{s=1} L(E, s).$$

Three key ingredients:

1. Modularity (in a strong geometric form);
2. Heegner points on modular curves and the Gross-Zagier theorem;
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Modularity

**Theorem (Geometric modularity)**

*There is a non-constant morphism*

\[ \pi_E : J_0(N) \rightarrow E, \]

*were \( J_0(N) \) is the Jacobian of \( X_0(N) \).*

The proof uses:

1. The modularity theorem (Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor);
2. The Tate conjecture for curves and abelian varieties over number fields (Serre, Faltings).
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Heegner points

\( K = \) imaginary quadratic field satisfying the

**Heegner hypothesis (HH):** There exists an ideal \( \mathfrak{N} \) of \( \mathcal{O}_K \) of norm \( N \), with \( \mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z} \).

**Definition**

The Heegner points on \( X_0(N) \) of level \( c \) attached to \( K \) are the points given by pairs \( (A, A[\mathfrak{N}]) \) with \( \text{End}(A) = \mathbb{Z} + c\mathcal{O}_K \).

They are defined over the ring class field of \( K \) of conductor \( c \).

\[
P_K := \pi_E((A_1, A_1[\mathfrak{N}]) + \cdots + (A_h, A_h[\mathfrak{N}]) - h(\infty)) \in E(K).
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The Gross-Zagier Theorem

Theorem (Gross-Zagier)

For all $K$ satisfying (HH), the $L$-series $L(E/K, s)$ vanishes to odd order at $s = 1$, and

$$L'(E/K, 1) = \langle P_K, P_K \rangle \langle f, f \rangle \pmod{\mathbb{Q}^\times}.$$

In particular, $P_K$ is of infinite order iff $L'(E/K, 1) \neq 0$. 
**Kolyvagin’s Theorem**

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**Theorem (Kolyvagin)**

*If $P_K$ is of infinite order, then $\text{rank}(E(K)) = 1$, and $\mathfrak{W}(E/K) < \infty$.***

---

- The Heegner point $P_K$ is part of a norm-coherent system of algebraic points on $E$;
- This collection of points satisfies the axioms of an *Euler system* (a *Kolyvagin system* in the sense of Mazur-Rubin) which can be used to bound the $p$-Selmer group of $E/K$. 
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Proof of BSD in analytic rank $\leq 1$

**Theorem (Gross-Zagier, Kolyvagin)**

If $\text{ord}_{s=1} L(E, s) \leq 1$, then $\mathcal{L}(E/\mathbb{Q})$ is finite and

$$\text{rank}(E(\mathbb{Q})) = \text{ord}_{s=1} L(E, s).$$

**Proof.**

1. Bump-Friedberg-Hoffstein, Murty-Murty $\Rightarrow$ there exists a $K$ satisfying (HH), with $\text{ord}_{s=1} L(E/K, s) = 1$.
2. Gross-Zagier $\Rightarrow$ the Heegner point $P_K$ is of infinite order.
3. Koyvagin $\Rightarrow$ $E(K) \otimes \mathbb{Q} = \mathbb{Q} \cdot P_K$, and $\mathcal{L}(E/K) < \infty$.
4. Explicit calculation $\Rightarrow$

the point $P_K$ belongs to

$$\begin{cases} 
E(\mathbb{Q}) & \text{if } L(E, 1) = 0, \\
E(K) & \text{if } L(E, 1) \neq 0.
\end{cases}$$
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Proof of BSD in analytic rank ≤ 1

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If \( \text{ord}_{s=1} L(E, s) \leq 1 \), then \( \mathcal{W}(E/\mathbb{Q}) \) is finite and

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Proof.

1. Bump-Friedberg-Hoffstein, Murty-Murty \( \Rightarrow \) there exists a \( K \) satisfying (HH), with \( \text{ord}_{s=1} L(E/K, s) = 1 \).
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Proof of BSD in analytic rank \( \leq 1 \)

**Theorem (Gross-Zagier, Kolyvagin)**

If \( \text{ord}_{s=1} L(E, s) \leq 1 \), then \( \mathcal{W}(E/Q) \) is finite and

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\[ P_K \text{ belongs to } \begin{cases} E(\mathbb{Q}) & \text{if } L(E, 1) = 0, \\ E(K)^{-} & \text{if } L(E, 1) \neq 0. \end{cases} \]
Proof of BSD in analytic rank $\leq 1$

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Totally real fields

The mathematical objects exploited by Gross-Zagier and Kolyvagin continue to be available when \( \mathbb{Q} \) is replaced by a *totally real field* \( F \) of degree \( n > 1 \).

**Definition**

An elliptic curve \( E/F \) is *modular* if there is an automorphic representation \( \pi(E) \) of \( \text{GL}_2(\mathbb{A}_F) \) attached to \( E \), or, equivalently, a Hilbert modular form \( G \in S_2(N) \) over \( F \) such that

\[
L(E/F, s) = L(G, s).
\]

Modularity is often known, and will be assumed from now on.
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Geometric modularity

Geometrically, the Hilbert modular form $G$ corresponds to a $(2^n$-dimensional$)$ subspace

$$\Omega_G \subset \Omega_{\text{har}}^n(V(\mathbb{C}))^G,$$

where $V$ is a suitable Hilbert modular variety of dimension $n$.

**Definition**
The elliptic curve $E/F$ is said to satisfy the Jacquet-Langlands hypothesis (JL) if either $[F : \mathbb{Q}]$ is odd, or there is at least one prime $\nu|N$ at which $\pi_\nu(E)$ is not in the principal series.

**Theorem (Geometric modularity)**
Suppose that $E/F$ is modular and satisfies (JL). There there exists a Shimura curve $X/F$ and a non-constant morphism

$$\pi_E : \text{Jac}(X) \longrightarrow E.$$
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Shimura curves, like modular curves, are equipped with a plentiful supply of CM points.

**Theorem (Zhang)**

Let $E/F$ be a modular elliptic curve satisfying hypothesis (JL). If $\operatorname{ord}_{s=1} L(E/F, s) \leq 1$, then $\mathcal{W}(E/F)$ is finite and

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$$\text{rank}(E(F)) = \text{ord}_{s=1} L(E/F, s).$$

In analytic rank zero one can dispense with (JL).

**Theorem (Matteo Longo)**

Let $E/F$ be a modular elliptic curve. If $L(E/F,1) \neq 0$, then $E(F)$ is finite and $\mathbb{W}(E/F)[p^\infty]$ is finite for almost all $p$.

**Proof.**

Congruences between modular forms $\Rightarrow$ the Galois representation $E[p^n]$ occurs in $J_n[p^n]$, where $J_n = \text{Jac}(X_n)$ and $X_n$ is a Shimura curve $X_n$ whose level may (and does) depend on $n$. Use CM points on $X_n$ to bound the $p^n$-Selmer group of $E$. 

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\[ \square \]
The Challenge that remains

When \( \text{ord}_{s=1} L(E/F, s) = 1 \) but hypothesis (JL) is not satisfied, produce the point in \( E(F) \) whose existence is predicted by BSD.

**Remark**: If \( E/F \) does not satisfy (JL), then its conductor is a square.

Prototypical case where (JL) fails to hold:

\[ F = \mathbb{Q}(\sqrt{N}), \text{ a real quadratic field,} \]

\[ \text{cond}(E/F) = 1. \]

I will focus on this case for simplicity.

**Fact**: \( E(F) \) has even analytic rank, so Longo's theorem applies.
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Consider the twist $E_M$ of $E$ by a quadratic extension $M/F$.

**Proposition**

1. If $M$ is totally real or CM, then $E_M$ has even analytic rank.
2. If $M$ is an ATR (Almost Totally Real) extension, then $E_M$ has odd analytic rank.

**Conjecture (on ATR twists)**

Let $M$ be an ATR extension of $F$ and let $E_M$ be the associated twist of $E$. If $L'(E_M/F, 1) \neq 0$, then $E_M(F)$ has rank one and $\mathfrak{w}(E_M/F) < \infty$.

Although BSD is much better understood in analytic rank one, the conjecture on ATR twists presents a genuine mystery!
ATR twists

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Although BSD is much better understood in analytic rank one, the conjecture on ATR twists presents a genuine mystery!
Some years ago, Adam Logan and I proposed a strategy for calculating a global point on $E_M(F)$, based on Abel-Jacobi images of ATR cycles.

Let $Y$ be the (open) Hilbert modular surface attached to $E/F$:

$$Y(\mathbb{C}) = \text{SL}_2(\mathcal{O}_F) \backslash (\mathcal{H}_1 \times \mathcal{H}_2).$$

To any $\mathcal{O}_F$-algebra embedding

$$\Psi : \mathcal{O}_M \to M_2(\mathcal{O}_F),$$

one can attach cycles $\Delta_\psi \subset Y(\mathbb{C})$ of real dimension one which “behave like Heegner points”.
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one can attach cycles $\Delta_\Phi \subset Y(\mathbb{C})$ of real dimension one which “behave like Heegner points”.
ATR cycles

\( \tau^{(1)}_\Psi := \text{fixed point of } \Psi(\mathcal{M}^\times) \cap \mathcal{H}_1; \)

\( \tau^{(2)}_\Psi, \tau^{(2)'}_\Psi := \text{fixed points of } \Psi(\mathcal{M}^\times) \cap (\mathcal{H}_2 \cup \mathbb{R}); \)

\( \tau_\Psi = \{\tau^{(1)}_\Psi\} \times \text{geodesic}(\tau^{(2)}_\Psi \to \tau^{(2)'}_\Psi). \)

\[ \Delta_\Psi = \tau_\Psi / \langle \Psi(\mathcal{O}^\times_\mathcal{M}) \rangle \subset \mathcal{Y}(\mathbb{C}). \]

**Key fact:** The cycles \( \Delta_\Psi \) are null-homologous.
ATR cycles

\[ \tau_{\psi}^{(1)} := \text{fixed point of } \psi(\mathcal{M}^\times) \cap \mathcal{H}_1; \]

\[ \tau_{\psi}^{(2)}, \tau_{\psi}^{(2)'} := \text{fixed points of } \psi(\mathcal{M}^\times) \cap (\mathcal{H}_2 \cup \mathbb{R}); \]

\[ \mathcal{Y}_\psi = \{ \tau_{\psi}^{(1)} \} \times \text{geodesic}(\tau_{\psi}^{(2)} \rightarrow \tau_{\psi}^{(2)'}). \]

\[ \Delta_\psi = \mathcal{Y}_\psi/\langle \psi(\mathcal{O}_M^\times) \rangle \subset \mathcal{Y}(\mathbb{C}). \]

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**ATR cycles**

\[ \tau_\Psi^{(1)} := \text{fixed point of } \Psi(M^\times) \cap \mathcal{H}_1; \]

\[ \tau_\Psi^{(2)}, \tau_\Psi^{(2)'} := \text{fixed points of } \Psi(M^\times) \cap (\mathcal{H}_2 \cup \mathbb{R}); \]

\[ \Upsilon_\Psi = \{\tau_\Psi^{(1)}\} \times \text{geodesic}(\tau_\Psi^{(2)} \to \tau_\Psi^{(2)'}). \]

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Oda’s periods

ATR cycles are similar to the modular symbols on Hilbert modular varieties of Mladen Dimitrov’s lecture, whose classes in homology encode special values of $L$-functions.

Since ATR cycles are null-homologous, one may hope to relate them to first derivatives.

For any 2-form $\omega_G \in \Omega_G$,

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P^2_{\Psi}(G) := \int_{\partial^{-1}\Delta_{\Psi}} \omega_G \in \mathbb{C}/\Lambda_G
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Conjecture (Oda)

For a suitable choice of $\omega_G$, we have $\mathbb{C}/\Lambda_G \sim E(\mathbb{C})$. In particular $P^2_{\Psi}(G)$ can then be viewed as a point in $E(\mathbb{C})$. 
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Conjecture (Logan, D, 2004)

The points \( P^?_{\Psi}(G) \) belong to \( E(H) \otimes \mathbb{Q} \), where \( H \) is the Hilbert class field of \( M \). The points \( P^?_{\Psi_1}(G), \ldots, P^?_{\Psi_h}(G) \) are conjugate to each other under \( \text{Gal}(H/M) \). Finally, the point

\[
P^?_M(G) := P^?_{\Psi_1}(G) + \cdots + P^?_{\Psi_h}(G)
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is of infinite order iff \( L'(E/M, 1) \neq 0 \).

This conjecture (in a sufficiently general and precise form) would imply the Conjecture on ATR twists. But we do not know how to tackle it.
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The current work with Rotger and Zhao: \( \mathbb{Q} \)-curves

**Definition**

A \( \mathbb{Q} \)-curve over \( F \) is an elliptic curve \( E/F \) which is \( F \)-isogenous to its Galois conjugate.

**Theorem (Ribet)**

Let \( E \) be a \( \mathbb{Q} \)-curve of conductor 1 over \( F = \mathbb{Q}(\sqrt{N}) \). Then there is an elliptic modular form \( f \in S_2(\Gamma_1(N), \varepsilon_F) \) with fourier coefficients in a quadratic (imaginary) field such that

\[
L(E/F, s) = L(f, s)L(f^\sigma, s).
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The Hilbert modular form \( G \) on \( GL_2(\mathbb{A}_F) \) is the Doi-Naganuma lift of \( f \). Modular parametrisation defined over \( F \):

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**Definition**

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The Hilbert modular form $G$ on $GL_2(\mathbb{A}_F)$ is the Doi-Naganuma lift of $f$. Modular parametrisation defined over $F$:

$$J_1(N) \longrightarrow E.$$
Recall: If $E$ is a $\mathbb{Q}$-curve, then $E/F$ has even analytic rank; the same is true for its twists by CM or totally real quadratic characters $\chi$ of $F$ with $\chi(\mathfrak{N}) = 1$.

Theorem (Victor Rotger, Yu Zhao, D)

Let $E$ be a $\mathbb{Q}$-curve of square conductor $\mathfrak{N}_E$ over a real quadratic field $F$, and let $M/F$ be an ATR extension of discriminant prime to $\mathfrak{N}_E$. If $L'(E_M/F, 1) \neq 0$, then $E_M(F)$ has rank one and $\mathcal{W}(E_M/F)$ is finite.
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Elliptic curves of conductor 1

**Pinch, Cremona:** For $N = \text{disc}(F)$ prime and $\leq 1000$, there are exactly 17 isogeny classes of elliptic curves of conductor 1 over $\mathbb{Q}(\sqrt{N})$,

$N = 29, 37, 41, 109, 157, 229, 257, 337, 349,$

$397, 461, 509, 509, 877, 733, 881, 997.$

All but two ($N = 509, 877$) are $\mathbb{Q}$-curves.
Some Galois theory

Let $\mathcal{M}$ = Galois closure of $M$ over $\mathbb{Q}$. Then $\text{Gal}(\mathcal{M}/\mathbb{Q}) = D_8$.

This group contains two copies of the Klein 4-group:

$$V_F = \langle \tau_M, \tau'_M \rangle, \quad V_K = \langle \tau_L, \tau'_L \rangle.$$
Some Galois theory

Suppose that \( F = \mathcal{M}^{\mathcal{V}_F} \), \( M = \mathcal{M}^{\mathcal{T}_M} \), \( M' = \mathcal{M}^{\mathcal{T}'_M} \),

and set \( K = \mathcal{M}^{\mathcal{V}_K} \), \( L = \mathcal{M}^{\mathcal{T}_L} \), \( L' = \mathcal{M}^{\mathcal{T}'_L} \).
Key facts about $K$ and $L$

Let \[
\begin{align*}
\chi_M &: \mathbb{A}_F^\times \rightarrow \pm 1 \text{ be the quadratic character attached to } M/F; \\
\chi_L &: \mathbb{A}_K^\times \rightarrow \pm 1 \text{ be the quadratic character attached to } L/K.
\end{align*}
\]

1. $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field, and satisfies (HH);
2. The central character $\chi_L|_{\mathbb{A}_Q^\times}$ is equal to $\varepsilon_F$.
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The Artin formalism

Let $E/F$ be a $\mathbb{Q}$-curve and let $f \in S_2(\Gamma_0(N), \varepsilon_F)$ be the associated elliptic cusp form.

$$L(E_M/F, s) = L(E/F, \chi_M, s) = L(f/F, \chi_M, s) = L(f \otimes \text{Ind}_F^Q \chi_M, s)$$

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In particular, $L'(E_M/F, 1) \neq 0$ implies that $L'(f/K, \chi_L, 1) \neq 0$. 
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A theorem of GZK-type

The following strikingly general recent generalisation of the GZK theorem applies to forms on $\Gamma_1(N)$ with non-trivial nebentype character.

Theorem (Ye Tian, Xinyi Yuan, Shou-Wu Zhang, Wei Zhang)

If $L'(f/K, \chi_L, 1) \neq 0$, then $A_f(L^-) \otimes \mathbb{Q}$ has dimension one over $T_f$, and therefore

$$\text{rank}(A_f(L^-)) = 2.$$ 

Furthermore $\mathcal{W}(A_f/L^-)$ is finite.

$$\text{rank}(A_f(L^-)) = \text{rank}(A_f(M^-)), \quad A_f(M^-) = E(M)^- \oplus E(M)^-.$$ 

Corollary

If $L'(E_M/F, 1) \neq 0$, then $\text{rank}(E_M(F)) = 1$ and $\mathcal{W}(E_M/F) < \infty$. 
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If $L'(E_M/F, 1) \neq 0$, then $\text{rank}(E_M(F)) = 1$ and $\mathcal{W}(E_M/F) < \infty$. 
A final question

Underlying this theorem is the construction of a “classical” Heegner point $P_M(f) \in E_M(F)$.

**Question**

*Is there a direct formula relating the ATR point $P_M^?(G)$ and the “classical” Heegner point $P_M(f)$ arising from $J_1(N)$?*

The study undertaken with Rotger and Zhao suggests a relation of the form

$$P_M^?(G) \overset{?}{=} \ell \cdot P_M(f), \quad \ell \in \mathbb{Q}^\times.$$

This statement resembles the period relations of Oda relating the periods of an elliptic cusp form with those of its Doi-Naganuma lift, and hence might be more tractable (both computationally, and theoretically) than my original conjecture with Logan.
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My usual apology

Sorry for running over time!
A Big Thank You to

Pierre Colmez,

Wee Teck Gan,

Eknath Ghate,

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Kenneth Ribet,\textsuperscript{1}

Vinayak Vatsal.\textsuperscript{1}

for organising this inspiring conference in such a lovely setting!

\textsuperscript{1}Even if he did not show up...