SCHOLAR: Conference in honor of Ram Murty's 60th birthday

From p-adic to Artin representations: a story in three vignettes

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Artin representations

Definition

An *Artin representation* is a continuous representation

$$\varrho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\mathbb{C}), \qquad G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

Artin L-function:

$$L(\varrho, s) = \prod_{\ell} \det((1 - \sigma_\ell \ell^{-s})|_{V_\varrho^{I_\ell}})^{-1}.$$

$\sigma_\ell =$ Frobenius element at ℓ;

$V_\varrho =$ complex vector space realising ϱ;

$I_\ell =$ inertia group at ℓ.

The Artin conjecture

Conjecture

The L-function $L(\varrho, s)$ extends to a holomorphic function of $s \in \mathbb{C}$ (except for a possible pole at $s = 1$).

• One-dimensional representations factor through abelian quotients, and their study amounts to *class field theory* for $\mathbb{Q}$:

$$L(\varrho, s) = L(\chi, s),$$

where $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is a *Dirichlet character*.

• This talk will focus mainly on two-dimensional representations which are *odd*: $\varrho(\sigma_\infty)$ has eigenvalues 1 and -1.

Modular forms of weight one

The role of Dirichlet characters in the study of odd two-dimensional Artin representations is played by *cusp forms of weight one*:

Definition

A cusp form of weight one, level N, and (odd) character χ is a holomorphic function $g : \mathcal{H} \longrightarrow \mathbb{C}$ satisfying

$$g(\frac{az+b}{cz+d}) = \chi(d)(cz+d)g(z).$$

Such a cusp form has a *fourier expansion*:

$$g = \sum a_n(g) q^n, \qquad q = e^{2\pi i z}.$$

The strong Artin conjecture

Conjecture

If ϱ is an odd, irreducible, two-dimensional representation of $G_{\mathbb{Q}}$, there is a cusp form g of weight one, level $N = cond(\varrho)$, and character $\chi = \det(\varrho)$, satisfying

$$L(\varrho, s) = L(g, s).$$

$$L(g, s) = \sum_n a_n(g) n^{-s}$$

is the *Hecke L-function* attached to g.

First vignette: the Deligne-Serre theorem

Theorem (Deligne-Serre)

Let g be a weight one eigenform. There is an odd two-dimensional Artin representation

$$\varrho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

satisfying

$$L(\varrho_g, s) = L(g, s).$$

First vignette, cont'd: congruences

The first step of the proof relies crucially on *congruences between modular forms*:

Proposition: For each prime ℓ, there exists an eigenform $g_\ell \in S_\ell(N, \chi)$ of weight ℓ satisfying

$$g \equiv g_\ell \pmod{\ell}.$$

Idea:

• Multiply g by the Eisenstein series $E_{\ell-1}$ of weight $\ell - 1$, to obtain a mod ℓ eigenform with the right fourier coefficients;

• lift this mod ℓ eigenform to an eigenform with coefficients in $\bar{\mathbb{Q}}$.

First vignette, cont'd: étale cohomology

It was already known, thanks to Deligne, how to associate Galois representations to eigenforms of weight $\ell \geq 2$: they occur in the *étale cohomology* of certain *Kuga-Sato varieties*.

$\mathcal{E} :=$ universal elliptic curve over $X_1(N)$;

$W_\ell(N) = \mathcal{E} \times_{X_1(N)} \cdots \times_{X_1(N)} \mathcal{E} \qquad (\ell - 2 \text{ times});$

$V_{g_\ell} := H^{\ell-1}_{\mathrm{et}}(W_\ell(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)[g_\ell]$.

Conclusion: For each ℓ there exists a mod ℓ representation

$$\varrho_\ell : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbf{F}}_\ell)$$

satisfying

$$\mathrm{trace}(\varrho_\ell(\sigma_p)) = a_p(g) \pmod{\ell}, \qquad \text{for all } p \nmid N\ell.$$

First vignette, cont'd: conclusion of the proof

Using *a priori* estimates on the size of $a_p(g)$, and some group theory, the size of the image of ϱ_ℓ is *bounded independently of ℓ.*

Hence the ϱ_ℓ's can be pieced together into a ϱ with finite image and values in $\mathrm{GL}_2(\mathbb{C})$.

First vignette: conclusion

Note the key role played in this proof by:

• Congruences between weight one forms and modular forms of higher weights;

• Geometric structures — Kuga-Sato varieties, and their associated étale cohomology groups — which allow the construction of associated ℓ-adic Galois representations.

Second vignette: the Strong Artin Conjecture

Theorem

Let ϱ be an odd, irreducible, two-dimensional Artin representation. There exists an eigen-cuspform g of weight one satisfying

$$L(g, s) = L(\varrho, s).$$

- This theorem is now completely proved, over $\mathbb{Q}$, thanks to the proof of the Serre conjectures by Khare and Wintenberger.

- Prior to that, significant progress on the conjecture was achieved based on a program of Taylor building on the fundamental *modularity lifting theorems* of Wiles.

- The "second vignette" is concerned with the broad outline of Taylor's approach.

Scond vignette: Classification of Artin representations

By projective image, in order of increasing arithmetic complexity:

A. Reducible representations (sums of Dirichlet characters).

B. Dihedral, induced from an imaginary quadratic field.

C. Dihedral, induced from a real quadratic field.

D. Tetrahedral case: projective image A_4.

E. Octahedral case: projective image S_4.

F. Icosahedral case: projective image A_5.

Second vignette: the status of the Artin conjecture

Cases A-C date back to Hecke, while D and E can be handled via techniques based on *solvable base change*.

The interesting case is the icosahedral case, where ϱ has projective image A_5.

Technical hypotheses: Asssume ϱ is unramified at 2, 3 and 5, and that $\varrho(\sigma_2)$ has distinct eigenvalues.

Second vignette: the Shepherd-Barron–Taylor construction

Theorem

There exists a principally polarised abelian surface A with $\mathbb{Z}[\frac{1+\sqrt{5}}{2}] \hookrightarrow End(A)$ such that

- $A[2] \simeq \overline{V_\varrho}$ *as* $G_{\mathbb{Q}}$*-modules;*
- $A[\sqrt{5}] \simeq E[5]$ *for some elliptic curve* E.

Second vignette: the propagation of modularity

Langlands-Tunnel: $E[3]$ is modular.

Wiles' modularity lifting, at 3: $T_3(E) := \lim_{\leftarrow,n} E[3^n]$ is modular.

Hence E is modular, hence $E[5] = A[\sqrt{5}]$ is as well.

Modularity lifting, at $\sqrt{5}$: $T_{\sqrt{5}}(A)$ is modular.

Hence A is modular, hence so is $A[2] = \overline{V_\varrho}$.

Modularity lifting, at 2: The representation ϱ is 2-adically modular, i.e., it corresponds to a 2-adic overconvergent modular form of weight one.

Second vignette: from overconvergent to classical forms

The theory of companion forms produces two distinct overconvergent 2-adic modular forms attached to ϱ. (Using the distinctness of the eigenvalues of $\varrho(\sigma_2)$.)

Buzzard-Taylor. A suitable linear combination of these forms can be extended to a classical form of weight one. (A key hypothesis on ϱ that is exploited is the triviality of $\varrho(I_2)$.)

This beautiful strategy has recently been extended to totally real fields by Kassaei, Sasaki, Tian, ...

Brief summary

A dominant theme in both vignettes is the rich interplay between Artin representations and ℓ-adic and mod ℓ representations, via congruences between the associated modular forms, (of weight one, and weight ≥ 2, where the geometric arsenal of étale cohomology becomes available.)

Third vignette: the Birch and Swinnerton-Dyer conjecture

Let E be an elliptic curve over $\mathbb{Q}$. Hasse-Weil-Artin L-series

$$L(E, \varrho, s) = L(V_p(E) \otimes V_\varrho, s).$$

Conjecture (BSD)

The L-series $L(E, \varrho, s)$ extends to an entire function of s and

$$\mathrm{ord}_{s=1} L(E, \varrho, s) = r(E, \varrho) := \dim_{\mathbb{C}} E(\bar{\mathbb{Q}})^{\varrho},$$

where

$$E(\bar{\mathbb{Q}})^{\varrho} = \hom_{G_{\mathbb{Q}}}(V_\varrho, E(\bar{\mathbb{Q}}) \otimes \mathbb{C}).$$

Remark: $r(E, \varrho)$ is the multiplicity with which the Artin representation V_ϱ appears in the Mordell-Weil group of E over the field cut out by ϱ.

Third vignette: the rank 0 case

A special case of the equivariant BSD conjecture is

Conjecture

If $L(E, \varrho, 1) \neq 0$, then $r(E, \varrho) = 0$.

• If ϱ is a quadratic character, it follows from the work of Gross-Zagier-Kolyvagin, combined with a non-vanishing result on L-series due to Bump-Friedberg Hoffstein and Murty-Murty.

• If ϱ is one-dimensional, it follows from the work of Kato.

• If ϱ is induced from a non-quadratic ring class character of an imaginary quadratic field, it follows from work of Bertolini, D., Longo, Nekovar, Rotger, Seveso, Vigni, Zhang,.... building on the fundamental breakthroughs of Gross-Zagier and Kolyvagin.

Third vignette: recent progress

Assume that

- $\varrho = \varrho_1 \otimes \varrho_2$, where ϱ_1 and ϱ_2 are odd irreducible Artin representations of dimension two.
- The conductors of E and ϱ are relatively prime.
- $\det(\varrho_1) = \det(\varrho_2)^{-1}$, and hence in particular ϱ is *self-dual*.

Theorem (D, Victor Rotger)

If $L(E, \varrho, 1) \neq 0$, then $r(E, \varrho) = 0$.

Third vignette: local and global Tate duality

The Mordell-Weil group injects into a global Galois cohomology group

$$E(\bar{\mathbb{Q}})^{\varrho} \longrightarrow H^1_f(\mathbb{Q}, V_p(E) \otimes V_{\varrho}).$$

Local and global duality, and the Poitou-Tate sequence: In order to bound $r(E, \varrho)$, it *suffices* to show that the natural map

$$H^1(\mathbb{Q}, V_p(E) \otimes V_{\varrho}) \longrightarrow \frac{H^1(\mathbb{Q}_p, V_p(E) \otimes V_{\varrho})}{H^1_f(\mathbb{Q}_p, V_p(E) \otimes V_{\varrho})}$$

is **surjective**.

Thus the problem of bounding $E(\bar{\mathbb{Q}})^{\varrho}$ translates into the problem of constructing global cohomology classes with "sufficiently singular" local behaviour at p.

Third vignette: modularity

Thanks to the modularity results alluded to in the first two vignettes, one can associate to $(E, \varrho_1, \varrho_2)$:

• An eigenform f of weight two, with $L(f, s) = L(E, s)$.

• Eigenforms g and h of weight one, with $L(g, s) = L(\varrho_1, s)$ and $L(h, s) = L(\varrho_2, s)$.

• We then have an identification

$$L(E, \varrho_1 \otimes \varrho_2, s) = L(f \otimes g \otimes h, s)$$

of the Hasse-Weil-Artin L-function with the Garret-Rankin triple product L-function attached to (f, g, h).

Third vignette: the theme of p-adic variation

Theorem (Hida)

There exist Hida families

$$\underline{g} = \sum_n \underline{a}_n(g,k)q^n, \qquad \underline{h} = \sum_n \underline{a}_n(h,k)q^n,$$

of modular forms, specialising to g and h in weight one.

The fourier coefficients $\underline{a}_n(g,k)$ and $\underline{a}_n(h,k)$ are rigid analytic functions on *weight space* $\mathcal{W} := \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$.

For each integer $k \geq 2$, we obtain a pair (g_k, h_k) of *classical forms* of higher weight k. These *converge* to (g,h) p-adically as $k \longrightarrow 1$ in $\mathcal{W}$.

Third vignette: generalised diagonal cycles

When $k \geq 2$, we can construct classes

$$\kappa(f, g_k, h_k) \in H^1(\mathbb{Q}, V_p(E) \otimes V_p(g_k) \otimes V_p(h_k)(k-1))$$

from the images of *generalised Gross-Kudla-Schoen cycles* in

$$\mathrm{CH}^k(X_0(N) \times W_k(N) \times W_k(N))_0.$$

p-**adic étale Abel-Jacobi map**:

$$\begin{aligned} &\mathrm{CH}^k(X_0(N) \times W_k(N) \times W_k(N))_0 \\ &\quad \longrightarrow \quad H^1(\mathbb{Q}, H^{2k-1}_{et}((X_0(N) \times W_k(N) \times W_k(N))_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(k)) \\ &\quad \longrightarrow \quad H^1(\mathbb{Q}, H^1_{\mathrm{et}}(X_0(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(1) \otimes H^{k-1}_{\mathrm{et}}(W_k(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)^{\otimes 2}(k-1)) \\ &\quad \longrightarrow \quad H^1(\mathbb{Q}, V_p(E) \otimes V_p(g_k) \otimes V_p(h_k)(k-1)). \end{aligned}$$

Third vignette: end of sketch of proof

The technical heart of the proof has two parts:

• The classes $\kappa(f, g_k, h_k)$ interpolate to a p-adic analytic family of cohomology classes, as k varies over $\mathcal{W}$. In particular, we can consider the p-adic limit

$$\kappa(f, g, h) := \lim_{k \longrightarrow 1} \kappa(f, g_k, h_k).$$

Theorem (Reciprocity law)

The class $\kappa(f, g, h)$ is non-cristalline, i.e., has non-zero image in $\frac{H^1(\mathbb{Q}_p, V_p(E) \otimes V_\varrho)}{H^1_f(\mathbb{Q}_p, V_p(E) \otimes V_\varrho)}$, if and only if $L(E, \varrho, 1) \neq 0$.

Application to ring class fields of real quadratic fields

Of special interest is the case where V_{ϱ_1} and V_{ϱ_2} are induced from finite order characters χ_1 and χ_2 (of mixed signature) of the same *real quadratic field* K:

$$V_{\varrho_1} \otimes V_{\varrho_2} = \mathrm{Ind}_K^{\mathbb{Q}}(\psi) \oplus \mathrm{Ind}_K^{\mathbb{Q}}(\tilde{\psi}), \qquad \psi = \chi_1\chi_2, \quad \tilde{\psi} = \chi_1\chi_2'.$$

The characters ψ and $\tilde{\psi}$ are ring class characters of K.

Theorem

Assume that (E, K) *satisfies the* analytic non-vanishing condition *of the next slide. Then, for all ring class characters* $\psi : \mathrm{Gal}(H/K) \longrightarrow \mathbb{C}^\times$ *of* K *of conductor prime to* N_E*,*

$$L(E/K, \psi, 1) \neq 0 \Rightarrow (E(H) \otimes \mathbb{C})^\psi = 0.$$

The analytic non-vanishing condition

Given an elliptic curve $E/\mathbb{Q}$ and a (real) quadratic field K, the non-vanishing condition is:

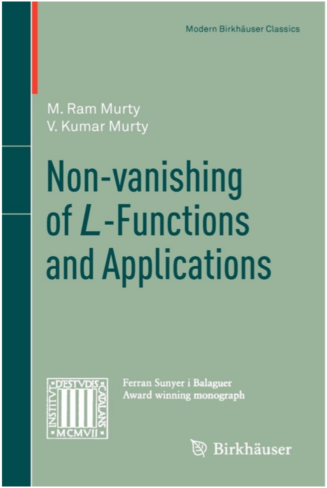
Non-vanishing condition: There exist even and odd quadratic twists E' of E such that

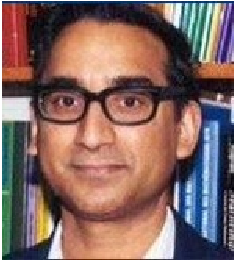
$$L(E'/K, 1) \neq 0.$$

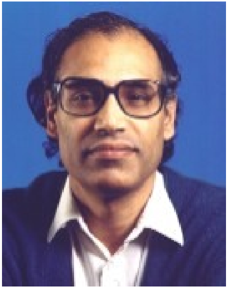
Question: When is this condition satisfied for (E, K)?

Theorem (Bump-Friedberg-Hoffstein, Murty, Murty). There exist infinitely many quadratic twists E' of E for which $L(E'/\mathbb{Q}, 1) \neq 0$ and also infinitely many for which $L'(E'/\mathbb{Q}, 1) \neq 0$.

Non-vanishing of L-series







Tetrahedral and Octahedral forms

Assume throughout that N_E is coprime to the discriminant of $P(x)$.

Theorem

Let P be a polynomial of degree 4 with Galois group A_4 and no real roots, and let K be any subfield of its splitting field. Then $L(E/K, 1) \neq 0 \Rightarrow E(K)$ is finite.

Theorem

Let P be a polynomial of degree 4 with Galois group S_4 and at least two non-real roots, and assume that $L(E, \epsilon, 1) \neq 0$, where ϵ is the quadratic character attached to the discriminant of P. Then, for any subfield K of the splitting field of P, $L(E/K, 1) \neq 0 \Rightarrow E(K)$ is finite.

An icosahedral application

Theorem

Let P be a polynomial of degree 5 with Galois group A_5 and a single real root, and let K be the quintic field generated by a root of P. Then

$$\mathrm{ord}_{s=1} L(E,s) = \mathrm{ord}_{s=1} L(E/K,s) \Rightarrow \mathrm{rank}(E(\mathbb{Q})) = \mathrm{rank}(E(K)).$$

Explanation: $\mathrm{Ind}_K^{\mathbb{Q}} 1 = 1 \oplus V_1 \otimes V_2$, where V_1 and V_2 are odd two-dimensional representations of the binary icosahedral group.

The method says nothing (as far as we can tell!) about the arithmetic of E over the field generated by a root of Lagrange's sextic resolvent of $P(x)$.

Happy 60th Birthday, Ram!