MOCK PLECTIC INVARIANTS

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To Jan Nekovář

Abstract. A p-arithmetic subgroup of $\mathbf{SL}_2(\mathbb{Q})$ like the Ihara group $\Gamma := \mathbf{SL}_2(\mathbb{Z}[1/p])$ acts by Möbius transformations on the Poincaré upper half plane $\mathcal{H}$ and on Drinfeld's p-adic upper half plane $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$. The diagonal action of Γ on the product is discrete, and the quotient $\Gamma\backslash(\mathcal{H}_p \times \mathcal{H})$ can be envisaged as a "mock Hilbert modular surface". According to a striking prediction of Nekovář and Scholl, the CM points on genuine Hilbert modular surfaces should give rise to "plectic Heegner points" that encode non-trivial regulators attached, notably, to elliptic curves of rank two over real quadratic fields. This article develops the analogy between Hilbert modular surfaces and their mock counterparts, with the aim of transposing the plectic philosophy to the mock Hilbert setting, where the analogous plectic invariants are expected to lie in the alternating square of the Mordell–Weil group of certain elliptic curves of rank two over $\mathbb{Q}$.

Contents

Introduction

The Hilbert modular group $\mathbf{SL}_2(\mathcal{O}_F)$ attached to a real quadratic field F, viewed as a discrete subgroup of $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{SL}_2(\mathbb{R})$ by ordering the real embeddings $\nu_1, \nu_2 \colon F \to \mathbb{R}$, acts discretely by Möbius transformations on the product $\mathcal{H} \times \mathcal{H}$ of two Poincaré upper half planes. The cohomology of the complex surface

$$\mathcal{S}_F := \mathbf{SL}_2(\mathcal{O}_F)\backslash(\mathcal{H} \times \mathcal{H})$$

is intimately tied with the arithmetic of elliptic curves with everywhere good reduction over F. More precisely, if $E_{/F}$ is such a (modular) elliptic curve, $E_j := E \otimes_{F,\nu_j} \mathbb{R}$ are the associated real elliptic curves for $j = 1, 2$, and π_E is the associated automorphic representation of $\mathbf{GL}_2(F)$, Oda's period conjecture predicts an isomorphism

$$H^2(\mathcal{S}_F, \mathbb{Q})[\pi_E] \simeq H^1(E_1, \mathbb{Q}) \otimes H^1(E_2, \mathbb{Q}) \tag{1}$$

of rational Hodge structures [Oda82]. This strong "geometric" form of modularity has implications for the arithmetic of $E_{/F}$ that are richer, more subtle and less well understood than those that arise from realising E as a quotient of the jacobian of a Shimura curve. For instance, let $K \subset \mathrm{M}_2(F) \subset M_2(\mathbb{R}) \times M_2(\mathbb{R})$ be a quadratic extension of F which is "Almost Totally Real" (ATR), i.e., satisfies

$$K \otimes_{F,\nu_1} \mathbb{R} = \mathbb{C}, \qquad K \otimes_{F,\nu_2} \mathbb{R} = \mathbb{R} \oplus \mathbb{R}.$$

Let $\tau_1 \in \mathcal{H}$ be the fixed point for the action of $\nu_1(K^\times) \subset \mathbf{GL}_2(\mathbb{R})$ on $\mathbb{P}_1(\mathbb{C})$, and let $\tau_2, \tau_2' \in \mathbb{R}$ be the fixed points of $\nu_2(K^\times)$. Denote by (τ_2, τ_2') the hyperbolic geodesic in $\mathcal{H}$ joining τ_2 to τ_2', and let $\gamma \subset \mathcal{S}_F$ be the simple closed geodesic contained in the image of

$$\{\tau_1\} \times (\tau_2, \tau_2') \subset \mathcal{H} \times \mathcal{H}.$$

The finiteness of $H_1(\mathcal{S}_F, \mathbb{Z})$ implies there there is an integer $m \geq 1$ and a smooth real two-dimensional region $\Pi \subset \mathcal{S}_F$ having $m\gamma$ as its boundary. Oda's period conjecture (1) implies that, for a suitable real analytic two-form $\omega \in \Omega^2(\mathcal{S}_F)[\pi_E]$, the complex integral

$$P_\gamma := \frac{1}{m} \int_\Pi \omega \ \in \ \mathbb{C} \tag{2}$$

is independent of the choice of Π up to elements in a suitable period lattice Λ_1 attached to E_1. Viewing (2) as an element of $\mathbb{C}/\Lambda_1 \cong E_1(\mathbb{C})$, the complex point P_γ is conjectured to be defined over an explicit ring class field of K, following a numerical recipe that is worked out and tested experimentally in [DL03] and [GM13].

Analogously, the *Ihara group* $\Gamma := \mathbf{SL}_2(\mathbb{Z}[1/p])$ acts by Möbius transformations on $\mathcal{H}$ and on Drinfeld's p-adic upper half plane $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$. Its diagonal action on the product $\mathcal{H}_p \times \mathcal{H}$ is discrete, and the quotient

$$\mathcal{S} := \Gamma \backslash (\mathcal{H}_p \times \mathcal{H})$$

can be envisaged as a "mock Hilbert modular surface", following a suggestive terminology of Barry Mazur [Maz01]. Fleshing out the analogy between $\mathcal{S}_F$ and $\mathcal{S}$ leads to fruitful perspectives on the arithmetic of elliptic curves (and modular abelian varieties) over $\mathbb{Q}$ with multiplicative reduction at p. Notably,

- the "exceptional zero conjecture" on derivatives of the p-adic L-functions of these elliptic curves formulated by Mazur, Tate and Teitelbaum [MTT86] and proved by Greenberg and Stevens [GS93] can be understood as the counterpart of (1) for $\mathcal{S}$;
- the mock analogues of the ATR points of (2) are the Stark–Heegner points of [Dar01] which are indexed by real quadratic geodesics on $\mathcal{S}$ and are conjecturally defined over ring class fields of real quadratic fields.

These two analogies are briefly explained in Sections 1 and 2 respectively.

A striking insight of Nekovář and Scholl ([Nek10], [Nek16], [NS16]) suggests that zero-dimensional CM cycles on $\mathcal{S}_F$ should give rise to "plectic Heegner points" involving non-trivial regulators for elliptic curves (over F) of rank two. At the moment, no precise numerical recipe is available to compute them, placing the conjectures of loc.cit. somewhat outside the scope of experimental verification. (The reader is nevertheless referred to [For23] for some results in that direction.) More recently, the second author and Lennart Gehrmann have transposed the plectic conjectures to the setting of quaternionic Shimura varieties uniformised by products of p-adic upper half planes [FG23(a)], where plectic Heegner points admit a concrete analytic description. In that context the plectic philosophy has been tested experimentally in [FGM22], and some partial evidence has been given in [FG23(b)], even though its theoretical underpinnings remain poorly understood. These non-archimedean perspectives suggest that it might be instructive to examine the plectic philosophy in the intermediate setting of mock Hilbert modular surfaces, whose periods involve a somewhat delicate mix of complex and p-adic integration.

The primary aim of this note is to develop the analogy between $\mathcal{S}_F$ and $\mathcal{S}$ and describe its most important arithmetic applications, with a special emphasis on the plectic framework where it had not been examined systematically before. The main new contribution, presented in Section 3, is the construction of *global* cohomology classes – referred to as "mock plectic invariants" – associated to elliptic curves over $\mathbb{Q}$ of conductor p and CM points on $\mathcal{S}$. These invariants generalize and upgrade the construction of plectic p-adic invariants of [FGM22], [FG23(a)] in the CM setting, and are related to the leading terms of anticyclotomic p-adic L-functions that were introduced and studied in [BD96]. Viewed in this way, the plectic philosophy is seen to be consistent with the anticyclotomic Birch and Swinnerton-Dyer conjecture in the somewhat exotic setting of loc.cit., where the twisted L-values that one wishes to interpolate *vanish identically* and it becomes necessary to p-adically interpolate the Heegner points themselves over the anticyclotomic tower, viewed as algebraic avatars of *first derivatives* of twisted L-series. The authors hope that this consistency provides some oblique evidence for the plectic philosophy of Nekovář and Scholl, while enriching the dictionary between Hilbert modular surfaces and their mock counterparts.

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1. Mock Hilbert modular forms and their periods

A *mock Hilbert modular form* on $\mathcal{S}$ (of parallel weight 2) should be thought of, loosely speaking, as a "holomorphic differential two-form on $\Gamma\backslash(\mathcal{H}_p \times \mathcal{H})$", i.e., a function $f(z_p, z_\infty)$ of the variables $z_p \in \mathcal{H}_p$ and $z_\infty \in \mathcal{H}$ which is rigid analytic in z_p, holomorphic in z_∞, and satisfies the transformation rule

$$\text{“}f\left(\frac{az_p+b}{cz_p+d}, \frac{az_\infty+b}{cz_\infty+d}\right) = (cz_p+d)^2(cz_\infty+d)^2 f(z_p,z_\infty)\text{”} \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{3}$$

The awkward mix of rigid and complex analysis inherent in this (non) definition prevents (3) from resting on a solid mathematical foundation. A few simple facts about rigid differentials on the Drinfeld upper half plane can nonetheless be made to conjure a concrete object that captures key features of (3).

1.1. Digression: rigid analytic differentials on $\mathcal{H}_p$.

Let $\mathcal{T} := \mathcal{T}_0 \sqcup \mathcal{T}_1$ denote the Bruhat–Tits tree of $\mathbf{SL}_2(\mathbb{Q}_p)$, whose set $\mathcal{T}_0$ of vertices is in bijection with homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$, two lattices being joined by an edge in $\mathcal{T}_1 \subset \mathcal{T}_0^2$ if they are represented by lattices contained one in the other with index p. There is a natural *reduction map*

$$\mathrm{red} : \mathcal{H}_p \longrightarrow \mathcal{T}$$

from $\mathcal{H}_p$ to $\mathcal{T}$, which maps the *standard affinoid*

$$\mathcal{A}_\circ := \{z \in \mathcal{O}_{\mathbb{C}_p} \text{ such that } |z-a| \geq 1, \text{ for all } a \in \mathbb{Z}_p\} \ \subset \ \mathbb{P}_1(\mathbb{C}_p) \tag{4}$$

to the vertex attached to the lattice $v_\circ = [\mathbb{Z}_p^2]$. The $p+1$ mod p residue discs in the complement of $\mathcal{A}_\circ$ are in natural bijection with $\mathbb{P}_1(\mathbb{F}_p)$ and contain the boundary annuli

$$\begin{aligned} \mathcal{W}_\infty &= \{z \in \mathbb{P}_1(\mathbb{C}_p) \text{ such that } 1 < |z| < p\}, \\ \mathcal{W}_j &= \{z \in \mathbb{P}_1(\mathbb{C}_p) \text{ such that } 1/p < |z-j| < 1\}, \text{ for } j = 0, \dots, p-1. \end{aligned} \tag{5}$$

The edges having $v_\circ$ as an endpoint are likewise in bijection with $\mathbb{P}_1(\mathbb{F}_p)$ by setting

$$e_\infty \leftrightarrow (\mathbb{Z}_p^2,\ \mathbb{Z}_p\cdot(1,0)+p\mathbb{Z}_p^2), \qquad e_j \leftrightarrow (\mathbb{Z}_p^2,\ \mathbb{Z}_p\cdot(j,1)+p\mathbb{Z}_p^2), \quad \text{for } j = 0,\dots,p-1. \tag{6}$$

The preimage of the singleton $\{e_j\}$ under the reduction map is the annulus $\mathcal{W}_j$, for each $j \in \mathbb{P}_1(\mathbb{F}_p)$. The properties

$$\mathrm{red}^{-1}(\{v_\circ\}) = \mathcal{A}_\circ, \qquad \mathrm{red}^{-1}(\{e_j\}) = \mathcal{W}_j, \quad \text{for all } j \in \mathbb{P}_1(\mathbb{F}_p),$$

combined with the requirement of compatibility with the natural actions of $\mathbf{SL}_2(\mathbb{Q}_p)$ on $\mathcal{H}_p$ and on $\mathcal{T}$, determine the reduction map uniquely. In particular, the preimage, denoted $\mathcal{A}_v$, of the vertex $v \in \mathcal{T}_0$ is an affinoid obtained by taking the complement in $\mathbb{P}(\mathbb{C}_p)$ of $(p+1)$ mod p residue discs with $\mathbb{Q}_p$-rational centers (relative to a coordinate on $\mathbb{P}(\mathbb{Q}_p)$ depending on v). Given an edge $e = (v_1, v_2) \in \mathcal{T}_1$, the affinoids $\mathcal{A}_{v_1}$ and $\mathcal{A}_{v_2}$ are glued to each other along the p-adic annulus attached to e, denoted $\mathcal{W}_e$. With just a modicum of artistic licence, the entire Drinfeld upper-half plane can be visualised as a tubular neighbourhood of $\mathcal{T}$, as in the figure below for $p = 2$.

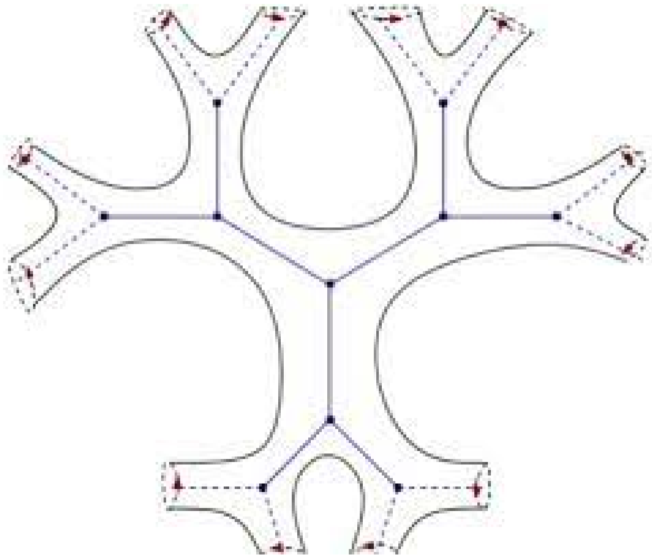

FIGURE 1. The Drinfeld upper half plane and the Bruhat-Tits tree

This picture suggests that $\mathcal{H}_p$, unlike its Archimedean counterpart, is far from being simply connected and that its first cohomology is quite rich. For each edge $e \in \mathcal{T}_1$, the de Rham cohomology of $\mathcal{W}_e$ is identified with $\mathbb{C}_p$ via the map that sends $\omega \in \Omega^1_{\mathrm{rig}}(\mathcal{W}_e)$ to its p-adic annular residue, denoted $\mathrm{res}_{\mathcal{W}_e}(\omega)$. This residue map is well-defined up to a sign, which is determined by fixing an *orientation* on $\mathcal{W}_e$, or, equivalently, viewing e as an *ordered* edge of $\mathcal{T}$, having a source and target. Let $\mathcal{E}(\mathcal{T})$ denote the set of such ordered edges, let $s, t : \mathcal{E}(\mathcal{T}) \to \mathcal{T}_0$ denote the source and target maps, and write $\bar{e}$ for the edge e with its source and target interchanged.

Definition 1.1. A *harmonic cocycle* on $\mathcal{T}$ is a $\mathbb{C}_p$-valued function

$$c : \mathcal{E}(\mathcal{T}) \to \mathbb{C}_p$$

satisfying the following properties:

- $c(\bar{e}) = -c(e)$, for all $e \in \mathcal{E}(\mathcal{T})$;
- for all vertices v of $\mathcal{T}$,

$$\sum_{s(e)=v} c(e) = \sum_{t(e)=v} c(e) = 0.$$

The $\mathbb{C}_p$-vector space of $\mathbb{C}_p$-valued harmonic cocycles on $\mathcal{T}$ is denoted $\mathrm{C}_{har}(\mathcal{T}, \mathbb{C}_p)$. The class of a rigid analytic differential $\omega \in \Omega^1_{\mathrm{rig}}(\mathcal{H}_p)$ in the de Rham cohomology of $\mathcal{H}_p$ is encoded in the $\mathbb{C}_p$-valued function c_ω on $\mathcal{E}(\mathcal{T})$ defined by

$$c_\omega(e) = \mathrm{res}_{\mathcal{W}_e}(\omega).$$

That c_ω is a harmonic cocycle follows directly from the residue theorem for rigid differentials. The oriented edges of $\mathcal{T}$ are also in natural bijection with the compact open balls in $\mathbb{P}_1(\mathbb{Q}_p)$, by assigning to $e \in \mathcal{E}(\mathcal{T})$ the ball U_e according to the following prescriptions:

$$U_{\bar{e}} \sqcup U_e = \mathbb{P}_1(\mathbb{Q}_p), \qquad U_{e_\infty} = \mathbb{P}_1(\mathbb{Q}_p) - \mathbb{Z}_p, \qquad U_{\gamma e} = \gamma U_e, \text{ for all } \gamma \in \Gamma,$$

where e_∞ is the distinguished edge of $\mathcal{E}(\mathcal{T})$ evoked in (6). The harmonic cocycle c_ω can therefore be parlayed into a $\mathbb{C}_p$-valued distribution μ_ω satisfying the defining property

$$\mu_\omega(U_e) = c_\omega(e),$$

where $U_e \subset \mathbb{P}_1(\mathbb{Q}_p)$ is the open ball corresponding to the ordered edge e. The distribution μ_ω lives in the dual space of locally constant $\mathbb{C}_p$-valued functions on $\mathbb{P}_1(\mathbb{Q}_p)$.

For this paragraph, and this paragraph only, let $\Gamma \subset \mathbf{SL}_2(\mathbb{Q}_p)$ be a group acting *discretely* on $\mathcal{H}_p$ and for which the quotient graph $\Gamma\backslash\mathcal{T}$ is finite. If the rigid differential ω is Γ-invariant, the harmonic cocycle c_ω takes on finitely many values and is therefore *p-adically bounded*. The distribution μ_ω then extends to a $\mathbb{C}_p$-valued measure, which can be integrated against continuous functions on $\mathbb{P}_1(\mathbb{Q}_p)$. The differential ω can then be obtained from μ_ω by the rule

$$\omega = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{\mathrm{d}z}{z-t}\, \mathrm{d}\mu_\omega(t), \tag{7}$$

a special case of Jeremy Teitelbaum's p-adic Poisson Kernel formula [Tei90] which recovers a rigid analytic modular form on $\mathcal{H}_p$ from its associated boundary distribution.

1.2. **Modular form-valued harmonic cocycles as mock residues.** Returning to the setting where $\Gamma = \mathbf{SL}_2(\mathbb{Z}[1/p])$ and to the dubious notion of a mock Hilbert modular form on $\Gamma\backslash(\mathcal{H}_p \times \mathcal{H})$ proposed in (3), the discussion in the previous section suggests at least what its system of p-adic annular residues ought to look like:

Definition 1.2. A *system of mock residues* is a harmonic cocycle

$$c : \mathcal{E}(\mathcal{T}) \longrightarrow \Omega^1(\mathcal{H})$$

with values in the space $\Omega^1(\mathcal{H})$ of holomorphic differentials on $\mathcal{H}$, satisfying

- $c(e)$ is a weight two cusp form on the stabiliser Γ_e of e in Γ, i.e., a holomorphic differential on the standard compactification of $\Gamma_e\backslash\mathcal{H}$;
- more generally, for all $\gamma \in \Gamma$ and all $e \in \mathcal{E}(\mathcal{T})$,

$$\gamma^* c(\gamma e) = c(e).$$

Roughly speaking, a system of mock residues is what one might expect to obtain from the p-adic annular residues of a mock Hilbert modular form of parallel weight two. But unlike (3), Definition 1.2 is completely rigorous. Since Γ acts transitively on the unordered edges of $\mathcal{T}$, and because the Hecke congruence group $\Gamma_0(p)$ is the stabiliser in Γ of the distinguished edge $e_\infty \in \mathcal{E}(\mathcal{T})$ of (6), the map $c \mapsto c(e_\infty)$ identifies the complex vector space $\mathcal{C}_{\mathrm{har}}(\mathcal{T}, \Omega^1(\mathcal{H}))^\Gamma$ of mock residues for Γ with the space $S_2(\Gamma_0(p))^{p\text{-new}}$ of weight two newforms of level p. It transpires that mock Hilbert modular forms – or at least, their systems of mock p-adic residues – are merely a slightly overwrought incarnation of classical modular forms of weight two on $\Gamma_0(p)$.

1.3. **$\mathbb{C}$-valued distributions.** Given a p-new weight two cusp form f on $\Gamma_0(p)$, denote by c_f the associated mock residue, and write $f_e := c_f(e) \in S_2(\Gamma_e)$. For any $x, y \in \mathcal{H}^* := \mathcal{H} \sqcup \mathbb{P}_1(\mathbb{Q})$, the assignment

$$e \mapsto 2\pi i \int_x^y f_e(z)\, \mathrm{d}z \tag{8}$$

is a $\mathbb{C}$-valued harmonic cocycle on $\mathcal{T}$, denoted $c_f[x, y]$. It determines a $\mathbb{C}$-valued distribution $\mu_f[x, y]$ on $\mathbb{P}_1(\mathbb{Q}_p)$, which can be integrated against locally constant complex-valued functions on $\mathbb{P}_1(\mathbb{Q}_p)$. In order to integrate $\mu_f[x, y]$ against Teitelbaum's p-adic Poisson kernel as in (7), the distribution $\mu_f[x, y]$ needs to be upgraded to a measure with suitable integrality properties.

1.4. **Modular symbols.** Suppose henceforth that f is a Hecke eigenform with rational fourier coefficients, and let $E_{/\mathbb{Q}}$ denote the corresponding strong Weil curve. The theory of modular symbols shows that the values of the harmonic cocycle $c_f[x, y]$ acquire good integrality properties when x, y belong to the boundary $\mathbb{P}_1(\mathbb{Q})$ of the extended upper half plane. More precisely, Manin and Drinfeld have shown that the values

$$\begin{aligned} c_f[r, s](e) &= 2\pi i \int_r^s f_e(z)\, \mathrm{d}z \qquad r, s \in \mathbb{P}_1(\mathbb{Q}) \\ &= 2\pi i \int_{\gamma r}^{\gamma s} f(z)\, \mathrm{d}z, \qquad \text{where } \gamma \in \Gamma \text{ satisfies } \gamma e = e_\infty \end{aligned}$$

belong to a lattice $\Lambda_f \subset \mathbb{C}$ which is commensurable with the period lattice Λ_E of E. Restricting the function $(x, y) \mapsto c_f[x, y]$ to $\mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$ leads to a modular symbol with values in the space of Λ_f-valued harmonic cocycles on $\mathcal{T}$. For economy of notation, the resulting Λ_f-valued measures on $\mathbb{P}_1(\mathbb{Q}_p)$ will continue to be denoted

$$\mu_f[r, s] \in \mathrm{Meas}(\mathbb{P}_1(\mathbb{Q}_p), \Lambda_f).$$

The Λ_f-valued measures $\mu_f[r, s]$ are intimately connected to special values of the Hasse–Weil L-series attached to E, via the formulae

$$\mu_f[0, \infty](\mathbb{Z}_p) = L(E, 1), \qquad \mu_f[0, \infty](\mathbb{Z}_p^\times) = (1 - a_p(E)) \cdot L(E, 1), \tag{9}$$

where $a_p(E) = 1$ or -1 depending on whether E has split or non-split multiplicative reduction at p. The Mazur–Swinnerton-Dyer p-adic L-function attached to E (viewed as taking values in $\mathbb{Q}_p \otimes \Lambda_f$) is the Mellin–Mazur transform of $\mu_f[0, \infty]$ restricted to $\mathbb{Z}_p^\times$:

$$L_p(E, s) = \int_{\mathbb{Z}_p^\times} \langle x \rangle^{s-1} d\mu_f[0, \infty](x). \tag{10}$$

More generally, if χ is a primitive Dirichlet character of conductor c prime to p, the twisted L-values $L(E, \chi, 1)$ can be obtained analogously from the measures $\mu_f[a/c, \infty]$:

$$\begin{aligned} \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \overline{\chi}(a) \cdot \mu_f[-a/c, \infty](\mathbb{Z}_p) &= \frac{c}{\tau(\chi)} \cdot L(E, \chi, 1), \\ \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \overline{\chi}(a) \cdot \mu_f[-a/c, \infty](\mathbb{Z}_p^\times) &= \frac{c}{\tau(\chi)} \cdot \big(1 - \overline{\chi}(p) a_p(E)\big) \cdot L(E, \chi, 1), \end{aligned}$$

where $\tau(\chi) = \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(a) e^{2\pi i \frac{a}{c}}$ is the Gauss sum attached to χ, while the Mazur–Swinnerton-Dyer p-adic L-function can be defined by setting

$$L_p(E, \chi, s) := \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(a) \int_{\mathbb{Z}_p^\times} \langle x \rangle^{s-1} d\mu_f[-a/c, \infty](x). \tag{11}$$

Even more importantly for the constructions that will follow, a system of $\Lambda_f \otimes \mathbb{C}_p$-valued rigid differentials on the p-adic upper half-plane can also be obtained by integrating the measures $\mu_f[r,s]$ against Teitelbaum's Poisson kernel:

$$\omega_f[r,s] \;:=\; \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{\mathrm{d}z}{z-t}\,\mathrm{d}\mu_f[r,s](t) \quad \in \quad \Omega^1_{\mathrm{rig}}(\mathcal{H}_p)\otimes\Lambda_f. \tag{12}$$

The assignment $\omega_f : (r,s) \mapsto \omega_f[r,s]$ determines a Γ-equivariant modular symbol with values in $\Omega^1_{\mathrm{rig}}(\mathcal{H}_p)\otimes\Lambda_f$, satisfying

$$\gamma^*\omega_f[\gamma r, \gamma s] = \omega_f[r,s], \qquad \text{for all } \gamma\in\Gamma.$$

1.5. **The Mazur–Tate–Teitelbaum conjecture.** Write C_p for the quadratic unramified extension of $\mathbb{Q}_p$, let $\mathcal{A}^\times$ denote the multiplicative group of non-zero rigid analytic functions on $\mathcal{H}_p$ endowed with the Γ-action induced by Möbius transformations, and let $\mathcal{A}^\times/C_p^\times$ be the quotient by the subgroup of constant $C_p^\times$-valued functions. The logarithmic derivative $F \mapsto \mathrm{d}F/F$ gives a Γ-equivariant map from $\mathcal{A}^\times/C_p^\times$ to $\Omega^1_{\mathrm{rig}}(\mathcal{H}_p)$, whose image contains the rigid differentials $\omega_f[r,s]$:

Lemma 1.3. *The differentials $\omega_f[r,s]$ are in the image of the logarithmic derivative map, i.e., there are elements $F_f[r,s]\in(\mathcal{A}^\times/C_p^\times)\otimes\Lambda_f$ satisfying*

$$\mathrm{dlog}\, F_f[r,s] = \omega_f[r,s]. \tag{13}$$

Proof. A *partitioning* of $\mathbb{P}_1(\mathbb{Q}_p)$ is a collection

$$\mathcal{C} = \{(C_1,t_1),\ldots,(C_m,t_m)\}, \tag{14}$$

where the C_j are compact open subsets of $\mathbb{P}_1(\mathbb{Q}_p)$ which are mutually disjoint and satisfy

$$\mathbb{P}_1(\mathbb{Q}_p) = C_1 \sqcup\cdots\sqcup C_m, \qquad t_j\in C_j \text{ for } j=1,\ldots,m.$$

The set of partitionings of $\mathbb{P}_1(\mathbb{Q}_p)$ is equipped with a natural partial ordering in which $\mathcal{C}\le\mathcal{C}'$ if each of the compact open subsets involved in $\mathcal{C}'$ is contained in one of the compact open subsets arising in $\mathcal{C}$. Let μ be a $\mathbb{Z}$-valued mesure of total measure zero on $\mathbb{P}_1(\mathbb{Q}_p)$. Each partitioning of $\mathbb{P}_1(\mathbb{Q}_p)$ gives rise to a system of degree zero divisors on $\mathbb{P}_1(\mathbb{Q}_p)$ by associating to the partitioning $\mathcal{C}$ of (14) the divisor

$$\mathscr{D}_{\mathcal{C}} := \sum_{j=1}^m \mu(C_j)\cdot[t_j].$$

Fix a base point $z_0\in\mathcal{H}_p(C_p)$ and let $F_{\mathcal{C}}$ be the unique rational function satisfying

$$\mathrm{Divisor}(F_{\mathcal{C}}) = \mathscr{D}_{\mathcal{C}}, \qquad F_{\mathcal{C}}(z_0)=1,$$

which exists because the divisor $\mathscr{D}_{\mathcal{C}}$ is supported on $\mathbb{P}_1(\mathbb{Q}_p)$. The limit

$$F_\mu := \lim_{\mathcal{C}} F_{\mathcal{C}} \in \mathcal{A}^\times$$

taken over any maximal chain in the set of partitionings, is a well-defined element of $\mathcal{A}^\times$ which depends only on μ and on the base point z_0, and whose image in $\mathcal{A}^\times/C_p^\times$ does not depend on the choice of z_0. Extending this construction to Λ_f-valued measures in the obvious way and applying it to the measures $\mu_f[r,s]$, it is readily verified that

$$F_f[r,s] := F_{\mu_f[r,s]} \in (\mathcal{A}^\times/C_p^\times)\otimes\Lambda_f$$

satisfies (13). $\square$

The assignment $(r,s) \mapsto F_f[r,s]$ defines a Γ-invariant modular symbol with values in the Γ-module $(\mathcal{A}^\times/C_p^\times) \otimes \Lambda_f$, i.e.,

$$F_f \in \mathrm{MS}\big((\mathcal{A}^\times/C_p^\times) \otimes \Lambda_f\big)^\Gamma,$$

where $\mathrm{MS}(\Omega)$ denotes the Γ-module of modular symbols with values in a Γ-module Ω. The obstruction to lifting F_f to $\mathrm{MS}(\mathcal{A}^\times \otimes \Lambda_f)^\Gamma$ is intimately tied with the p-adic uniformisation of the elliptic curve E which has multiplicative reduction at p. Namely, let $q \in \mathbb{Q}_p^\times$ be the p-adic Tate period of E. The following theorem is a consequence of the conjecture of Mazur, Tate and Teitelbaum [MTT86] and its proof by Greenberg and Stevens [GS93]:

Theorem 1.4. *There exists a lattice $\Lambda_f' \supset \Lambda_f$ such that the modular symbol F_f can be lifted to a Γ-invariant modular symbol with values in $(\mathcal{A}^\times/q^{\mathbb{Z}}) \otimes \Lambda_f'$.*

Sketch of proof. The functor $\mathrm{MS}(-)$ of modular symbols is exact, and taking Γ-cohomology gives the exact sequence

$$\mathrm{MS}\big((\mathcal{A}^\times/q^{\mathbb{Z}}) \otimes \Lambda_f\big)^\Gamma \xrightarrow{\ \eta\ } \mathrm{MS}\big((\mathcal{A}^\times/C_p^\times) \otimes \Lambda_f\big)^\Gamma \xrightarrow{\ \delta\ } H^1\big(\Gamma, \mathrm{MS}((C_p^\times/q^{\mathbb{Z}}) \otimes \Lambda_f)\big)$$

where $\ker(\eta)$ is annihilated by 12 because the abelianization of Γ is finite of exponent dividing 12 ([Ser80], II, 1.4). The obstruction to lifting F_f to $\mathrm{MS}\big((\mathcal{A}^\times/q^{\mathbb{Z}}) \otimes \Lambda_f\big)^\Gamma$ is encoded by its image, denoted c_f, in $H^1\big(\Gamma, \mathrm{MS}((C_p^\times/q^{\mathbb{Z}}) \otimes \Lambda_f)\big)$, which is represented by the 1-cocycle

$$\tilde{c}_f(\gamma) = \frac{\gamma \cdot \tilde{F}_f}{\tilde{F}_f}$$

where $\tilde{F}_f \in \mathrm{MS}(\mathcal{A}^\times \otimes \Lambda_f)$ is any lift of F_f. Let $\log_q$ be the branch the p-adic logarithm satisfying $\log_q(q) = 0$ which induces a map $C_p^\times/q^{\mathbb{Z}} \longrightarrow C_p$ with finite kernel. The claim of the theorem then reduces to the equality

$$\log_q(c_f) = 0$$

in $H^1\big(\Gamma, \mathrm{MS}(C_p \otimes \Lambda_f)\big)$, or equivalently to

$$\log(c_f) = \frac{\log(q)}{\mathrm{ord}_p(q)} \cdot \mathrm{ord}_p(c_f).$$

Now, [Dar01, Corollary 3.3 & Lemma 3.4] imply that $\mathrm{ord}_p(c_f)$ is non-trivial and that the two classes $\log(c_f)$ and $\mathrm{ord}_p(c_f)$ are proportional. The factor of proportionality is obtained by producing a suitable triple $(\gamma, r, s) \in \Gamma \times \mathbb{P}_1(\mathbb{Q})^2$ such that

$$\log(c_f)(\gamma)[r,s] = \frac{\log(q)}{\mathrm{ord}_p(q)} \cdot \mathrm{ord}_p(c_f)(\gamma)[r,s] \qquad \& \qquad \mathrm{ord}_p(c_f)(\gamma)[r,s] \neq 0, \tag{15}$$

and for which $b(\gamma)[r,s] = 0$ for any one-coboundary on Γ with values in $\mathrm{MS}(\mathbb{C}_p^\times)$. The latter property is satisfied when $\gamma \in \Gamma$ fixes r and s. The stabiliser in Γ of any pair $(r,s) \in \mathbb{P}_1(\mathbb{Q})^2$ is generated (up to torsion) by a hyperbolic matrix $\gamma_{r,s}$ which has powers of p as its eigenvalues and fixes the differential $\omega_f[r,s]$. The multiplicative period

$$J_f[r,s] := \tilde{c}_f(\gamma_{r,s})[r,s](z) = \frac{\tilde{F}_f[r,s](\gamma_{r,s}z)}{\tilde{F}_f[r,s](z)}, \quad \text{satisfying} \quad \log(J_f[r,s]) = \int_z^{\gamma_{r,s}z} \omega_f[r,s],$$

does not depend on the base point $z \in \mathcal{H}_p(C_p)$ and belongs to $\mathbb{Q}_p^\times \otimes \Lambda_f$ [Dar01, Prop. 2.7]. When $(r,s) = (0,\infty)$, the period is related to the central critical value $L(E,1)$ and to the first derivative of the Mazur–Swinnerton-Dyer p-adic L-function $L_p(E,s)$ attached to E in (10):

$$\mathrm{ord}_p(J_f[0,\infty]) = \delta_p(E) \cdot L(E,1), \qquad \log(J_f[0,\infty]) = \delta_p(E) \cdot L_p'(E,1),$$

where $\delta_p(E) = 1$ if $a_p(E) = 1$ and $\delta_p(E) = 0$ if $a_p(E) = -1$ (cf. [Dar01, §2.2 & 2.3]). As the Mazur–Tate–Teitelbaum conjecture asserts that

$$L'_p(E, 1) = \frac{\log(q)}{\mathrm{ord}_p(q)} \cdot L(E, 1),$$

we deduce that $J_f[0, \infty]$ belongs to $q^{\mathbb{Z}} \otimes \Lambda'_f$, after letting

$$\Lambda'_f := \frac{1}{t}\Lambda_f, \text{ where } t = \#(C_p^\times)_{\mathrm{tors}} \cdot \mathrm{ord}_p(q).$$

More generally, the valuations and logarithms of the periods $J_f[\infty, a/c]$ with $\gcd(a, c) = 1$ can be expressed in terms the special values (resp. derivatives) of partial L-series (resp. partial p-adic L-series) whose linear combinations give all the twisted values $L(E, \chi, 1)$ and $L'_p(E, \chi, 1)$ defined in (11), as χ ranges over all primitive Dirichlet characters of conductor c for which $\chi(p) = a_p(E)$ [Dar01, §2.2 & 2.3]. The Mazur–Tate–Teitelbaum conjecture for these L-series and the non-vanishing result of [Dar01, Lemma 2.17] implies that the collection of all $J_f[r, s]$ are contained in Λ'_f and generate a non-trivial lattice. Therefore (15) can be achieved and the theorem follows. □

Explicitly, Theorem 1.4 ensures the existence of a lattice $\Lambda'_f \supset \Lambda_f$ such that F_f is a Γ-invariant modular symbol with values in $(\mathcal{A}^\times/q^{\mathbb{Z}}) \otimes \Lambda'_f$ satisfying

- $\mathrm{dlog}\, F_f[r, s] = \omega_f[r, s]$, for all $r, s \in \mathbb{P}_1(\mathbb{Q})$;
- $F[\gamma r, \gamma s](\gamma z) = F[r, s](z) \pmod{q^{\mathbb{Z}} \otimes \Lambda'_f}$, for all $\gamma \in \Gamma,\ r, s \in \mathbb{P}_1(\mathbb{Q}),$ and $z \in \mathcal{H}_p$.

The statement that the (multiplicative) periods $J_f[r, s]$ of the "mock Hilbert modular form" attached to E lie in a lattice commensurable with $q^{\mathbb{Z}} \otimes \Lambda_E$ resonates with Oda's period conjecture for Hilbert modular surfaces. The emergence of the Tate period in what had, up to now, been a rather formal sequence of constructions provides the first inkling that the point of view of mock Hilbert modular surfaces opens genuinely new perspectives on arithmetic questions related to f and its associated elliptic curve E.

2. Stark–Heegner points

A *real multiplication* (RM) point on $\mathcal{H}_p$ is an element $\tau \in \mathcal{H}_p$ which also lies in a real quadratic field K. Its *associated order* is the subring

$$\mathcal{O}_\tau := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2\big(\mathbb{Z}[1/p]\big) \text{ satisfying } a\tau + b = c\tau^2 + d\tau \right\} \tag{16}$$

of the matrix ring $\mathrm{M}_2\big(\mathbb{Z}[1/p]\big)$. This commutative ring is identified with a $\mathbb{Z}[1/p]$-order in K by sending a matrix in (16) to its automorphy factor $c\tau + d$. Global class field theory associates to any $\mathbb{Z}[1/p]$-order $\mathcal{O} \subset K$ an abelian extension $H_{\mathcal{O}}$ (resp. $H^+_{\mathcal{O}}$) of K whose Galois group over K is identified with the Picard group (resp. the narrow Picard group) of projective $\mathcal{O}$-modules (resp. of projective $\mathcal{O}$-modules endowed with an orientation at ∞):

$$\mathrm{Gal}(H_{\mathcal{O}}/K) = \mathrm{Pic}(\mathcal{O}), \qquad \mathrm{Gal}(H^+_{\mathcal{O}}/K) = \mathrm{Pic}^+(\mathcal{O}).$$

The stabiliser in Γ of the RM point $\tau \in \mathcal{H}_p$ is identified with the group of norm one elements in $\mathcal{O}_\tau$. Since p is non-split in $K = \mathbb{Q}(\tau)$, this stabiliser is of rank one. The choice of a fundamental unit of K, which is fixed once and for all, determines a generator γ_τ of the stabiliser of τ modulo torsion. The *Stark–Heegner point* attached to τ is the element

$$P_\tau := F[r, \gamma_\tau r](\tau) \in (\mathbb{C}_p^\times/q^{\mathbb{Z}}) \otimes \Lambda'_f = E(\mathbb{C}_p) \otimes \Lambda'_f.$$

The definition of P_τ ostensibly rests on the choice of an auxiliary base point $r \in \mathbb{P}_1(\mathbb{Q})$ but is ultimately independent of that choice. After choosing real and imaginary generators Ω_f^+ and Ω_f^- of $\Lambda'_f \cap \mathbb{R}$ and $\Lambda'_f \cap i\mathbb{R}$ and writing

$$P_\tau = P_\tau^+ \cdot \Omega_f^+ + P_\tau^- \cdot \Omega_f^-,$$

the invariants P_τ^+ and $P_\tau^- \in E(\mathbb{C}_p)$ are conjectured to satisfy the following [Dar01]:

Conjecture 2.1. *The points P_τ^+ and P_τ^- are defined over the ring class field $H_{\mathcal{O}_\tau}$ and the narrow ring class field $H^+_{\mathcal{O}_\tau}$ respectively. The point P_τ^- is in the minus part for the action of complex conjugation on $E(H^+_{\mathcal{O}_\tau})$.*

The points $P_\tau^\pm$ are expected to behave in most key respects just like classical Heegner points over ring class fields of imaginary quadratic fields; in particular they should satisfy an analogue of the Gross–Zagier formula. Stark–Heegner points are the "mock" counterpart of the "ATR points" on elliptic curves over real quadratic fields, arising from topological one-cycles on a genuine Hilbert modular surface, that were alluded to in equation (2) of the introduction. The properties of the points $P_\tau^\pm$ predicted in Conjecture 2.1 are poorly understood, just as they are for their ATR counterparts, in spite of the theoretical evidence obtained in [BD09], [LV14], [LMY20], and [BDRSV21] for instance.

Remark 2.2. The construction of Stark–Heegner points has been generalised to various different settings over the years, notably in [Tr06], [Gr09], [GMS15], [GMM20], and [FG23(a)].

Conjecture 2.1 is consistent with the Birch and Swinnerton-Dyer conjecture, since the sign in the functional equation for $L(E/K, s)$ is always -1 for E an elliptic curve (or modular abelian variety) of conductor p and K a real quadratic field in which p is inert. The same is true as well for the L-functions $L(E/K, \chi, s)$ twisted by ring class characters χ of prime-to-p conductor. It follows that

$$\mathrm{ord}_{s=1} L(E/K, \chi, s) \geq 1, \qquad \text{for all } \chi : \mathrm{Gal}(H_{\mathcal{O}_\tau}/K) \longrightarrow \mathbb{C}^\times,$$

and hence that

$$\mathrm{ord}_{s=1} L(E/H_\tau, s) \geq [H_{\mathcal{O}_\tau} : K].$$

The Stark–Heegner point construction gives a conjectural analytic recipe for the systematic supply of non-trivial global points over ring class fields of K whose existence is predicted by the Birch and Swinnerton-Dyer conjecture.

3. Mock plectic invariants

A remarkable insight of Nekovář and Scholl suggests that zero-dimensional CM cycles on Hilbert modular surfaces should encode determinants of global points for elliptic curves of rank two called "plectic Heegner points". This suggests that the CM points on $\mathcal{H}_p \times \mathcal{H}$ are just as interesting arithmetically as the RM points on $\mathcal{H}_p$ that lead to Stark–Heegner points. The goal of this last chapter is to describe the "mock plectic invariants" attached to CM zero-cycles on the mock Hilbert surface $\mathcal{S}$ and to explore the relevance of these invariants for the arithmetic of elliptic curves of rank two.

Let K be a quadratic imaginary field, viewed simultaneously as a subfield of $\mathbb{C}_p$ and $\mathbb{C}$, and embedded diagonally in $\mathbb{C}_p \times \mathbb{C}$. A point $\tau = (\tau_p, \tau_\infty) \in (\mathcal{H}_p \times \mathcal{H}) \cap K$ is called a *CM point* on $\mathcal{S}$ attached to K. For simplicity, it shall be assumed henceforth that its associated order, defined as in (16), is the maximal $\mathbb{Z}[1/p]$-order in the imaginary quadratic field K, that this order has class number one, and that the prime p in inert in K, leaving aside the slightly more delicate case where p is ramified.

In contrast with the setting for Conjecture 2.1 and the discussion following it, the sign in the functional equation for $L(E/K, \chi, s)$ is now systematically equal to 1, for any ring class character χ of K of prime-to-p conductor. The Birch and Swinnerton-Dyer conjecture therefore predicts that $E(K)$ has *even rank.* A systematic supply of Heegner points over K or over ring class fields of conductor prime to p is therefore not expected to arise in this setting. Rather, the "plectic Stark-Heegner point" attached to τ will be used to prove the implication

$$L(E/K,1) \neq 0 \quad \Rightarrow \quad E(K) \text{ is finite,}$$

and it conjecturally remains non-trivial when $\mathrm{ord}_{s=1}L(E/K,s) = 2$.

3.1. **$E(\mathbb{C})$-valued harmonic cocycles.** To parlay the system c_f of mock residues attached to f into a rigorous evaluation of the plectic invariant attached to τ, it is natural to replace the Γ-stable subset $\mathbb{P}_1(\mathbb{Q}) \subset \mathcal{H}^*$ of Section 1.4 by the Γ-orbit

$$\Sigma := \Gamma\tau_\infty$$

of τ_∞ in $\mathcal{H}$. For each pair $(x,y) \in \Sigma^2$, one obtains a $\mathbb{C}$-valued harmonic cocycle on $\mathcal{T}$ via (8), denoted $c_f[x,y]$. The collection of $c_f[x,y]$ as x,y vary over Σ satisfies the Γ-equivariance property

$$c_f[\gamma x, \gamma y](\gamma e) = c_f[x,y](e), \qquad \text{for all } \gamma \in \Gamma, \ x,y \in \Sigma, \text{ and } e \in \mathcal{E}(\mathcal{T}).$$

Recall that Λ_f is a lattice in $\mathbb{C}$ containing all the periods of the form $\int_r^s f(z)dz$ with $r,s \in \mathbb{P}_1(\mathbb{Q})$, and that E has been replaced by the isogenous curve with period lattice Λ_f, which is possible by the Manin-Drinfeld theorem. The theory of modular symbols can be invoked to obtain a Γ-equivariant collection $\{c_f[x]\}_{x\in\Sigma}$ of harmonic cocycles, indexed by a single $x \in \Sigma$, but with values in $\mathbb{C}/\Lambda_f = E(\mathbb{C})$, satisfying

$$c_f[x,y] = c_f[y] - c_f[x] \pmod{\Lambda_f}, \qquad \text{for all } x,y \in \Sigma.$$

This is done by setting

$$c_f[x](e) = \int_{i\infty}^{x} f_e(z)\,\mathrm{d}z \;=\; \int_{i\infty}^{\gamma x} f(z)\,\mathrm{d}z \pmod{\Lambda_f}, \qquad \text{where } \gamma e = e_\infty \text{ for } \gamma \in \Gamma. \tag{17}$$

As in Section 1.3, the harmonic cocycle $c_f[x]$ gives rise to an $E(\mathbb{C})$-valued distribution on $\mathbb{P}_1(\mathbb{Q}_p)$, denoted $\mu_f[x]$, which can only be integrated against locally constant $\mathbb{Z}$-valued functions on $\mathbb{P}_1(\mathbb{Q}_p)$.

3.2. **The isogeny tree of a CM curve.** The eventual upgrading of $\mu_f[x]$ to a measure is based on the observation that the values in (17) can be interpreted as Heegner points on $E_{/\mathbb{Q}}$ attached to CM points of p-power conductor on the modular curve $X_0(p)$. These points are defined over the anticyclotomic extension

$$K_\infty = \bigcup_{n=0}^{\infty} K_n,$$

where K_n is the ring class field of K of conductor p^n. This field is totally ramified at the (unique) prime of K above p. Write

$$G_n = \mathrm{Gal}(K_n/K), \qquad G_\infty = \mathrm{Gal}(K_\infty/K) = \varprojlim G_n.$$

Global class field theory identifies G_∞ with $K_{p,1}^\times$ the group of norm one elements,

$$\mathrm{rec}\colon K_{p,1}^\times \xrightarrow{\sim} G_\infty.$$

Let A be the elliptic curve over $\overline{\mathbb{Q}}$ with complex multiplication by the maximal order $\mathcal{O}_K$. It is unique up to isomorphism over $\overline{\mathbb{Q}}$ and has a model over $\mathbb{Q}$ because of the running class number one assumption. Let $\mathcal{T}_A$ be the p-isogeny graph of A, whose vertices are elliptic curves over $\overline{\mathbb{Q}}$ related to A by a cyclic isogeny of p-power degree, and whose edges correspond to p-isogenies. This graph is a tree of valency $(p+1)$ with a distinguished vertex v_A attached to A. Since the elliptic curves that are p-power isogenous to A are all defined over K_∞, the

Galois group $G_\infty = K_{p,1}^\times$ acts on $\mathcal{T}_A$ in the natural way. This action fixes v_A and transitively permutes all the vertices (or edges) that lie at a fixed distance from v_A. More precisely, for every $n \geq 0$ the subgroup $\mathscr{U}_n \leq K_{p,1}^\times$ attached to the ring class field K_n under the Galois correspondence stabilizes all the vertices of $\mathcal{T}_A$ at distance n from v_A. Choose a sequence of adjacent vertices $\{v_n\}_{n\geq 0}$ satisfying

$$\mathscr{U}_n = \mathrm{Stab}_{K_{p,1}^\times}(v_n).$$

For every $n \geq 1$ the oriented edge $e_n = (v_{n-1}, v_n)$ from v_{n-1} to v_n satisfies

$$\mathscr{U}_n = \mathrm{Stab}_{K_{p,1}^\times}(e_n).$$

The vertices (resp. edges) of $\mathcal{T}_A$ at distance n from v_A are in bijection with the $G_n = K_{p,1}^\times/\mathscr{U}_n$-orbit of v_n (resp. e_n). It is convenient to interpret each vertex of $\mathcal{T}_A$ as a point on the j-line $X_0(1)$, and to view each edge as a point on the modular curve $X_0(p)$, the coarse moduli space of pairs of elliptic curves related by a p-isogeny. For $n \geq 1$ let $P_n \in X_0(p)(K_n)$ be the point corresponding to the ordered edge e_n. Since $\mathscr{U}_n/\mathscr{U}_{n+1}$ acts simply transitively on the set of edges at distance $n+1$ from v_A having v_n as an endpoint, it follows that

$$\mathrm{Tr}_{K_{n+1}/K_n}(P_{n+1}) = U_p(P_n) \qquad \forall\ n \geq 1.$$

Remark 3.1. Recall we previously defined $a_p(E) = 1$ or -1 depending on whether E has split or non-split multiplicative reduction at p. If we write $y_n \in E(K_n)$ for the Heegner point arising from the divisor $a_p(E)^n \cdot (P_n - \infty)$ on $X_0(p)$ through the modular parametrization $\varphi_E \colon X_0(p) \to E$ (normalized by $\varphi_E(\infty) = 0_E$), then the collection $\{y_n \in E(K_n)\}_{n\geq 1}$ is trace-compatible.

Fix a trivialisation $H_1(A(\mathbb{C}), \mathbb{Z}_p) \simeq \mathbb{Z}_p^2$. The choice of a complex embedding $\iota_\infty : K_\infty \hookrightarrow \mathbb{C}$ together with Shimura's reciprocity law determine a $K_{p,1}^\times$-equivariant graph isomorphism

$$j_A : \mathcal{T}_A \xrightarrow{\sim} \mathcal{T}$$

which sends the vertex attached to A' to the lattice $H_1(A'(\mathbb{C}), \mathbb{Z}_p) \subseteq H_1(A(\mathbb{C}), \mathbb{Q}_p) = \mathbb{Q}_p^2$, after viewing A' as a curve over $\mathbb{C}$ via ι_∞. In particular, j_A maps the distinguished vertex v_A to $v_\circ$. The identification j_A allows the harmonic cocycle $c_f[\tau_\infty]$ to be viewed as taking values in $E(K_\infty)$. More precisely, [Dar01, Lemma 1.5] and equation (17) give

$$c_f[\tau_\infty](\alpha \cdot e_n) = y_n^{\mathrm{rec}(\alpha)}, \qquad \forall\ n \geq 1,\ \alpha \in K_{p,1}^\times. \tag{18}$$

The general case of $\tau = \gamma^{-1}\tau_\infty \in \Sigma$ is dealt with by the formula

$$c_f[\tau](e) = c_f[\tau_\infty](\gamma e) \qquad \forall\ e \in \mathcal{E}(\mathcal{T}).$$

3.3. **Measures and the Poisson transform.** In order to integrate continuous function with respect to the measure $\mu_f[\tau_\infty]$, the value group $E(K_\infty)$ needs to be p-adic completed. There is a natural map from the (infinitely generated) Mordell-Weil group $E(K_\infty)$ to its p-adic completion

$$\widehat{E(K_\infty)} := \lim_{\leftarrow, n} E(K_\infty) \otimes \mathbb{Z}/p^n\mathbb{Z},$$

which will be shown to be injective.

Viewing $\mu_f[x]$ (for $x \in \Sigma$) as an $\widehat{E(K_\infty)}$-valued measure on $\mathbb{P}_1(\mathbb{Q}_p)$, the Teitelbaum transform of $\mu_f[x]$ gives a collection of elements

$$\omega_f[x] := \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{\mathrm{d}z}{z-t}\, \mathrm{d}\mu_f[x](t) \quad \in \quad \Omega^1_{\mathrm{rig}}(\mathcal{H}_p)\widehat{\otimes}\widehat{E(K_\infty)}. \tag{19}$$

For any $x \in \Sigma$, let

$$\iota_x : K \longrightarrow M_2(\mathbb{Q})$$

be the algebra embedding that sends K to the fraction field of the order $\mathcal{O}_x$. The group Γ acts on $\Omega^1_{\mathrm{rig}}(\mathcal{H}_p)\widehat{\otimes}\widehat{E(K_\infty)}$ by translation on $\mathcal{H}_p$, and the Galois group G_∞ acts via its natural action on $\widehat{E(K_\infty)}$.

Proposition 3.2. *The $\widehat{E(K_\infty)}$-valued rigid differentials $\omega_f[x]$ satisfy the following properties:*

- *For all $\gamma \in \Gamma$, $x \in \Sigma$*
$$\gamma^*\omega_f[\gamma x] = \omega_f[x].$$
- *For all $\alpha \in K_{p,1}^\times$,*
$$\iota_x(\alpha)^*(\omega_f[x]) = \mathrm{rec}(\alpha)\omega_f[x].$$

The second part of this proposition is particularly noteworthy: setting $x = \tau_\infty$, it relates the action of the p-adic torus $\iota_\tau(K_p^\times)$ on $\mathcal{H}_p$, which fixes τ_p, to the Galois action on the elements $\omega_f[\tau] \in \Omega^1_{\mathrm{rig}}\hat{\otimes}\widehat{E}(K_\infty)$, and is just a reformulation of the Shimura reciprocity law.

3.4. **The mock plectic invariant.** Following the same ideas as in the proof of Lemma 1.3, a well-defined system of multiplicative primitives
$$F_f[x] \in (\mathcal{A}^\times/\mathbb{C}_p^\times)\hat{\otimes}\widehat{E}(K_\infty)$$
can be attached to the elements $\omega_f[x]$, satisfying
$$\mathrm{dlog}(F_f[x]) = \omega_f[x], \qquad \text{for all } x \in \Sigma.$$
The torus $\iota_\tau(K_p^\times)$ has *two* fixed points τ_p, $\overline{\tau}_p$ acting on $\mathcal{H}_p$, they are interchanged by the action of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$. This circumstance leads to the definition of the multiplicative Nekovář–Scholl *mock plectic invariant* attached to the CM point τ, by setting
$$\mathcal{Q}^\times(\tau) := \frac{F_f[\tau_\infty](\tau_p)}{F_f[\tau_\infty](\overline{\tau}_p)} \quad \in \quad \widehat{E(K_\infty)}\widehat{\otimes}K_{p,1}^\times.$$
Since p is inert in K, the group $K_{p,1}^\times$ consists of p-adic units and the invariant $\mathcal{Q}^\times(\tau)$ is almost completely determined by its p-adic logarithm
$$\mathcal{Q}(\tau) := \log \mathcal{Q}^\times(\tau) = \int_{\overline{\tau}_p}^{\tau_p} \omega_f[\tau_\infty] \quad \in \quad \widehat{E(K_\infty)}\widehat{\otimes}K_p.$$

Lemma 3.3. *The mock plectic invariant $\mathcal{Q}(\tau)$ belongs to $\left(\widehat{E(K_\infty)}\widehat{\otimes}K_p\right)^{G_\infty}$.*

Proof. For all $\alpha \in K_{p,1}^\times$, we have
$$\mathrm{rec}(\alpha)\mathcal{Q}(\tau) = \int_{\overline{\tau}_p}^{\tau_p} \mathrm{rec}(\alpha)\omega_f[\tau] = \int_{\overline{\tau}_p}^{\tau_p} \iota_\tau(\alpha)^*\omega_f[\tau] = \mathcal{Q}(\tau),$$
where the penultimate equality follows from the second assertion in Proposition 3.2, and the last from the change of variables formula and the fact that $\iota_\tau(K_p^\times)$ fixes both τ_p and $\overline{\tau}_p$. □

3.5. **Anticyclotomic p-adic L-functions.** We will now give a formula for the mock plectic invariant $\mathcal{Q}(\tau)$ in terms of the first derivatives of certain "anticyclotomic p-adic L-functions" in the sense of [BD97].

Recall that the CM point $\tau = (\tau_p, \tau_\infty)$ determines an embedding $K_p \subseteq \mathrm{M}_2(\mathbb{Q}_p)$ and hence an action of $K_p^\times$ on $\mathbf{P}_1(\mathbb{Q}_p)$. Let
$$A : \mathbb{P}_1(\mathbb{Q}_p) \longrightarrow K_{p,1}^\times, \qquad A(x) = \frac{x - \tau_p}{x - \overline{\tau}_p} \tag{20}$$
be the Möbius transformation that sends $(\tau_p, \overline{\tau}_p, \infty)$ to $(0, \infty, 1)$, and let $\mu_{f,K}$ be the pushforward of the measure $\mu_f[\tau_\infty]$ to $K_{p,1}^\times$ via A:
$$\mu_{f,K} := A_*\mu_f[\tau_\infty].$$

The *anticyclotomic p-adic L-function* attached to (E, K) is the Mazur-Mellin transform of the measure $\mu_{f,K}$:

$$L_p(E, K, s) := \int_{K_{p,1}^\times} \langle\alpha\rangle^{s-1} d\mu_{f,K}(\alpha). \tag{21}$$

It can be viewed as a p-adic analytic function from $1 + p\mathbb{Z}_p$ to $\widehat{E(K_\infty)}\hat{\otimes}K_p$.

Theorem 3.4. *The p-adic L-function $L_p(E, K, s)$ vanishes at $s = 1$ and*

$$\mathcal{Q}(\tau) = L_p'(E, K, 1).$$

Proof. The vanishing of $L_p(E, K, 1)$ follows from the fact that $\mu_f[\tau_\infty]$, and hence also $\mu_{f,K}$, have total measure zero. By the definition of $\mathcal{Q}(\tau)$ combined with (19),

$$\mathcal{Q}(\tau) = \int_{\bar{\tau}_p}^{\tau_p} \omega_f[\tau_\infty] = \int_{\bar{\tau}_p}^{\tau_p} \left(\int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_f[\tau_\infty](t) \right) dz.$$

Interchanging the order of integration and integrating with respect to z gives

$$\mathcal{Q}(\tau) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \log\left(\frac{\tau_p - t}{\bar{\tau}_p - t}\right) d\mu_f[\tau_\infty](t) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \log A(t) \cdot d\mu_f[\tau_\infty](t).$$

The change of variables $\alpha = A(t)$ can be used to rewrite this last expression as an integral over $K_{p,1}^\times$,

$$\mathcal{Q}(\tau) = \int_{K_{p,1}^\times} \log(\langle\alpha\rangle) \cdot d\mu_{f,K}(\alpha) = L_p'(E, K, 1),$$

where the last equality follows directly from (21). □

As explained in [BD97], the quantity $L_p'(E, K, 1)$ is directly related to the *Kolyvagin derivative* of the norm-compatible collection $y_n = c_f[\tau_\infty](e_n) \in E(K_n)$ of Heegner points, where $\{e_n\}_{n\geq 1} \subset \mathcal{E}(\mathcal{T})$ is the sequence of adjacent edges that was fixed in Section 3.2. More precisely, for every $n \geq 1$, the collection $\{\alpha \cdot U_{e_n}\}_{\alpha \in K_{p,1}^\times}$ is a finite covering of $\mathbb{P}_1(\mathbb{Q}_p)$ by compact open subsets, which are equal to the cosets $\{\alpha \cdot \mathscr{U}_n\}_{\alpha\in K_{p,1}^\times}$ under the homeomorphism (20). Rewriting $L_p'(E, K, 1)$ as the limit when $n \to \infty$ of the Riemann sums attached to these coverings, equation (18) gives

$$\begin{aligned}
\mathcal{Q}(\tau) &= \int_{K_{p,1}^\times} \log(\alpha) d\mu_{f,K}(\alpha) \\
&= \lim_{n\to\infty} \sum_{\alpha\in K_{p,1}^\times/\mathscr{U}_n} c_f[\tau_\infty](\alpha \cdot e_n) \otimes \log(\langle\alpha\rangle) \\
&= \lim_{n\to\infty} \sum_{\alpha\in K_{p,1}^\times/\mathscr{U}_n} y_n^{\mathrm{rec}(\alpha)} \otimes \log(\langle\alpha\rangle)
\end{aligned} \tag{22}$$

(see [BD98, §6] for related discussions).

3.6. **Kolyvagin's cohomology classes.** After choosing a topological generator of $K_{p,1}^\times$, Lemma 3.3 allows the mock plectic invariant $\mathcal{Q}(\tau)$ to be viewed (non-canonically) as an element of $\widehat{E(K_\infty)}$ fixed by G_∞. Consider the natural injective map

$$E(K) \otimes \mathbb{Z}_p \longrightarrow \widehat{E(K_\infty)}^{G_\infty}.$$

It need not be surjective in general: the group $\widehat{E(K_\infty)}$ fails to satisfy the principle of Galois descent. Moreover, while it is relatively straightforward to establish the infinitude of

$\widehat{E(K_\infty)}^{G_\infty}$, the Birch and Swinnerton-Dyer conjecture predicts the finitude of $E(K)$ when $L(E/K,1) \neq 0$. Even when $L(E/K,s)$ vanishes at $s=1$, and hence has a zero of order ≥ 2, establishing that $E(K)\otimes\mathbb{Z}_p$ is infinite seems very hard to do unconditionally.

A useful handle on the group $\widehat{E(K_\infty)}^{G_\infty}$ is obtained by relating it to Galois cohomology. Let $H^1(K_m, E[p^n])$ be the first Galois cohomology of K_m with values in the module of p^n-division points of E, and let

$$H^1(K_m, T_p(E)) = \lim_{\leftarrow, n} H^1(K_m, E[p^n]),$$

the inverse limit being taken relative to the multiplication by p maps $E[p^{n+1}] \longrightarrow E[p^n]$.

Lemma 3.5. *The module $E[p^n](K_m)$ is trivial for all m and n.*

Proof. The mod p Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ attached to E contains a non-trivial unipotent element in the image of inertia at p, its determinant is surjective, and it is irreducible since $p \geq 11$. It follows that the mod p Galois representation is surjective, and in particular its image is non-solvable. Therefore $E[p](L)$ is trivial for any solvable extension L of $\mathbb{Q}$. The lemma follows. $\square$

Lemma 3.6. *The restriction map*

$$H^1(K, E[p^n]) \longrightarrow H^1(K_m, E[p^n])^{G_m}$$

is an isomorphism.

Proof. The kernel and cokernel in the inflation-restriction sequence

$$H^1(G_m, E[p^n](K_m)) \longrightarrow H^1(K, E[p^n]) \xrightarrow{\text{res}} H^1(K_m, E[p^n])^{G_m} \longrightarrow H^2(G_m, E[p^n](K_m))$$

are trivial, by Lemma 3.5, and the claim follows. $\square$

Denote by δ_n the mod p^n Kummer map

$$\delta_n : E(K_m)/p^n E(K_m) \longrightarrow H^1(K_m, E[p^n]),$$

and by

$$\delta_\infty : E(K_m)\otimes\mathbb{Z}_p \longrightarrow H^1(K_m, T_p(E))$$

the map induced on the inverse limits. Let $P_\infty = \{P_n\}_{n\geq 1} \in \widehat{E(K_\infty)}^{G_\infty}$, where

$$P_n \in (E(K_{m_n}) \otimes \mathbb{Z}/p^n\mathbb{Z})^{G_{m_n}}$$

for suitable integers $m_n \geq 1$. Lemma 3.6 shows that each $\delta_n(P_n) \in H^1(K_{m_n}, E[p^n])^{G_{m_n}}$ is the restriction of a unique class $\kappa_n \in H^1(K, E[p^n])$. The κ_n are compatible under the natural multiplication by p maps from $E[p^{n+1}]$ to $E[p^n]$, and the assignment $(P_n)_{n\geq 1} \mapsto (\kappa_n)_{n\geq 1}$ determines a canonical inclusion

$$\widehat{E(K_\infty)}^{G_\infty} \hookrightarrow H^1(K, T_p(E)).$$

It will be convenient for the rest of this note to identify $\mathfrak{Q}(\tau)$ with its image in $H^1(K, T_p(E))$ under this map.

3.7. **Local properties of the mock plectic invariant.** For each place v of K, let

$$\mathrm{res}_v : H^1(K, T_p(E)) \longrightarrow H^1(K_v, T_p(E))$$

be the local restriction map to a decomposition group at v. The local cohomology group $H^1(K_p, T_p(E))$ is equipped with a natural two-step filtration

$$0 \longrightarrow H^1_f(K_p, T_p(E)) \longrightarrow H^1(K_p, T_p(E)) \longrightarrow H^1_{\mathrm{sing}}(K_p, T_p(E)) \longrightarrow 0,$$

where $H^1_f(K_p, T_p(E)) := \delta_\infty(E(K_p)\otimes\mathbb{Z}_p)$.

Definition 3.7. The *pro-p-Selmer group* of E is the group, denoted $H^1_f(K, T_p(E))$, of global classes $\kappa \in H^1(K, T_p(E))$ satisfying

$$\mathrm{res}_p(\kappa) \in H^1_f(K_p, T_p(E)).$$

The reader will note that in the setting of pro-p Selmer groups, no local conditions need to be imposed at the places $v \neq p$. This is because the natural map $\delta_\infty : E(K_v) \otimes \mathbb{Z}_p \longrightarrow H^1(K_v, T_p(E))$ is an isomorphism. The only possible obstruction for a global class to lie in the pro-p Selmer group therefore occurs at the place p. Let

$$\mathcal{Q}_p(\tau) := \mathrm{res}_p(\mathcal{Q}(\tau)) \in H^1(K_p, T_p(E))$$

denote the restriction of the global class $\mathcal{Q}(\tau)$ to the decomposition group at p. The first important result about the mock plectic invariant is that it is non-Selmer precisely when $L(E/K, 1) \neq 0$.

Theorem 3.8. *The class $\mathcal{Q}_p(\tau)$ lies in $H^1_f(K_p, T_p(E))$ if and only if $L(E/K, 1) = 0$.*

Sketch of proof. Recall that the ring class field K_n of K of conductor p^n is a cyclic Galois extension of K with Galois group $G_n \cong K^\times_{p,1}/\mathcal{U}_n$. The Kolyvagin derivative of the Heegner point $y_n \in E(K_n)$ is defined as

$$D_n y_n := \sum_{\alpha \in K^\times_{p,1}/\mathcal{U}_n} y_n^{\mathrm{rec}(\alpha)} \otimes \alpha \quad \in \quad E(K_n) \otimes K^\times_{p,1}/\mathcal{U}_n$$

and is fixed by the action of G_n because $\mathrm{Tr}_{K_n/K}(y_n) = 0$ (cf. [BD96, (8)] and [Gro91, Proposition 3.6]). Then, just as in equation (22) one sees that

$$\mathcal{Q}^\times(\tau) = \lim_{\leftarrow, n} D_n y_n.$$

By choosing a topological generator of $K^\times_{p,1}$ we can consider the image of $\mathcal{Q}^\times(\tau)$ in $H^1(K, T_p(E))$ and study its local properties via the following diagram taken from [Gro91, (4.2)]:

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & & 0 \arrow[d] & H^1(G_n, E(K_n)) \arrow[d, "\mathrm{Inf}"] \\
E(K_n)^{\mathrm{Tr}_{K_n/K}=0} \arrow[dr, "D_n"] & & H^1(K, E[p^{n-1}]) \arrow[r] \arrow[d, "\mathrm{Res}"] & H^1(K, E) \arrow[d, "\mathrm{Res}"] \\
0 \arrow[r] & \big(E(K_n)/p^{n-1}\big)^{G_n} \arrow[r, "\delta"] & H^1(K_n, E[p^{n-1}])^{G_n} \arrow[r] \arrow[d] & H^1(K_n, E)^{G_n} \\
 & & 0 & .
\end{tikzcd}$$

By construction the image of $D_n y_n$ in $H^1(K, E)$ belongs to the image of $H^1(G_n, E(K_n))$ under inflation and it can be represented by the following 1-cocycle [McC91, Lemma 4.1]:

$$G_n \ni \sigma \mapsto -\frac{(\sigma - 1)D_n(y_n)}{p^{n-1}}, \tag{23}$$

where $\frac{(\sigma-1)D_n(y_n)}{p^{n-1}}$ denotes the unique p^{n-1}-th root of $(\sigma - 1)D_n(y_n)$ in $E(K_n)$. Now, we can use the explicit description (23) to study the image $\partial_p \mathcal{Q}^\times(\tau)$ of $\mathcal{Q}^\times(\tau)$ in $H^1_{\mathrm{sing}}(K_p, T_p(E))$.

Denote by Φ_n the group of connected components of the special fiber of the Néron model of E over the p-adic completion $K_{n,p}$ of K_n. By [BD97, Lemma 6.7] the reduction map $E(K_{n,p}) \to \Phi_n$ produces an injection

$$H^1(G_n, E(K_{n,p}))[p^{n-1}] \hookrightarrow \Phi_n[p^{n-1}],$$

where $H^1(G_n, \Phi_n) = \mathrm{Hom}_{\mathbb{Z}}(G_n, \Phi_n)$ is identified with Φ_n by evaluation at a generator of G_n. A direct computation using (23) and [Gro91, (3.5)] then shows that $\partial_p \mathcal{Q}^\times(\tau)$ vanishes exactly when the compatible collection of Heegner points $\{y_n\}_{n\geq 1}$ maps to a finite order element in the projective limit $\{\Phi_n\}_{n\geq 1}$. By adapting the proof of [BD97, Theorem 5.1] one sees that this happens precisely when the L-value $L(E/K, 1)$ vanishes. □

Interestingly, when $L(E/K, 1) \neq 0$, the mock plectic invariant $\mathcal{Q}(\tau)$ suffices to prove that the Mordell-Weil group of E' – the quadratic twist of E attached to K – is finite.

Theorem 3.9. *If $L(E/K, 1) \neq 0$, then $E'(\mathbb{Q})$ is finite.*

Sketch of proof. We follow the proof of [BD97, Corollary 7.2]. By Theorem 3.8 we know that $\mathcal{Q}(\tau)$ is a global class ramified only at p. The claim is then that the localization map $E'(\mathbb{Q}) \otimes \mathbb{Q}_p \hookrightarrow E'(\mathbb{Q}_p) \otimes \mathbb{Q}_p$ – which is always injective – is the zero morphism.

Under our assumptions, the hypothesis $L(E/K, 1) \neq 0$ also implies that E has split multiplicative reduction at p, hence the class $\mathcal{Q}(\tau)$ lives in the minus-eigenspace for the action of complex conjugation [BD97, Prop. 6.5] and its image in $H^1_{\mathrm{sing}}(K_p, V_p(E))$ satisfies $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_p(E')) = \mathbb{Q}_p \cdot \mathcal{Q}_p(\tau)$. The claim then follows because local Tate duality induces the identification

$$E'(\mathbb{Q}_p) \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_p(E'))^\vee \qquad Q \mapsto \langle Q, - \rangle_p,$$

and Poitou–Tate duality implies that any point $P \in E(K) \otimes \mathbb{Q}_p$ satisfies

$$0 = \sum_\ell \langle \mathrm{res}_\ell(P), \mathrm{res}_\ell(\mathcal{Q}(\tau)) \rangle_\ell = \langle \mathrm{res}_p(P), \mathcal{Q}_p(\tau) \rangle_p.$$

(See [BD97, Prop. 6.8].) □

Remark 3.10. To obtain the finiteness of $E(K)$ from $L(E/K, 1) \neq 0$ one also needs to consider tame deformations of mock plectic invariants (cf. [BD97, Rem. p.132]).

3.8. **Elliptic curves of rank two.** When $L(E/K, 1) = 0$, the local class $\mathcal{Q}_p(\tau)$ belongs to $H^1_f(K_p, T_p(E)) \otimes K_p$, i.e., the global class $\mathcal{Q}(\tau)$ lies in the pro-p Selmer group of E. It is not expected to be trivial in general: in fact, as we now proceed to explain it should provide a non-trivial Selmer class in settings where $L(E/K, 1) = L'(E/K, 1) = 0$ but $L''(E/K, 1) \neq 0$.

Consider the p-adic L-function $L_p(E, K, s)$ whose first derivative computes $\mathcal{Q}(\tau)$ according to Theorem 3.4. The conjecture [BD96, Conj. 4.1] of Birch and Swinnerton-Dyer type concerning this p-adic L-function suggests an expression for the mock plectic invariant and sheds some light on when it can be expected to be non-trivial. More precisely, set $E^+ := E$, denote by E^- the quadratic twist of E attached to K, and define

$$\delta^\pm = \begin{cases} 1 & \text{if } a_p(E^\pm) = +1 \\ 0 & \text{if } a_p(E^\pm) = -1. \end{cases}$$

Then [BD96, Conjecture 4.1] asserts that the order of vanishing of $L_p(E, K, s)$ at $s = 1$ is

$$\varrho = \max\{r_{\mathrm{alg}}(E^\pm/\mathbb{Q}) + \delta^\pm\} - 1,$$

which satisfies $2\varrho \geq r_{\mathrm{alg}}(E/K)$. Since under our assumptions the global root number over K always equals $\varepsilon(E/K) = +1$, while over $\mathbb{Q}$ depends on the reduction type at the prime p, the BSD conjecture suggests the following:

Conjecture 3.11. *Suppose that $L(E/K,1)=0$, then*

$$\mathcal{Q}(\tau)\neq 0 \qquad \Longleftrightarrow \qquad \begin{cases} r_{\mathrm{alg}}(E^+/\mathbb{Q})=0, \quad r_{\mathrm{alg}}(E^-/\mathbb{Q})=2 & \text{if } \ a_p(E)=+1,\\ r_{\mathrm{alg}}(E^+/\mathbb{Q})=1, \quad r_{\mathrm{alg}}(E^-/\mathbb{Q})=1 & \text{if } \ a_p(E)=-1.\end{cases}$$

In order to make explicit the relation between the mock plectic invariant $\mathcal{Q}(\tau)$ and global points in $E(K)$, denote by $\log_E^{a_p}\colon E(K_p)\to K_p$ the composition of the p-adic logarithm of E with the endomorphism $(1-a_p(E)\cdot\sigma_p)$ of $E(K_p)$, where $\sigma_p\in\mathrm{Gal}(K_p/\mathbb{Q}_p)$ denotes the non-trivial involution.

Conjecture 3.12. *Suppose that $L(E/K,1)=0$, then $\mathcal{Q}(\tau)$ is in the image of the regulator $\wedge^2E(K)\to H^1(K,T_p(E))\otimes_{\mathbb{Q}_p}K_p$ given by*

$$P\wedge Q\mapsto \delta_\infty(P)\otimes\log_E^{a_p}(Q)-\delta_\infty(Q)\otimes\log_E^{a_p}(P).$$

Remark 3.13. Given the analogy between mock plectic invariants and the plectic p-adic invariants of [FGM22], [FG23(a)], we invite the reader to compare Conjectures 3.11 & 3.12 with [FGM22, Conjectures 1.5 & 1.3].

The fact that the plectic invariant is forced to lie in a specific eigenspace for complex conjugation and can sometimes vanish for trivial reasons (for example, when $E(\mathbb{Q})$ has rank two) suggests that it is only "part of the story" and represents the projection of a more complete invariant which should be non-trivial in all scenarios when $E(K)$ has rank two. The authors believe that a full mock analogue of plectic Stark–Heegner points can be obtained by exploiting the p-adic deformations of f arising from Hida theory, and hope to treat this idea in future work. These "mock plectic points" should control the arithmetic of elliptic curves of rank two over quadratic imaginary fields, and be unaffected by the degeneracies of anticyclotomic height pairings that plague [BD96, Conjecture 4.1].

The approach of this article might also be exploited to upgrade the plectic Heegner points of [FG23(a)] from tensor products of p-adic points to global cohomology classes. Such an improvement would take care of the degeneracies of the original construction (evoked, for example, in [FG23(a), Remark 1.1]) and, in light of Theorems 3.8 and 3.9, it would help explain the arithmetic meaning of plectic Stark–Heegner points in the regime considered in [FG23(a), Remark 1.3]. (See also the paragraph before [FGM22, Conjecture 1.6].)

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