p-ADIC GREEN'S FUNCTIONS FOR REAL QUADRATIC GEODESICS

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Abstract. The quotient $\Gamma\backslash(\mathfrak{H} \times \mathfrak{H}_p)$ of the product of a Poincaré and a Drinfeld upper half plane by a discrete p-arithmetic subgroup Γ of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$ is equipped with an infinite supply of closed geodesic cycles of real dimension one, which are indexed by ideals in orders in real quadratic fields in which the prime p is non-split. This article lays the foundations for an arithmetic intersection theory of such cycles by defining a p-adic Green's function generalising the "differences of real quadratic singular moduli" explored in [DV1]. When the second cohomology group of Γ is trivial, the values of this p-adic Green's function are conjectured to be p-adic logarithms of algebraic numbers belonging to a suitable compositum of ring class fields of real quadratic fields. For general Γ, they should encode the analytic contribution to the p-adic height pairing between *Stark–Heegner points* which are conjecturally defined over the same ring class fields.

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Introduction

Let Γ be a p-arithmetic subgroup of an indefinite quaternion algebra B over $\mathbb{Q}$. After fixing real and p-adic splittings of B, the group Γ acts on both the Poincaré upper half plane $\mathfrak{H}$ and the Drinfeld p-adic upper half-plane $\mathfrak{H}_p$ by Möbius transformations, and its action on the product $\mathfrak{H} \times \mathfrak{H}_p$ is discrete. The quotient topological space $\Gamma\backslash(\mathfrak{H} \times \mathfrak{H}_p)$ is endowed with a plentiful supply of closed cycles of real dimension one, which are indexed by the set $\mathfrak{H}_p^{\mathrm{RM}}$ of algebra embeddings into B of real quadratic fields in which the prime p is non-split. These *RM cycles* are analogous to CM points on Shimura curves attached to definite quaternion algebras, and seem to have similar arithmetic implications, notably, for the construction of singular moduli contained in class fields of real quadratic fields [DV1] and of rational points on elliptic curves defined over these class fields [Dar].

The present work lays the foundations for an arithmetic intersection theory of RM cycles by defining a *p-adic Green's function*

$$G_p : (\Gamma\backslash\mathfrak{H}_p^{\mathrm{RM}}) \times (\Gamma\backslash\mathfrak{H}_p^{\mathrm{RM}}) \longrightarrow \mathbb{C}_p$$

on pairs of distinct elements of $\Gamma\backslash\mathfrak{H}_p^{\mathrm{RM}}$. When the cycle $\alpha_1 \in \Gamma\backslash\mathfrak{H}_p^{\mathrm{RM}}$ is *principal*, i.e. arises as the divisor of a rigid meromorphic cocycle J_{α_1}, the quantity $G_p(\alpha_1,\alpha_2)$ is the p-adic logarithm of the special value $J_{\alpha_1}[\alpha_2]$ in the sense of [DV1], whose algebraicity is predicted by the conjectures of *loc.cit.* and partially established in [DV4]. In the general case, $G_p(\alpha_1,\alpha_2)$ is expected to encode information about p-adic height pairings between the *Stark–Heegner points* attached to α_1 and α_2.

Although the quotient $\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p)$ is not an algebraic variety in any meaningful sense, it is suggestive to view it as "mock Hilbert modular surface" over $\mathbb{Q}$, and more specifically as the generic fiber of an arithmetic threefold $\mathcal{X}^?$ fibered over $\mathrm{Spec}(\mathbb{Z})$. In this perspective, RM cycles, which are of real dimension one, might be envisaged as "algebraic cycles of dimension $1/2$" on the generic fiber, extending to arithmetic cycles of middle dimension $3/2$ on $\mathcal{X}^?$. Such analogies are pure metaphysics in the sense of Weil's essay [We], but the concrete arithmetic intersection theory for RM cycles that emerges from them suggests that the seemingly dubious notion of middle dimensional cycles on mock Hilbert modular arithmetic threefolds deserves to be further examined and better understood.

The notion of a p-adic Green's function on RM geodesic cycles supplies the conceptual framework for understanding the calculations in [DV4], where certain *height pairings* between RM cycles are realised as the Fourier coefficients of modular generating series, leading to a p-adic Gross–Zagier formula for Stark–Heegner points and a proof of the algebraicity of certain non-trivial expressions involving the RM values of rigid meromorphic cocycles. The importance of the Green's function lies in the fact that it represents the *analytic contribution* to the p-adic height pairing between RM geodesic cycles.

This article concludes in § 3.11 with an elementary formula for the *trace to* $\mathbb{Q}_p$ of $G_p(\alpha_1,\alpha_2)$ under certain conditions on Γ, which include the case $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$. We proceed to describe this formula in the latter scenario, when the RM cycles can also be indexed by primitive integral binary quadratic forms whose discriminant D satisfies

(1) $$D > 0, \qquad \text{and } p \text{ is not split in } K := \mathbb{Q}(\sqrt{D}).$$

In this bijection, the form $F(x,y) = ax^2+bxy+cy^2$ corresponds to the embedding

$$K \hookrightarrow M_2(\mathbb{Q})\ ;\ \sqrt{D} \mapsto \begin{pmatrix} b & -2c \\ 2a & -b \end{pmatrix}.$$

Let $F_1(x,y) := a_1x^2+b_1xy+c_1y^2$ and $F_2(x,y) := a_2x^2+b_2xy+c_2y^2$ be a pair of primitive integral binary quadratic forms, whose discriminants D_1 and D_2 are coprime. The roots

$$\tau_1 = \frac{-b_1+\sqrt{D_1}}{2a_1}, \qquad \tau_2 = \frac{-b_2+\sqrt{D_2}}{2a_2}, \tag{2}$$

are viewed as real numbers by choosing the positive square roots in the expression (2). The oriented hyperbolic geodesic (α_j) on $\mathfrak{H}$ joining τ_j to its algebraic conjugate τ_j' is preserved by the stabiliser Γ_j of τ_j in $\mathrm{SL}_2(\mathbb{Z})$ acting naturally on $\mathfrak{H}\cup\mathbb{R}$ by Möbius transformations. The choice of an orientation on $\mathfrak{H}$ determines a topological intersection number on $\mathfrak{H}$:

$$(\alpha_1) \frown (\alpha_2) \in \{-1,0,1\}. \tag{3}$$

After fixing embeddings of the fields $K_j = \mathbb{Q}(\sqrt{D_j})$ into $\overline{\mathbb{Q}}_p$, the roots τ_j and τ_j' can also be viewed as elements of $\mathfrak{H}_p$ by condition (1) on their discriminants. Consider

$$g(\tau_1,\tau_2) := \frac{(\tau_1-\tau_2)(\tau_1'-\tau_2')}{(\tau_1-\tau_1')(\tau_2-\tau_2')} \in \bar{\mathbb{Q}}_p \tag{4}$$

which is the cross-ratio of the roots τ_1,τ_2,τ_1' and τ_2', a point pair invariant for the action of $\mathrm{SL}_2(\mathbb{Q}_p)$ on $\mathfrak{H}_p$, and belongs to the field

$$F := \mathbb{Q}\left(\sqrt{D_1D_2}\right) \quad \subset \quad L := K_1K_2 = \mathbb{Q}\left(\sqrt{D_1},\sqrt{D_2}\right).$$

Note that p splits in F when $p \nmid D_1D_2$, and ramifies otherwise. Write F_p and L_p for the completions of F and L in $\bar{\mathbb{Q}}_p$. Let M_n be the set of 2×2 matrices with integer entries and determinant p^{2n}, a set that is preserved under both left and right multiplication by $\mathrm{SL}_2(\mathbb{Z})$. The expression

$$\mathbb{G}_p^{(n)}(\tau_1,\tau_2) = \log_p\left(\prod_{b\in\Gamma_1\backslash M_n/\Gamma_2} g(\tau_1,b\tau_2)^{(\alpha_1)\frown b(\alpha_2)}\right)$$

involves only finitely many non-trivial factors, and belongs to F_p, while the Green's function value $G_p(\tau_1,\tau_2)$ can be viewed as an element of L_p. The following is a special case of the main result of § 3.11.

Theorem. *The sequence $\mathbb{G}_p^{(n)}(\tau_1,\tau_2)$ converges to a p-adic limit as n tends to ∞, and*

$$\mathbb{G}_p(\tau_1,\tau_2) := \lim_{n\longrightarrow\infty}\mathbb{G}_p^{(n)}(\tau_1,\tau_2) = \mathrm{Trace}_{F_p}^{L_p}\, G_p(\tau_1,\tau_2).$$

Remark 1. It is natural to contemplate multiplicative refinements of this theorem. The calculations of § 3.11 show the existence of the limit

$$\mathbb{J}_p(\tau_1,\tau_2) := \lim_{n\to\infty}\left(\prod_{b\in\Gamma_1\backslash M_n/\Gamma_2} g(\tau_1,b\tau_2)^{(\alpha_1)\frown b(\alpha_2)}\right)^{12}. \tag{5}$$

It would be interesting to give a more direct proof of this convergence.

When $p = 2, 3, 5, 7$, or 13 is a prime for which the modular curve $X_0(p)$ has genus zero, the expression (5) is the norm to F_p of the differences of real quadratic singular moduli explored in [DV1], and is therefore expected to be algebraic, and to lie in the compositum $\mathrm{H}_1\,\mathrm{H}_2$ of the ring class fields attached to the orders of discriminants D_1 and D_2.

The significance of the finer multiplicative invariant (5) for general p remains to be explored. It is expected to be transcendental, and its logarithm is expected to relate to the non-archimedean Green's functions (in the sense of § 2) of the Stark–Heegner points on quotients of $J_0(p)$ with multiplicative reduction, attached to α_1 and α_2. These points are conjecturally defined over ring class fields of K_1 and K_2.

Remark 2. In spite of ostensible similarities, there is a very important difference with the infinite products of cross ratios studied in [DV2], which instead had the shape

$$\prod_{b\in\Gamma_1\backslash M_n/\Gamma_2}\left(\frac{(\tau_1-b\tau_2)(\tau_1'-b\tau_2')}{(\tau_1-b\tau_2')(\tau_1'-b\tau_2)}\right)^{(\alpha_1)\frown b(\alpha_2)} = \prod_{b\in\Gamma_1\backslash M_n/\Gamma_2}\left(\frac{g(\tau_1,b\tau_2)}{g(\tau_1,b\tau_2)-1}\right)^{(\alpha_1)\frown b(\alpha_2)}$$
$$= \prod_{b\in\Gamma_1\backslash M_n/\Gamma_2}\left(\frac{g(\tau_1,b\tau_2)}{g(\tau_1',b\tau_2)}\right)^{(\alpha_1)\frown b(\alpha_2)}$$

The p-adic convergence of this expression as $n\to\infty$, proved more elementarily in [DV2, Section 3], follows immediately from the stronger results in the present paper.

In defining the Green's function on RM geodesic cycles, we strive to emphasise their strong analogy with p-adic Green's functions on Mumford curves, as described by Gross [Gr86] and Werner [Wer]. This description is recalled in § 2, with a narrative that foreshadows the extension to the real quadratic case, contained in § 3.

1. Preliminaries

Let $\mathfrak{H}_p$ denote the Drinfeld p-adic upper half-plane, a rigid analytic space whose underlying set of $\mathbb{C}_p$-valued points is identified with $\mathbb{P}_1(\mathbb{C}_p)-\mathbb{P}_1(\mathbb{Q}_p)$. The projective line $\mathbb{P}_1(\mathbb{C}_p)$ is endowed with a p-adic metric by setting

$$d((x_1:y_1),(x_2:y_2)) = \left|\det\begin{pmatrix} x_1 & x_2\\ y_1 & y_2\end{pmatrix}\right|_p,$$

where the homogenous coordinates are always chosen to be primitive vectors in $\mathcal{O}_{\mathbb{C}_p}^2$. The space $\mathfrak{H}_p$ can be expressed as an increasing union

$$\mathfrak{H}_p = \bigcup_{n=0}^{\infty}\mathfrak{H}_p^{(n)}, \qquad \mathfrak{H}_p^{(0)}\subset\mathfrak{H}_p^{(1)}\subset\mathfrak{H}_p^{(2)}\subset\cdots\subset\mathfrak{H}_p^{(n)}\subset\cdots\subset\mathfrak{H}_p$$

of the *affinoid subsets*

$$\mathfrak{H}_p^{(n)} := \{z \in \mathbb{P}_1(\mathbb{C}_p) \quad \text{s.t.} \quad d(z,a) \geq p^{-n} \quad \text{for all } a \in \mathbb{P}_1(\mathbb{Q}_p)\}.$$

The region $\mathfrak{H}_p^{(n)}$ is somewhat analogous to a Siegel domain in the Poincaré upper half-plane, but its cohomology is richer, since it is the complement in $\mathbb{P}_1(\mathbb{C}_p)$ of the $(p+1)p^{n-1}$ disjoint mod p^n residue discs with $\mathbb{Q}_p$-rational centers.

A *formal divisor* on $\mathfrak{H}_p$ is a formal, possibly infinite $\mathbb{Z}$-linear combination

$$\mathscr{D} = \sum_{z \in \mathfrak{H}_p} m_z \cdot [z], \qquad m_z \in \mathbb{Z}$$

of elements of $\mathfrak{H}_p$. It is called a *divisor* if its support is finite, and a *locally finite divisor* if its support intersects each affinoid $\mathfrak{H}_p^{(n)}$ in a finite set. Given any subset X of $\mathfrak{H}_p$, write

$$\mathscr{D} \cap X := \sum_{z \in X} m_z \cdot [z], \qquad \mathscr{D}^{(n)} := \mathscr{D} \cap \mathfrak{H}_p^{(n)}.$$

If $\mathscr{D}$ is locally finite and X is an affinoid subset of $\mathfrak{H}_p$ (or simply, is contained in $\mathfrak{H}_p^{(n)}$ for some n) then $\mathscr{D} \cap X$ is a divisor with support on X. The groups of degree zero divisors on $\mathfrak{H}_p$, (finite) divisors, and locally finite divisors, are respectively denoted by

$$\mathrm{Div}_0(\mathfrak{H}_p) \subset \mathrm{Div}(\mathfrak{H}_p) \subset \mathrm{Div}^\dagger(\mathfrak{H}_p).$$

Let $\mathcal{T} := \mathcal{V} \sqcup \mathcal{E}$ be (the combinatorial realisation of) the Bruhat-Tits tree of $\mathrm{SL}_2(\mathbb{Q}_p)$, consisting of a collection $\mathcal{V}$ of vertices indexed by homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$, and a collection $\mathcal{E}$ of edges joining vertices that correspond to pairs of lattices that are, after suitable rescaling, contained one in the other with index p. The homothety class of the lattice $\mathbb{Z}_p^2$ corresponds to a distinguished vertex $v_\circ \in \mathcal{V}$, and a vertex of $\mathcal{T}$ is said to be *even* if its distance from this distinguished vertex is even, and is said to be *odd* otherwise.

The Drinfeld half plane is equipped with a natural *reduction map*

$$\mathrm{red} : \mathfrak{H}_p \longrightarrow \mathcal{T}.$$

The inverse images of a vertex v is called its *standard affinoid*, denoted $\mathcal{W}_v$. All standard affinoids are identified with complements in $\mathbb{P}_1(\mathbb{C}_p)$ of the $\mathbb{F}_p$-rational mod p residue discs relative to a suitable coordinate on $\mathbb{P}_{1/\mathbb{Q}_p}$. These affinoids are glued together along the p-adic annuli which are the inverse image under the reduction map of the edges of $\mathcal{T}$. Let $\mathcal{W}_v^+$ be the union of the affinoid $\mathcal{W}_v$ with the $(p+1)$ annuli attached to the edges having v as an endpoint; it is a connected "wide open space" in the terminology of R. Coleman. The p-adic upper half plane can be expressed as a disjoint union

$$\mathfrak{H}_p = \bigcup_{v \in \mathcal{V}} \Omega_v, \qquad \text{where } \Omega_v = \begin{cases} \mathcal{W}_v^+ & \text{if } v \text{ is even;} \\ \mathcal{W}_v & \text{if } v \text{ is odd.} \end{cases}$$

The group $\mathrm{SL}_2(\mathbb{Q}_p)$ acts on $\mathfrak{H}_p$ by Möbius transformations, and induces a unique action on $\mathcal{T}$ for which the reduction map is equivariant. The subgroup $\mathrm{SL}_2(\mathbb{Z}_p)$ fixes a vertex $v_\circ \in \mathcal{V}$, and the affinoid $\mathfrak{H}_p^{(n)}$ is the preimage under the reduction map of the induced subgraph of $\mathcal{T}$ on the vertices of distance $\leq n$ from $v_\circ$. The subsets Ω_v are permuted under the action of $\mathrm{SL}_2(\mathbb{Q}_p)$, which preserves the parity of vertices. (For more background on $\mathcal{T}$ and its connection to the Drinfeld half plane, the reader is invited to consult [DT].)

Write $\mathbb{Z}^{\mathcal{V}}$ for the group of $\mathbb{Z}$-valued functions on $\mathcal{V}$. The *degree map*

$$\mathrm{Deg} : \mathrm{Div}^\dagger(\mathfrak{H}_p) \to \mathbb{Z}^{\mathcal{V}} \tag{6}$$

sending a locally finite divisor $\mathscr{D}$ to the function

$$v \mapsto \deg(\mathscr{D} \cap \Omega_v)$$

plays the role of the usual degree in the setting of locally finite divisors. A locally finite divisor which is in the kernel of the degree map is said to be *of strong degree zero*, and the group of such divisors is denoted $\mathrm{Div}_0^\dagger(\mathfrak{H}_p)$.

Let $\mathbb{Q}_p^{\mathrm{nr}}$ be the maximal unramified extension of $\mathbb{Q}_p$. It is worth noting that the reduction map sends $\mathfrak{H}_p^{\mathrm{nr}} := \mathbb{P}_1(\mathbb{Q}_p^{\mathrm{nr}}) - \mathbb{P}_1(\mathbb{Q}_p)$ to $\mathcal{V}$. A divisor that is supported on $\mathfrak{H}_p^{\mathrm{nr}}$ is said to be *unramified*, and the group of divisors and locally finite divisors that are unramified are denoted $\mathrm{Div}(\mathfrak{H}_p^{\mathrm{nr}})$ and $\mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}})$ respectively. Likewise, $\mathrm{Div}_0(\mathfrak{H}_p^{\mathrm{nr}})$ and $\mathrm{Div}_0^\dagger(\mathfrak{H}_p^{\mathrm{nr}})$ are the relevant subgroups of divisors of degree zero and locally finite divisors of strong degree zero. These groups of divisors are related by two short exact sequences:

$$0 \to \mathrm{Div}_0(\mathfrak{H}_p) \longrightarrow \mathrm{Div}(\mathfrak{H}_p) \longrightarrow \mathbb{Z} \to 0, \tag{7}$$

$$0 \to \mathrm{Div}_0^\dagger(\mathfrak{H}_p) \longrightarrow \mathrm{Div}^\dagger(\mathfrak{H}_p) \longrightarrow \mathbb{Z}^{\mathcal{V}} \to 0. \tag{8}$$

The field $\mathscr{M}$ of *rigid meromorphic functions* on $\mathfrak{H}_p$ is a suitable p-adic completion of the field $\mathcal{R}$ of rational functions on $\mathbb{P}^1$. More precisely, a subset of $\mathfrak{H}_p$ is called a *good affinoid* if it is the preimage under the reduction map of a finite closed subgraph of $\mathcal{T}$. A $\mathbb{C}_p$-valued function on $\mathfrak{H}_p$ is *rigid analytic* if its restriction to each good affinoid subset $\mathcal{X} \subset \mathfrak{H}_p$ is a uniform limit with respect to the sup norm of elements of $\mathcal{R}$ (having poles outside $\mathcal{X}$). Note that for f to be rigid analytic, it is enough to require that its restriction to each $\mathfrak{H}_p^{(n)}$ be expressible as such a limit. The set $\mathscr{A}$ of rigid analytic functions on $\mathfrak{H}_p$ is a ring and its fraction field, denoted $\mathscr{M}$, is the field of *rigid meromorphic* functions on $\mathfrak{H}_p$.

Given a pair $(\mathscr{D}_1, \mathscr{D}_2)$ of degree zero divisors on $\mathbb{P}_1(\mathbb{C}_p)$ with disjoint supports, the *Weil symbol* $[\mathscr{D}_1; \mathscr{D}_2] \in \mathbb{C}_p^\times$ is defined as the value at $\mathscr{D}_1$ of any rational function having divisor $\mathscr{D}_2$. The following lemma extends the Weil symbol to a canonical pairing

$$\mathrm{Div}_0(\mathfrak{H}_p) \times \mathrm{Div}_0^\dagger(\mathfrak{H}_p) \longrightarrow \mathbb{C}_p^\times .$$

Lemma 3. *Let $\mathscr{D}_1$ be a degree 0 divisor on $\mathfrak{H}_p$ and let $\mathscr{D}_2 \in \mathrm{Div}_0^\dagger(\mathfrak{H}_p)$ be a locally finite divisor of strong degree zero. Then the sequence $[\mathscr{D}_1; \mathscr{D}_2^{(n)}]$ converges to a well-defined limit*

$$[\mathscr{D}_1; \mathscr{D}_2] := \lim_{n \longrightarrow \infty} [\mathscr{D}_1; \mathscr{D}_2^{(n)}].$$

This extended Weil symbol is equivariant, i.e.,

$$[\gamma\, \mathscr{D}_1; \gamma\, \mathscr{D}_2] = [\mathscr{D}_1; \mathscr{D}_2] \quad \textit{for all } \gamma \in \mathrm{SL}_2(\mathbb{Q}_p).$$

Proof. We need to show that the sequence $[\mathscr{D}_1; \mathscr{D}_2^{(n)}]$ forms a multiplicative Cauchy sequence in n. Of course, it is enough to show this for all large enough n, so assume without loss of generality that the divisor $\mathscr{D}_1$ is supported in $\mathfrak{H}_p^{(m)}$ and that $n > m$. Then we have

$$\mathscr{D}_2^{(n)} - \mathscr{D}_2^{(n-1)} = \sum_{d(v, v_\circ) = n} \mathscr{D}_2(v), \qquad \mathscr{D}_2(v) := \mathscr{D}_2 \cap \mathrm{red}^{-1}(v), \tag{9}$$

where the sum is taken over all the vertices of $\mathcal{T}$ whose distance from $v_\circ$ is equal to n. Since $\mathscr{D}_2$ is of strong degree 0, the divisors $\mathscr{D}_2(v)$ are of degree 0, and therefore

$$\big|\, [\mathscr{D}_1; \mathscr{D}_2(v)] - 1 \,\big|_p \leq p^{m-n}, \qquad \text{for all } v \text{ with } d(v, v_\circ) = n. \tag{10}$$

It follows from (9) and (10) that

$$\Big|\, [\mathscr{D}_1, \mathscr{D}_2^{(n)} - \mathscr{D}_2^{(n-1)}] - 1 \,\Big|_p \leq p^{m-n}, \qquad \text{for all } n > m,$$

and the convergence follows. In fact, upon replacing the affinoid cover $\{\mathfrak{H}_p^{(n)}\}_{n \geq 0}$ by an arbitrary increasing union of good affinoid subsets

$$\mathfrak{H}_p = \bigcup_{n=1}^{\infty} \mathcal{X}_n, \qquad \mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots \mathcal{X}_n \subset \cdots$$

the same argument shows that

$$[\mathscr{D}_1; \mathscr{D}_2] = \lim_{n \to \infty} [\mathscr{D}_1; \mathscr{D}_2 \cap \mathcal{X}_n]. \tag{11}$$

The Γ-invariance of the extended Weil symbol now follows from the Γ-invariance properties of the original Weil symbol:

$$\begin{aligned}
[\gamma\, \mathscr{D}_1; \gamma\, \mathscr{D}_2] &= \lim_{n \to \infty} [\gamma\, \mathscr{D}_1; (\gamma\, \mathscr{D}_2) \cap \mathfrak{H}_p^{(n)}] \\
&= \lim_{n \to \infty} [\mathscr{D}_1; \mathscr{D}_2 \cap (\gamma^{-1}\, \mathfrak{H}_p^{(n)})] \\
&= \lim_{n \to \infty} [\mathscr{D}_1; \mathscr{D}_2^{(n)}] \qquad \text{by (11)} \\
&= [\mathscr{D}_1; \mathscr{D}_2].
\end{aligned}$$

□

The following corollary is the basis for our construction of p-adic Green functions, both in the classical CM case discussed in § 2 and the RM case discussed in § 3.

Corollary 4. *Let $\mathscr{D} \in \mathrm{Div}_0^\dagger(\mathfrak{H}_p)$ be a locally finite divisor of strong degree zero, and let $\xi_p \in \mathfrak{H}_p$ be a base point which is disjoint from $\mathscr{D}$. The rational functions*

$$f_{\mathscr{D}}^{(n)}(z) := [(z) - (\xi_p); \mathscr{D}^{(n)}]$$

converge uniformly to a rigid meromorphic function $f_{\mathscr{D}}$ with divisor $\mathscr{D}$. For all $b \in \Gamma$,

$$f_{b\mathscr{D}}(bz) = f_{\mathscr{D}}(z) \pmod{\mathbb{C}_p^\times}.$$

2. Green's functions on CM points in $\mathfrak{H}_p$

This section describes p-adic Green's functions on Shimura curves, inspired by the treatments in [Gr86] and [Wer]. The aim of this largely expository discussion is to motivate what follows, and to highlight the key aspects of the strong analogy with the Néron symbols for RM geodesic cycles that will be defined in § 3.

2.1. **Shimura curves.** We place ourselves in the arithmetic setup of Shimura curves associated to maximal orders in quaternion algebras over $\mathbb{Q}$. At primes p of bad reduction, these curves have totally degenerate reduction, and admit an arithmetic uniformisation by $\mathfrak{H}_p$ described by Cerednik–Drinfeld [Cer, Dri].

Throughout this section 2, and this section only, it shall be assumed that $R \subset B$ is a maximal order in a *definite* quaternion algebra B of discriminant D_B. Let p be a prime that does not divide D_B, and fix a p-adic splitting

$$B \otimes \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p) \tag{12}$$

which identifies $R \otimes \mathbb{Z}_p$ with the standard maximal order $M_2(\mathbb{Z}_p) \subset M_2(\mathbb{Q}_p)$. The group $(B \otimes \mathbb{Q}_p)^\times$ acts on $\mathfrak{H}_p$ by Möbius transformations, and on $\mathcal{T}$, via the choice of splitting (12). Let $v_\circ$ denote the unique vertex of $\mathcal{T}$ whose stabiliser in $(B \otimes \mathbb{Q}_p)^\times$ is $(R \otimes \mathbb{Z}_p)^\times$.

The group $\Gamma = R[1/p]_1^\times$ of norm one elements of $R[1/p]$ can be expressed as an increasing union of finite sets. Indeed, let $R[d]$ be the set of elements in R of norm d, and define $\Gamma_n := p^{-n}R[p^{2n}]$, then we have

$$\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n, \qquad \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n \subset \cdots \subset \Gamma. \tag{13}$$

The finite subgroup $\Gamma_0 = R_1^\times$ acts naturally on Γ_n by left (or right) multiplication, and the quotients Γ_n/Γ_0 are in bijection with the set of vertices of $\mathcal{T}$ at distance $\leq n$ from $v_\circ$.

The group Γ acts discretely on $\mathfrak{H}_p$ by Möbius transformations. The theorem of Cerednik–Drinfeld identifies the quotient $\Gamma \backslash \mathfrak{H}_p$ with the $\mathbb{C}_p$-points of a Shimura curve X, viewed as a rigid analytic space. We denote the resulting quotient map by

$$\pi : \mathfrak{H}_p \longrightarrow \Gamma \backslash \mathfrak{H}_p = X(\mathbb{C}_p).$$

2.2. **Summary of the construction.** Before delving into the details, a brief summary may be helpful. Given a CM divisor α on $X(\mathbb{C}_p)$, there are well-defined classes

$$\mathscr{D}_\alpha \in \mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p)), \qquad \widehat{\mathscr{D}}_\alpha \in \mathrm{H}^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p)),$$

satisfying $\pi_*(\mathscr{D}_\alpha) = \alpha$, and $\widehat{\mathscr{D}}_\alpha = \pi^*(\alpha)$. The extended Weil symbol induces a pairing

$$[\ ,\ ] : \mathrm{H}_0(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p)) \times \mathrm{H}^0(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p)) \longrightarrow \mathbb{C}_p^\times .$$

The Néron symbol described below refines the logarithm of this pairing by allowing

(1) $\mathrm{H}_0(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p))$ to be replaced by its image in $\mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p))$, denoted
$$\mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p))_0$$
and consisting of the divisors on $\mathfrak{H}_p$ whose image on X has degree 0;

(2) $\mathrm{H}^0(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p))$ to be replaced by pullbacks of degree 0 divisors on X, denoted
$$\mathrm{H}^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p))_0$$
in what follows. It is the group of locally finite divisors on $\mathfrak{H}_p$ whose restriction to any fundamental region for the action of Γ on $\mathfrak{H}_p$ is of degree 0.

This leads to a bi-additive function

$$[\ ,\ ]_{\mathrm{Neron}} : \mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p))_0 \times \mathrm{H}^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p))_0 \longrightarrow \mathbb{C}_p$$

satisfying

$$[\mathscr{D}_1, \widehat{\mathscr{D}}_2]_{\mathrm{Neron}} = \log_p([\mathscr{D}_1, \widehat{\mathscr{D}}_2])$$

whenever both sides are defined. The Green's function is then defined on CM divisors α and β on $\Gamma\backslash\mathfrak{H}_p$ by the rule

$$G_p(\alpha, \beta) = [\mathscr{D}_\alpha, \widehat{\mathscr{D}}_\beta]_{\mathrm{Neron}}.$$

2.3. **The homology of divisors.** Taking the Γ-homology of the short exact sequence in (7) leads to the long exact homology sequence

$$(14) \qquad \cdots \longrightarrow \mathrm{H}_1(\Gamma, \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}_0(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p)) \xrightarrow{j} \mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p)) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

The kernel of the map deg in this sequence is the group $\mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p))_0$. The connecting map δ sends $b \in \mathrm{H}_1(\Gamma, \mathbb{Z}) = \Gamma_{\mathrm{ab}}$ to the class of the degree zero divisor $(b\xi_p) - (\xi_p)$, where $\xi_p \in \mathfrak{H}_p$ is an arbitrarily chosen base point.

Let $S_k^{\mathrm{new}}(D)$ be the space of newforms of weight k and level D.

Lemma 5. *The group* $\mathrm{H}_1(\Gamma, \mathbb{Z}) = \Gamma_{\mathrm{ab}}$ *is isomorphic modulo torsion to the character group of the torus uniformising the Jacobian of* X*, and has rank* $g = \mathrm{genus}(X)$*. It is annihilated by any Hecke operator that kills* $S_2^{\mathrm{new}}(pD_B)$*.*

Proof. The p-adic period pairing recalled in § 2.5 below (cf. Remark 8) canonically identifies the $\mathbb{C}_p$-points of the Jacobian of X with the quotient of $\mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)$ by a lattice. The first assertion follows from this. The second is a consequence of the Jacquet–Langlands correspondence between forms on $B^\times$ and on $\mathrm{GL}_{2,\mathbb{Q}}$. □

If the divisor $\alpha \in \mathrm{Div}(\Gamma\backslash \mathfrak{H}_p)$ is of degree zero, then $\mathscr{D}_\alpha \in \mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p))$ can be lifted to a class in $\mathrm{H}_0(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p))$, but this lift is *not unique*: any two lifts differ by a class in $\delta(\mathrm{H}_1(\Gamma, \mathbb{Z}))$. An important ingredient in the definition of the Néron symbol is the construction, after tensoring with $\mathbb{Q}$, of a canonical right inverse of the surjective map

$$j : \mathrm{H}_0(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p)) \longrightarrow \mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p))_0$$

from (14), leading to a direct sum decomposition

$$\mathrm{H}_0(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p)) \otimes \mathbb{Q} \quad = \quad \delta(\mathrm{H}_1(\Gamma, \mathbb{Q})) \;\oplus\; \mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p))_0 \otimes \mathbb{Q}\,.$$

This map is constructed in § 2.6 using the non-degeneracy of the p-adic period pairing.

2.4. **The cohomology of locally finite divisors.** The Γ-cohomology of the short exact sequence in (8) gives rise to another long exact sequence,

$$0 \to \mathrm{H}^0(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p)) \longrightarrow \mathrm{H}^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p)) \xrightarrow{\mathrm{Deg}} \mathrm{H}^0(\Gamma, \mathbb{Z}^{\mathcal{V}}) \longrightarrow \cdots \tag{15}$$

Lemma 6. *The target of the degree map* Deg *is isomorphic to the space of unramified automorphic forms on $B^\times$. It is annihilated by any Hecke operator that kills $S_2^{\mathrm{new}}(D_B)$.*

Proof. The set $\mathcal{V}$ is identified with $(B \otimes \mathbb{Q}_p)^\times/(R \otimes \mathbb{Z}_p)^\times$. The group Γ preserves the subsets $\mathcal{V}^{\mathrm{even}}$ and $\mathcal{V}^{\mathrm{odd}}$ of vertices that are at even and odd distance from $v_\circ$, respectively. The quotient $\Gamma\backslash \mathcal{V}$ is thus in natural bijection with two copies of

$$R[1/p]^\times\backslash(B \otimes \mathbb{Q}_p)^\times/(R \otimes \mathbb{Z}_p)^\times = B^\times\backslash(B \otimes \mathbb{A}_\mathbb{Q})^\times/(R \otimes \hat{\mathbb{Z}})^\times,$$

where the equality follows from strong approximation for $R[1/p]^\times$. The set of functions on the rightmost double coset space is the space of automorphic forms on $B^\times$ of level 1, so the first assertion follows. The second is a consequence of the Jacquet–Langlands correspondence between forms on $B^\times$ and on $\mathrm{GL}_{2,\mathbb{Q}}$. □

2.5. **The p-adic period pairing.** Consider the natural maps

$$\begin{array}{rcccl} \Pi_\Gamma & : & \mathrm{H}_0(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p)) & \longrightarrow & \mathrm{H}^0(\Gamma, \mathscr{M}^\times/\,\mathbb{C}_p^\times), \\ \Sigma_\Gamma & : & \mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p)) & \longrightarrow & \mathrm{H}^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p)), \end{array}$$

defined by choosing a base point $\xi_p \in \mathfrak{H}_p$ and setting

$$\Pi_\Gamma(\mathscr{D})(z) = \prod_{b\in\Gamma} [(z) - (\xi_p); b\,\mathscr{D}], \qquad\qquad \Sigma_\Gamma(\mathscr{D}) = \sum_{b\in\Gamma} b\,\mathscr{D}\,. \tag{16}$$

They fit into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{H}_1(\Gamma,\mathbb{Z}) & \xrightarrow{\delta} & \mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p)) & \xrightarrow{j} & \mathrm{H}_0(\Gamma,\mathrm{Div}(\mathfrak{H}_p)) & \xrightarrow{\deg} & \mathbb{Z} \longrightarrow 0 \\ & \downarrow{\scriptstyle \Pi_\Gamma} & & \downarrow{\scriptstyle \Pi_\Gamma} & & \downarrow{\scriptstyle \Sigma_\Gamma} & & \\ 0 \longrightarrow & \mathrm{H}^0(\Gamma,\mathscr{A}^\times/\,\mathbb{C}_p^\times) & \longrightarrow & \mathrm{H}^0(\Gamma,\mathscr{M}^\times/\,\mathbb{C}_p^\times) & \longrightarrow & \mathrm{H}^0(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p)) & \longrightarrow & \cdots \end{array} \tag{17}$$

where the top row is (14) and the bottom row is the long exact sequence in cohomology associated to the short exact sequence of Γ-modules

$$1 \to \mathscr{A}^\times/\,\mathbb{C}_p^\times \longrightarrow \mathscr{M}^\times/\,\mathbb{C}_p^\times \longrightarrow \mathrm{Div}^\dagger(\mathfrak{H}_p) \to 1.$$

Let

$$\mathrm{per} : \mathrm{H}^0(\Gamma,\mathscr{A}^\times/\,\mathbb{C}_p^\times) \longrightarrow \mathrm{H}^1(\Gamma,\mathbb{C}_p^\times)$$

be the connecting homomorphism in the cohomology of the short exact sequence

$$1 \to \mathbb{C}_p^\times \longrightarrow \mathscr{A}^\times \longrightarrow \mathscr{A}^\times/\,\mathbb{C}_p^\times \to 1.$$

Composing the leftmost map Π_Γ in (17) with this injective period homomorphism yields a map

$$\eta_\Gamma := \mathrm{per} \circ \Pi_\Gamma : \mathrm{H}_1(\Gamma,\mathbb{Z}) \longrightarrow \mathrm{H}^1(\Gamma,\mathbb{C}_p^\times)$$

which induces pairings

$$\langle\ ,\ \rangle_\Gamma : \mathrm{H}_1(\Gamma,\mathbb{Z}) \times \mathrm{H}_1(\Gamma,\mathbb{Z}) \longrightarrow \mathbb{C}_p^\times\,. \tag{18}$$

This is the p-adic period pairing that arises in the Mumford–Schottky theory of p-adic uniformisation of $X(\mathbb{C}_p)$. The following theorem is well-known.

Proposition 7. *The $\mathbb{Z}$-valued pairing $\mathrm{ord}_p(\langle\ ,\ \rangle_\Gamma)$ is positive definite and hence non-degenerate.*

Proof. See [MD, §4] for example. □

Remark 8. It follows that $\eta_\Gamma(\Gamma_{\mathrm{ab}})$ is a lattice in $\mathrm{H}^1(\Gamma,\mathbb{C}_p^\times)$. The p-adic torus

$$\mathrm{Jac}_X(\mathbb{C}_p) := \mathrm{H}^1(\Gamma,\mathbb{C}_p^\times)/\eta_\Gamma(\Gamma_{\mathrm{ab}})$$

is identified with (the $\mathbb{C}_p$-points of) the Jacobian of X, and the map

$$\mathrm{H}_0(\Gamma,\mathrm{Div}(\mathfrak{H}_p))_0 \longrightarrow \mathrm{H}^1(\Gamma,\mathbb{C}_p^\times)/\eta_\Gamma(\Gamma_{\mathrm{ab}})$$

sending a divisor $\mathscr{D}$ to the periods of the rigid analytic function $\Pi_\Gamma(\tilde{\mathscr{D}})$ on $\mathfrak{H}_p$ (where $\tilde{\mathscr{D}}$ is any lift of $\mathscr{D}$ to $\mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))$) realises the p-adic Abel–Jacobi map.

2.6. **Unitary divisors.** A natural $\mathbb{Q}$-vector space complement of

$$\delta(\mathrm{H}_1(\Gamma,\mathbb{Z}))_{\mathbb{Q}} := \delta(\mathrm{H}_1(\Gamma,\mathbb{Z}))\otimes\mathbb{Q}$$

in $\mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))$ can be defined by exploiting the map

$$\mathrm{ord}_p\circ\mathrm{per}\circ\Pi_\Gamma : \mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))\longrightarrow \mathrm{H}^1(\Gamma,\mathbb{Z}).$$

Proposition 7 implies that the kernel

$$\mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))^\sharp_{\mathbb{Q}} := \ker\left(\mathrm{ord}_p\circ\mathrm{per}\circ\Pi_\Gamma : \mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))_{\mathbb{Q}}\longrightarrow \mathrm{H}^1(\Gamma,\mathbb{Q})\right) \tag{19}$$

is complementary to $\delta(\Gamma_{\mathrm{ab}})\otimes\mathbb{Q}$. Following the terminology in [Gr86, §10], a class in $\mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))^\sharp_{\mathbb{Q}}$ is said to be *unitary*. Given $\mathscr{D}\in\mathrm{H}_0(\Gamma,\mathrm{Div}(\mathfrak{H}_p))_0\otimes\mathbb{Q}$, its unique unitary lift shall be denoted

$$\mathscr{D}^\sharp\in\mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))^\sharp_{\mathbb{Q}}.$$

While explicit unitary lifts are not usually apparent, Lemma 9 below provides a useful exception. Let

$$\mathrm{Div}_{00}(\mathfrak{H}_p^{\mathrm{nr}}) := \mathrm{Div}(\mathfrak{H}_p^{\mathrm{nr}})\cap\mathrm{Div}_0^\dagger(\mathfrak{H}_p)\ \subset\ \mathrm{Div}_0(\mathfrak{H}_p^{\mathrm{nr}}).$$

The 0-cycles in $\mathrm{Div}_{00}(\mathfrak{H}_p^{\mathrm{nr}})$ admit a natural geometric interpretation: their pushforwards to the Shimura curve X are unramified divisors whose restriction to each irreducible component of the special fiber of X at p is of degree zero.

Lemma 9. *The image of the natural map*

$$\mathrm{H}_0(\Gamma,\mathrm{Div}_{00}(\mathfrak{H}_p^{\mathrm{nr}}))_{\mathbb{Q}}\longrightarrow\mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))_{\mathbb{Q}}$$

is contained in $\mathrm{H}_0(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))^\sharp_{\mathbb{Q}}$.

Proof. This follows from the fact that if $\mathscr{D}$ is represented by a divisor in $\mathrm{Div}_{00}(\mathfrak{H}_p^{\mathrm{nr}})$, the quantities

$$[\mathscr{D};(b\gamma\xi_p)-(b\xi_p)],\qquad b\in\Gamma$$

are p-adic units, as can be seen by reducing to the case where $\mathscr{D}$ is supported on $\mathrm{red}^{-1}(v)$ for a single vertex v of $\mathcal{T}$, and choosing the base point ξ_p to reduce to a vertex that is not Γ-equivalent to v. (The latter is always possible since there are always at least two distinct Γ-orbits for the action of Γ on $\mathcal{V}$.) □

2.7. **Rigid meromorphic functions and principal divisors.** Let $\mathscr{M}^\times$ denote the multiplicative group of rigid meromorphic functions on $\mathfrak{H}_p$. A locally finite divisor in the image of the natural divisor map

$$\mathrm{H}^0(\Gamma,\mathscr{M}^\times)\longrightarrow\mathrm{H}^0(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p)) \tag{20}$$

is called a *principal divisor*, and the group of such divisors is denoted $P^0(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p))$. Any principal divisor on $\mathfrak{H}_p$ is the pullback under π of an element in the group $P(X)$ of principal divisors on $X(\mathbb{C}_p) = \Gamma\backslash\,\mathfrak{H}_p$. Let $P^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))$ denote the group of *unramified* principal divisors on $\Gamma\backslash\,\mathfrak{H}_p$.

Note $\mathrm{H}^0(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p))$ and $P^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))$ are both contained in $\mathrm{H}^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p))_0$. The following lemma plays an important role in the definition of the Néron symbol.

Lemma 10. *The groups* $\mathrm{H}^0(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p))$ *and* $P^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))$ *together generate a finite index subgroup of* $\mathrm{H}^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p))_0$.

Proof. Let Ξ denote the natural image of the group $P^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))$ in $\mathbb{Z}^{\Gamma\backslash\mathcal{V}}$ under the map Deg of (15). By Cerednik–Drinfeld § 2.1, the quotient group

$$\frac{\mathrm{H}^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))}{P^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))} = \frac{\mathrm{Div}_0(X(\mathbb{Q}_p^{\mathrm{nr}}))}{P(X(\mathbb{Q}_p^{\mathrm{nr}}))}$$

is annihilated by any Hecke operator that kills $S_2^{\mathrm{new}}(pD_B)$, whereas the target $\mathbb{Z}^{\Gamma\backslash\mathcal{V}}$ of the degree map is annihilated by any Hecke operator that kills $S_2^{\mathrm{new}}(D_B)$. Since the spectra of these two spaces of newforms are disjoint, one can choose a Hecke operator T that annihilates $\mathrm{Div}_0(X)/P(X)$ while acting invertibly on $\mathbb{Q}^{\Gamma\backslash\mathcal{V}}$. The quotient $\mathbb{Z}^{\Gamma\backslash\mathcal{V}}/\Xi$ is then annihilated by T, since it is a homomorphic image of $\mathrm{Div}_0(X)/P(X)$, and therefore it must be finite. The lemma follows. ☐

Remark 11. The proof of Lemma 10 may strike the reader as being somewhat overwrought, invoking the theory of Cerednik-Drinfeld and of automorphic forms on definite quaternion algebras to prove what is *in fine* a general fact about rigid meromorphic functions on Mumford curves. It is presented here to motivate the almost identical proof of the analogous Lemma 23 in the setting of indefinite quaternion algebras, where the geometry of Mumford curves is ostensibly inapplicable.

Given an unramified principal divisor that also lies in $\mathrm{H}^0(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p))$, the following lemma recovers a function with that divisor in terms of the multiplicative Weil symbol.

Lemma 12. *Suppose that*

$$\mathscr{D} \in \mathrm{H}^0(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p^{\mathrm{nr}})) \cap P^0(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))$$

is an unramified principal divisor of strong degree zero. Then the rigid meromorphic function

$$f_{\mathscr{D}}(z) := [(z) - (\xi_p); \mathscr{D}]$$

is a Γ*-invariant rigid meromorphic function having* $\mathscr{D}$ *as its divisor. Its natural image in* $\mathrm{H}^0(\Gamma, \mathscr{M}^\times)/\,\mathbb{C}_p^\times$ *does not depend on the choice of* $\xi_p \in \mathfrak{H}_p$.

Proof. The rigid meromorphic function $f_{\mathscr{D}}$ on $\mathfrak{H}_p$ is Γ-invariant up to multiplicative constants, by Corollary 4. To compute its periods, let $\mathscr{D}_0 \in \mathrm{H}_0(\Gamma, \mathrm{Div}_{00}(\mathfrak{H}_p^{\mathrm{nr}}))$ be any divisor satisfying

$$\mathscr{D} = \pi^*\pi_*(\mathscr{D}_0).$$

Lemma 9 implies that the periods

$$\begin{aligned} f_{\mathscr{D}}(\gamma z)/f_{\mathscr{D}}(z) &= [(\xi_p)-(\gamma\xi_p);\mathscr{D}] = \prod_{b\in\Gamma}[(\xi_p)-(\gamma\xi_p); b\,\mathscr{D}_0] \\ &= \prod_{b\in\Gamma}[\mathscr{D}_0;(b\xi_p)-(b\gamma\xi_p)] \end{aligned}$$

are p-adic units, for all $\gamma\in\Gamma$. Let $F_{\mathscr{D}}$ be a Γ-invariant function having $\mathscr{D}$ as its divisor. Then the ratio $F_{\mathscr{D}}/f_{\mathscr{D}}$ is a rigid analytic function on $\mathfrak{H}_p$ whose periods are p-adic units. They must therefore be trivial, since the image of the connecting homomorphism

$$\mathrm{H}^0(\Gamma,\mathscr{A}^\times/\mathbb{C}_p^\times)\longrightarrow \mathrm{H}^1(\Gamma,\mathbb{C}_p^\times)$$

is a lattice in $\mathrm{H}^1(\Gamma,\mathbb{C}_p^\times)$. The ratio $\mathbb{F}_{\mathscr{D}}/f_{\mathscr{D}}$ is therefore Γ-invariant and hence constant by Liouville's theorem. □

2.8. p**-adic Néron symbols and Green's functions.** The Néron symbol can now be defined. Denote the set of pairs of divisors with disjoint supports by

$$\mathrm{H}_0(\Gamma,\mathrm{Div}(\mathfrak{H}_p))_0 \mathbin{\widehat{\times}} \mathrm{H}^0(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p))_0 \subset \mathrm{H}_0(\Gamma,\mathrm{Div}(\mathfrak{H}_p))_0\times \mathrm{H}^0(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p))_0.$$

Theorem 13. *There is a unique bi-additive function*

$$[\ ,\ ]_{\mathrm{Neron}}:\mathrm{H}_0(\Gamma,\mathrm{Div}(\mathfrak{H}_p))_0 \mathbin{\widehat{\times}} \mathrm{H}^0(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p))_0\longrightarrow\mathbb{C}_p$$

satisfying, for all $(\mathscr{D}_1,\widehat{\mathscr{D}}_2)$ in its domain, that

(1) if $\widehat{\mathscr{D}}_2$ is of strong degree zero,

$$[\mathscr{D}_1,\widehat{\mathscr{D}}_2]_{\mathrm{Neron}} = \log_p[\mathscr{D}_1^\sharp;\widehat{\mathscr{D}}_2];$$

(2) if $\widehat{\mathscr{D}}_2=(f)$ is uramified and principal,

$$[\mathscr{D}_1,\widehat{\mathscr{D}}_2]_{\mathrm{Neron}} = \log_p f(\mathscr{D}_1).$$

Proof. The uniqueness of a symbol with these two properties follows from Lemma 10 since $\mathbb{C}_p$ is uniquely divisible. To check existence, note that if $\widehat{\mathscr{D}}_2$ is a Γ-invariant locally finite divisor, then any pair $(\mathscr{P},\widehat{\mathscr{D}}_2^0)$ satisfying

$$\mathscr{P}\in P^0(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}})),\qquad \widehat{\mathscr{D}}_2^0\in\mathrm{H}^0(\Gamma,\mathrm{Div}_0^\dagger(\mathfrak{H}_p)),\qquad \widehat{\mathscr{D}}_2=\mathscr{P}+\widehat{\mathscr{D}}_2^0$$

is well defined up to replacing $(\mathscr{P},\widehat{\mathscr{D}}_2^0)$ with $(\mathscr{P}+\delta,\widehat{\mathscr{D}}_2^0-\delta)$, where

$$\delta\in\mathrm{H}^0(\Gamma,\mathrm{Div}_0^\dagger(\mathfrak{H}_p))\cap P^0(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))$$

is the divisor of a Γ-invariant rigid meromorphic function f. The resulting Néron symbol $[\mathscr{D}_1,\widehat{\mathscr{D}}_2]_{\mathrm{Neron}}$ is then changed by

$$\begin{aligned} \log_p f(\mathscr{D}_1)-\log_p[\mathscr{D}_1^\sharp;\delta] &= \log_p f(\mathscr{D}_1)-\log_p f_\delta(\mathscr{D}_1^\sharp) \\ &= \log_p f(\mathscr{D}_1)-\log_p f_\delta(\mathscr{D}_1)=0, \end{aligned}$$

where
(1) $f_\delta(\mathscr{D})$ is the rigid meromorphic function constructed in Lemma 12;
(2) the second equality follows from the fact that $f_\delta(\mathscr{D})$ depends only on the image of $\mathscr{D}$ in $\mathrm{H}_0(\Gamma, \mathrm{Div}(\mathfrak{H}_p))_0$;
(3) the penultimate expression vanishes because f and f_δ have the same divisor and hence differ by an element of $\mathrm{H}^0(\Gamma, \mathscr{A}^\times) = \mathbb{C}_p^\times$, and therefore these functions coincide on degree zero divisors.

□

With the Néron symbol in hand, the Green's function G_p can now be defined:

Definition 14. *The* Green's function

$$G_p : \mathrm{Div}_0(\Gamma\backslash\, \mathfrak{H}_p) \,\widehat{\times}\, \mathrm{Div}_0(\Gamma\backslash\, \mathfrak{H}_p) \longrightarrow \mathbb{C}_p$$

is defined by setting

$$G_p(\alpha_1, \alpha_2) = [\mathscr{D}_{\alpha_1}, \widehat{\mathscr{D}}_{\alpha_2}]_{\mathrm{Neron}}.$$

3. Green's functions on RM cycles in $\mathfrak{H} \times \mathfrak{H}_p$

3.1. **The set-up.** It will be assumed throughout § 3 that R is the (unique, up to conjugation) maximal order in an *indefinite* quaternion algebra B of discriminant D_B, and that $\Gamma = R[1/p]_1^\times$. Because $(B \otimes \mathbb{R})_1^\times \simeq \mathrm{SL}_2(\mathbb{R})$ is non compact, the group Γ no longer acts discretely on $\mathfrak{H}_p$. It can be expressed as an increasing union of sets just as in (13), but the subgroup $\Gamma_0 = R_1^\times$ is now an infinite arithmetic subgroup of Γ, acting discretely on $\mathfrak{H}$ by Möbius transformations. The quotient $\Gamma_0\backslash\,\mathfrak{H}$ (suitably compactified when B is the split quaternion algebra $M_2(\mathbb{Q})$) is identified with the complex points of a Shimura curve, whose space of regular differentials is isomorphic to $S_2^{\mathrm{new}}(D_B)$ as a Hecke module.

On the other hand, the quotients $\Gamma_0\backslash\,\mathfrak{H}_p$ or $\Gamma\backslash\,\mathfrak{H}_p$ ostensibly lack a clear connection to geometric objects like Shimura curves, and $\mathrm{H}^0(\Gamma_0, \mathrm{Div}^\dagger(\mathfrak{H}_p))$ is trivial. The group Γ does act discretely on $\mathfrak{H} \times \mathfrak{H}_p$, and it is suggestive to view the quotient $\Gamma\backslash(\mathfrak{H} \times \mathfrak{H}_p)$ as a "mock Hilbert modular surface" endowed with a supply of closed cycles of real dimension one which will now be described.

Let K be a real quadratic field in which all the primes dividing pD_B are non-split, let $\mathcal{O}$ be a (not necessarily maximal) order in K, and let $\alpha : K \longrightarrow B$ be an algebra embedding, satisfying $\alpha(K) \cap R = \alpha(\mathcal{O})$. Let $\tau_\alpha \in \mathfrak{H}_p$ denote the fixed point of $\alpha(K^\times)$ acting on $\mathfrak{H}_p$, normalised by requiring that $K^\times$ act on the tangent space of τ_α via the chosen embedding of K into $\mathbb{C}_p$. The stabiliser of α in Γ or Γ_0 is a group of rank one, generated up to torsion by the *automorph*

$$\gamma_\alpha \in \Gamma_\alpha := \mathrm{Stab}_\Gamma(\alpha).$$

Choose a base point $\xi_\infty \in \mathfrak{H}$ and let Z_α denote the image of the one-chain

$$\tilde{\mathscr{D}}_\alpha := [\xi_\infty, \gamma_\alpha\xi_\infty] \times \{\tau_\alpha\} \subset \mathfrak{H} \times \mathfrak{H}_p \tag{21}$$

in the quotient space $\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p)$, where $[z_1,z_2]$ represents the hyperbolic geodesic segment on $\mathfrak{H}$ joining z_1 to z_2. The RM point $\tau_\alpha\in\Gamma\backslash\mathfrak{H}_p^{\mathrm{RM}}$ thus corresponds to the closed cycle Z_α of real dimension one in the quotient space $\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p)$. Let $Z_1(\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p))$ denote the free abelian group generated by these cycles, as α varies over all embeddings into R of real quadratic orders in which all primes dividing pD_B are non-split.

Theorem 13 will be generalised to the indefinite setting by passing to higher cohomology, in order to define a Green's function on $Z_1(\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p))$ that mixes topological intersections at the archimedean place with rigid analytic function theory at p.

3.2. **Cohomological preliminaries.** Recall that, if M is any Γ-module, then

$$\mathrm{H}_1(\Gamma,M):=Z_1(\Gamma,M)/B_1(\Gamma,M),\qquad \mathrm{H}^1(\Gamma,M):=Z^1(\Gamma,M)/B^1(\Gamma,M),$$

where the groups appearing in the definition of $\mathrm{H}_1(\Gamma,M)$ are

$$\begin{array}{rcl} Z_1(\Gamma,M) & = & \left\{\sum\sigma_i\otimes m_i\in\mathbb{Z}[\Gamma]\otimes M\ :\ \sum m_i-\sigma_i^{-1}m_i=0\right\}, \\ \cup & & \\ B_1(\Gamma,M) & = & \left\langle\sigma\tau\otimes m-\sigma\otimes m-\tau\otimes\sigma^{-1}m\right\rangle, \end{array}$$

and those appearing in the definition of $\mathrm{H}^1(\Gamma,M)$ are

$$\begin{array}{rcl} Z^1(\Gamma,M) & = & \{f:\Gamma\to M\ :\ f(\sigma\tau)=f(\sigma)+\sigma f(\tau)\text{ for all }\sigma,\tau\in\Gamma\}, \\ \cup & & \\ B^1(\Gamma,M) & = & \{f:\Gamma\to M\ :\ f(\sigma)=\sigma m-m,\ \text{for some }m\in M\}. \end{array}$$

If $\mathbb{D}$ and $\mathbb{D}'$ are two left Γ-modules equipped with a Γ-equivariant pairing

$$[\ ,\ ]:\mathbb{D}\times\mathbb{D}'\longrightarrow\mathbb{C}_p^\times,$$

then there is an induced pairing

$$[\ ,\ ]:\mathrm{H}_1(\Gamma,\mathbb{D})\times\mathrm{H}^1(\Gamma,\mathbb{D}')\longrightarrow\mathbb{C}_p^\times\ ;\quad\left[\sum\sigma_i\otimes\delta_i,f\right]=\sum[\delta_i,f(\sigma_i)].$$

Applying this general fact to the extended Weil symbol $[\ ,\ ]$ on the modules $\mathbb{D}=\mathrm{Div}_0(\mathfrak{H}_p)$ and $\mathbb{D}'=\mathrm{Div}_0^\dagger(\mathfrak{H}_p)$, one obtains from the extended Weil symbol an induced pairing

$$[\ ,\ ]:\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))\times\mathrm{H}^1(\Gamma,\mathrm{Div}_0^\dagger(\mathfrak{H}_p))\longrightarrow\mathbb{C}_p^\times$$

denoted in the same way by a slight abuse of notation. The idea will be to upgrade the p-adic logarithm of this pairing to a Néron symbol

$$[\ ,\ ]_{\mathrm{Neron}}:\mathrm{H}_1(\Gamma,\mathrm{Div}(\mathfrak{H}_p))\ \hat{\times}\ \mathrm{H}^1(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p))\longrightarrow\mathbb{C}_p,$$

and, analogous to Definition 14, to parlay this symbol into a Green's function

$$G_p:Z_1(\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p))\ \hat{\times}\ Z_1(\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p))\ \longrightarrow\ \mathbb{C}_p\,.$$

3.3. **RM cycles and divisor-valued homology classes.** This section and the next explain how an RM cycle $Z_\alpha \in Z_1(\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p))$ attached to an embedding α of a real quadratic order into R gives rise to natural classes

$$\mathscr{D}_\alpha \in \mathrm{H}_1(\Gamma, \mathrm{Div}(\mathfrak{H}_p)), \qquad \hat{\mathscr{D}}_\alpha \in \mathrm{H}^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p)).$$

Definition 15. *The class in* $\mathrm{H}_1(\Gamma, \mathrm{Div}(\mathfrak{H}_p))$ *associated to* α *is defined by*

$$\mathscr{D}_\alpha = \gamma_\alpha \otimes [\tau_\alpha].$$

The assignment $\alpha \mapsto \mathscr{D}_\alpha$ extends by linearity to a map

$$Z^1(\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p)) \longrightarrow H_1(\Gamma, \mathrm{Div}(\mathfrak{H}_p)).$$

It will sometimes be useful to attach to $\mathscr{D}_\alpha$ a finite linear combination $\tilde{\mathscr{D}}_\alpha$ of (non-closed) geodesic segments in $\mathfrak{H}\times\mathfrak{H}_p$ satisfying $\pi_*(\tilde{\mathscr{D}}_\alpha) = Z_\alpha$. This is done by choosing a base point $\xi_\infty \in \mathfrak{H}$, and defining $\tilde{\mathscr{D}}_\alpha$ as in (21).

3.4. **RM cycles and divisor-valued cohomology classes.** The construction of the class

$$\widehat{\mathscr{D}}_\alpha \in \mathrm{H}^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p))$$

attached to $\alpha \in \Gamma\backslash\,\mathfrak{H}_p^{\mathrm{RM}}$, which we now describe, is slightly more involved. Recall that $(\alpha) \subset \mathfrak{H}$ denotes the oriented open geodesic on $\mathfrak{H}$ attached to α. For each $b \in \Gamma/\Gamma_\alpha$, we obtain a well-defined point and open geodesic

$$b\tau_\alpha \in \mathfrak{H}_p, \qquad b(\alpha) \subset \mathfrak{H}\,.$$

Choose a base point $\xi_\infty \in \mathfrak{H}$ in the complement of the Γ-translates of all the geodesics in $\Gamma\cdot(\alpha)$. The choice of ξ_∞ ensures that the open geodesics $b(\alpha)$ in $\mathfrak{H}$ intersect properly with the closed geodesic segment $[\xi_\infty, \gamma\xi_\infty]$. After choosing an orientation on $\mathfrak{H}$, which is fixed henceforth, we obtain a well-defined intersection number

$$[\xi_\infty, \gamma\xi_\infty] \frown b(\alpha) \in \{-1, 0, 1\}.$$

For any $\gamma \in \Gamma$, consider the formal divisor defined by

$$\widehat{\mathscr{D}}_\alpha(\gamma) := \sum_{b\in\Gamma/\Gamma_\alpha} ([\xi_\infty, \gamma\xi_\infty] \frown b(\alpha)) \cdot [b\tau_\alpha]. \tag{22}$$

Lemma 16. *The formal divisor* $\widehat{\mathscr{D}}_\alpha(\gamma)$ *is locally finite. The assignment* $\gamma \mapsto \widehat{\mathscr{D}}_\alpha(\gamma)$ *defines a one-cocycle on* Γ *with values in* $\mathrm{Div}^\dagger(\mathfrak{H}_p)$ *whose image in* $\mathrm{H}^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p))$ *does not depend on the choice of base point* ξ_∞ *that was made to define it.*

Proof. To prove the first assertion it suffices to show that for each $n \geq 0$, the divisor

$$\widehat{\mathscr{D}}_\alpha^{(n)}(\gamma) := \widehat{\mathscr{D}}_\alpha(\gamma) \cap \mathfrak{H}_p^{(n)} = \sum_{b\in\Gamma_n/\Gamma_\alpha} ([\xi_\infty, \gamma\xi_\infty] \frown b(\alpha)) \cdot [b\tau_\alpha]$$

has finite support. But the union of geodesic cycles

$$\sum_{b\in\Gamma_n/\Gamma_\alpha} b(\alpha)$$

on $\mathfrak{H}$ is invariant under Γ_0, and is the pull-back to $\mathfrak{H}$ of a finite union, indexed by elements of the finite set $\Gamma_0\backslash\Gamma_n/\Gamma_\alpha$, of geodesics on $\Gamma_0\backslash\mathfrak{H}$ under the natural projection $\pi:\mathfrak{H}\to\Gamma_0\backslash\mathfrak{H}$. Let ξ denote this closed geodesic on $\Gamma_0\backslash\mathfrak{H}$, satisfying

$$\pi^*(\xi)=\sum_{b\in\Gamma_n/\Gamma_\alpha} b(\alpha). \tag{23}$$

The support of $\widehat{\mathscr{D}}_\alpha^{(n)}(\gamma)$ is contained in the intersection of $\pi^*(\xi)$ with the closed geodesic segment $[\xi_\infty,\gamma\xi_\infty]$. But π induces a bijection between this set and the intersection of the closed geodesic ξ and $\pi_*([\xi_\infty,\gamma\xi_\infty])$ in $\Gamma_0\backslash\mathfrak{H}$. Since the latter intersection is finite, the first assertion follows. The second assertion is the result of a standard calculation verifying that $\widehat{\mathscr{D}}_\alpha$ satisfies the cocycle relation, and is left to the reader. □

The assignment $\alpha\mapsto\widehat{\mathscr{D}}_\alpha$ in (22) extends by linearity to a homomorphism

$$Z_1(\Gamma\backslash(\mathfrak{H}\times\mathfrak{H}_p))\longrightarrow \mathrm{H}^1(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p)).$$

3.5. **The homology of divisors.** The Γ-homology of the short exact sequence in (7) leads to the long exact sequence

$$\cdots\longrightarrow \mathrm{H}_2(\Gamma,\mathbb{Z})\xrightarrow{\delta}\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))\xrightarrow{j}\mathrm{H}_1(\Gamma,\mathrm{Div}(\mathfrak{H}_p))\longrightarrow \mathrm{H}_1(\Gamma,\mathbb{Z})\longrightarrow\cdots \tag{24}$$

Since $\mathrm{H}_1(\Gamma,\mathbb{Z})$ is finite, one has, after tensoring with $\mathbb{Q}$,

$$\cdots\longrightarrow \mathrm{H}_2(\Gamma,\mathbb{Q})\xrightarrow{\delta}\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))_{\mathbb{Q}}\xrightarrow{j}\mathrm{H}_1(\Gamma,\mathrm{Div}(\mathfrak{H}_p))_{\mathbb{Q}}\to 0, \tag{25}$$

which is formally similar to (14), with the notable difference that the cohomological degrees have increased by 1.

Just like the group $\mathrm{H}_1(\Gamma,\mathbb{Z})$ arising in (14), the group $\mathrm{H}_2(\Gamma,\mathbb{Z})$ is intimately connected to $S_2^{\mathrm{new}}(pD_B)$. More precisely, let $R_0(p)$ be an Eichler order of level p in the quaternionic order R and let $\Gamma_0^{(p)}:=R_0(p)_1^\times$. The following lemma is analogous to Lemma 5.

Lemma 17. *The group $\mathrm{H}_2(\Gamma,\mathbb{Q})$ is isomorphic to the p-new part of $\mathrm{H}_1(\Gamma_0^{(p)}\backslash\mathfrak{H},\mathbb{Q})$. In particular, it is annihilated by any Hecke operator that kills $S_2^{\mathrm{new}}(pD_B)$.*

Proof. Let $\vec{\mathcal{E}}$ be the set of ordered edges of $\mathcal{T}$, and let $\mathbb{Z}[\vec{\mathcal{E}}]$ and $\mathbb{Z}[\mathcal{V}]$ be the set of finite linear combinations of elements of $\vec{\mathcal{E}}$ and $\mathcal{V}$ respectively. The $\mathbb{Z}$-linear map $d:\mathbb{Z}[\vec{\mathcal{E}}]\longrightarrow\mathbb{Z}[\mathcal{V}]$ satisfying $d([v_1,v_2])=[v_2]-[v_1]$ fits into a short exact sequence

$$0\to\mathbb{Z}[\vec{\mathcal{E}}]\xrightarrow{d}\mathbb{Z}[\mathcal{V}]\xrightarrow{\deg}\mathbb{Z}\to 0$$

of Γ-modules. Taking its homology and invoking Shapiro's lemma shows that $\mathrm{H}_2(\Gamma,\mathbb{Z})$ maps to the p-new part of $\mathrm{H}_1(\Gamma_0^{(p)},\mathbb{Z})$ with finite kernel and co-kernel. Since the group $\Gamma_0^{(p)}$ gives the (complex!) uniformisation of the Shimura curve X_{p,D_B} attached to D_B and auxiliary Eichler level structure at p, the result follows. □

In conclusion, any class $\mathscr{D}_\alpha \in \mathrm{H}_1(\Gamma,\mathrm{Div}(\mathfrak{H}_p))_\mathbb{Q}$ attached to an RM cycle Z_α can be lifted to a class in $\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))_\mathbb{Q}$, but this lift is *not unique*: any two lifts differ by a class in $\delta(\mathrm{H}_2(\Gamma,\mathbb{Z}))$. An important ingredient in the definition of the Néron symbol on RM cycles is the construction of a canonical right inverse of the surjective map j in (25), leading to a direct sum decomposition

$$\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))_\mathbb{Q} \;=\; \delta(\mathrm{H}_2(\Gamma,\mathbb{Q})) \,\oplus\, \mathrm{H}_1(\Gamma,\mathrm{Div}(\mathfrak{H}_p))_\mathbb{Q}.$$

This shall be carried out in § 3.8.

3.6. **The cohomology of locally finite divisors.** The Γ-cohomology of the short exact sequence (8) leads to the long exact sequence, analogous to (15):

$$0 \to \mathrm{H}^1(\Gamma,\mathrm{Div}_0^\dagger(\mathfrak{H}_p)) \longrightarrow \mathrm{H}^1(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p)) \stackrel{\mathrm{Deg}}{\longrightarrow} \mathrm{H}^1(\Gamma,\mathbb{Z}^\mathcal{V}) \longrightarrow \cdots \tag{26}$$

The following is the counterpart of Lemma 6 for indefinite quaternion algebras:

Lemma 18. *The target of the degree map Deg is equal to*

$$\mathrm{H}^1(\Gamma_0,\mathbb{Z})^2 \simeq \mathrm{H}^1(X_{D_B}(\mathbb{C}),\mathbb{Z}),$$

where X_{D_B} is the Shimura curve that is uniformised by Γ_0 over $\mathbb{C}$. In particular, this target is annihilated by any Hecke operator that kills $S_2^{\mathrm{new}}(D_B)$.

Proof. The identification of $\mathrm{H}^1(\Gamma,\mathbb{Z}^\mathcal{V})$ with $\mathrm{H}^1(\Gamma_0,\mathbb{Z})^2$ follows from Shapiro's lemma, since there are precisely two Γ-orbits in $\mathcal{V}$ and the vertex stabilisers are isomorphic to Γ_0. The last assertion is a consequence of the Jacquet–Langlands correspondence. □

3.7. **The p-adic period pairing.** The goal of this section is to define a period pairing

$$\langle\ ,\ \rangle_\Gamma : \mathrm{H}_2(\Gamma,\mathbb{Z}) \times \mathrm{H}_2(\Gamma,\mathbb{Z}) \longrightarrow \mathbb{C}_p^\times \tag{27}$$

playing the role of (18) in the setting of indefinite quaternion algebras.

Two base points η_∞ and $\xi_\infty \in \mathfrak{H}$ are said to be *in generic position* (relative to the group Γ) if η_∞ does not lie on any geodesic segments of the form $[\gamma_1\xi_\infty,\gamma_2\xi_\infty]$ with $\gamma_1,\gamma_2\in\Gamma$, and likewise ξ_∞ lies on no geodesic segment of the form $[\gamma_1\eta_\infty,\gamma_2\eta_\infty]$. This implies that, for any $r,s\in\Gamma\eta_\infty$ and any $t,u\in\Gamma\xi_\infty$, the geodesic segments $[r,s]$ and $[t,u]$ must always intersect transversally (if at all).

Lemma 19. *A pair (ξ_∞,η_∞) of base points in $\mathfrak{H}$ in generic position relative to Γ exists.*

Proof. Let Q_B be the quadric over $\mathbb{Q}$ whose points over a field E are given by

$$Q_B(E) := \{x \in B \otimes E \text{ such that } \mathrm{Trace}(x) = \mathrm{Nrd}(x) = 0\}/E^\times.$$

Identifying $\mathfrak{H}$ with $Q_B(\mathbb{C}) - Q_B(\mathbb{R})$, the action of $B^\times$ on $\mathfrak{H}$ becomes independent of the choice of a real splitting of B, and we can write $\mathfrak{H}(E) := Q_B(E) - Q_B(\mathbb{R})$ for any subfield E of $\mathbb{C}$. Let $E_1, E_2 \subset \mathbb{C}$ be two linearly disjoint abelian CM extensions of $\mathbb{Q}$. Choose $\xi_\infty \in \mathfrak{H}(E_1)$ in such a way that its Galois conjugates in $\mathfrak{H}$ do not all lie on a common geodesic. Such a property is readily achieved once the degree of E_1 is large enough. Make a similar choice for η_∞, with E_1 replaced by E_2. Given any $\gamma_1, \gamma_2 \in \Gamma$, the defining equation for the geodesic through $\gamma_1\eta_\infty$ and $\gamma_2\eta_\infty$ involves only addition, multiplication, and complex conjugation, which commutes with the automorphisms of E_1E_2. It follows that

$$\xi_\infty \in [\gamma_1\eta_\infty, \gamma_2\eta_\infty] \quad \Rightarrow \quad \xi_\infty^\sigma \in [\gamma_1\eta_\infty^\sigma, \gamma_2\eta_\infty^\sigma], \quad \text{for all } \sigma \in \mathrm{Gal}(E_1E_2/\mathbb{Q}).$$

In particular, if ξ_∞ lies on a geodesic segment of the form $[\gamma_1\eta_\infty, \gamma_2\eta_\infty]$, then the same has to be true of all of its conjugates by $\mathrm{Gal}(E_1E_2/E_2) = \mathrm{Gal}(E_1/\mathbb{Q})$, contradicting the choice of ξ_∞. The same argument applied to η_∞, with E_1 replaced by E_2, leads to the conclusion that $(\xi_\infty, \eta_\infty)$ are in general position. ☐

Recall that $\mathscr{A}^\times$ and $\mathscr{M}^\times$ denote the multiplicative group of rigid analytic and meromorphic functions on $\mathfrak{H}_p$. Fix a pair $(\eta_\infty, \xi_\infty)$ of base points in $\mathfrak{H}$ in generic position relative to Γ, and choose a base point $\xi_p \in \mathfrak{H}_p$. These choices will be used to define natural maps, in analogy with (16),

$$\begin{array}{rcccl} \Pi_\Gamma & : & \mathrm{H}_1(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p)) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathscr{M}^\times / \mathbb{C}_p^\times), \\ \Sigma_\Gamma & : & \mathrm{H}_1(\Gamma, \mathrm{Div}(\mathfrak{H}_p)) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p)). \end{array} \tag{28}$$

The map Π_Γ sends the element $\mathscr{D} = \sum_i \gamma_i \otimes \mathscr{D}_i \in \mathrm{H}_1(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p))$ to the cocycle

$$\Pi_\Gamma(\mathscr{D})(\gamma)(z) = \prod_i \prod_{b\in\Gamma} [(z) - (\xi_p); b\,\mathscr{D}_i]^{[\eta_\infty, \gamma\eta_\infty] \frown [b\xi_\infty, b\gamma_i\xi_\infty]}. \tag{29}$$

The infinite product defining $\Pi_\Gamma(\mathscr{D})(\gamma)$ as a function of z converges uniformly on affinoid subsets to a rigid meromorphic function and is independent of the choice of base point $\xi_p \in \mathfrak{H}_p$, up to multiplication by $\mathbb{C}_p^\times$. Furthermore, the assignment $\gamma \mapsto \Pi_\Gamma(\mathscr{D})(\gamma)$ defines a one-cocycle on Γ with values in $\mathscr{M}^\times / \mathbb{C}_p^\times$ whose image in $\mathrm{H}^1(\Gamma, \mathscr{M}^\times / \mathbb{C}_p^\times)$ does not depend on the choices of complex base points η_∞ and ξ_∞ that were made to define the cocycle $\Pi_\Gamma(\mathscr{D})$.

The map Σ_Γ sends the element $\mathscr{D}$ (where the $\mathscr{D}_i$ are no longer assumed to be necessarily of degree zero) to the cocycle

$$\Sigma_\Gamma(\mathscr{D})(\gamma) = \sum_i \sum_{b\in\Gamma} ([\eta_\infty, \gamma\eta_\infty] \frown [b\xi_\infty, b\gamma_i\xi_\infty]) \cdot b\,\mathscr{D}_i\,.$$

These maps fit into the commutative diagram with exact rows, analogous to (17),

(30)
$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{H}_2(\Gamma,\mathbb{Z}) & \stackrel{\delta}{\longrightarrow} & \mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p)) & \stackrel{j}{\longrightarrow} & \mathrm{H}_1(\Gamma,\mathrm{Div}(\mathfrak{H}_p)) & \longrightarrow \cdots \\ & \downarrow \Pi_\Gamma & & \downarrow \Pi_\Gamma & & \downarrow \Sigma_\Gamma & \\ 0 \longrightarrow & \mathrm{H}^1(\Gamma,\mathscr{A}^\times/\,\mathbb{C}_p^\times) & \longrightarrow & \mathrm{H}^1(\Gamma,\mathscr{M}^\times/\,\mathbb{C}_p^\times) & \longrightarrow & \mathrm{H}^1(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p)) & \longrightarrow \cdots \end{array}$$

where the top row is (24) and the second row arises from the cohomology of the short exact sequence

$$1 \to \mathscr{A}^\times\,/\,\mathbb{C}_p^\times \longrightarrow \mathscr{M}^\times\,/\,\mathbb{C}_p^\times \longrightarrow \mathrm{Div}^\dagger(\mathfrak{H}_p) \to 1.$$

Composing the leftmost map Π_Γ in (30) with the period homomorphism

(31)
$$\mathrm{per}: \mathrm{H}^1(\Gamma,\mathscr{A}^\times\,/\,\mathbb{C}_p^\times) \longrightarrow \mathrm{H}^2(\Gamma,\mathbb{C}_p^\times)$$

arising from cohomology of the short exact sequence $1 \to \mathbb{C}_p^\times \longrightarrow \mathscr{A}^\times \longrightarrow \mathscr{A}^\times\,/\,\mathbb{C}_p^\times \to 1$ yields a homomorphism

$$\eta_\Gamma := \mathrm{per}\circ\Pi_\Gamma : \mathrm{H}_2(\Gamma,\mathbb{Z}) \longrightarrow \mathrm{H}^2(\Gamma,\mathbb{C}_p^\times)$$

which induces the period pairing in (27) . This pairing is entirely analogous to the p-adic period pairing (18) arising in the Mumford-Schottky theory of p-adic uniformisation of Shimura curves when B is a definite quaternion algebras. The following extends Proposition 7 to the setting of indefinite quaternion algebras.

Proposition 20. *The $\mathbb{Z}$-valued pairing $\mathrm{ord}_p(\langle\ ,\ \rangle_\Gamma)$ is non-degenerate.*

Proof. The appendix of [DV3] explains that the natural image of $\mathrm{H}^1(\Gamma,\mathscr{A}^\times\,/\,\mathbb{C}_p^\times)$ in $\mathrm{H}^2(\Gamma,\mathbb{C}_p^\times)$ is a lattice in this p-adic torus, and the proposition follows from this. ☐

Remark 21. The quotient

$$\mathrm{H}^2(\Gamma,\mathbb{C}_p^\times)/\eta_\Gamma(\mathrm{H}_2(\Gamma,\mathbb{Z}))$$

appears to uniformise an abelian variety which is isogenous to (two copies of) new part of $J_0(pD_B)$. This striking assertion is a reformulation of the "exceptional zero conjecture" of Mazur, Tate, and Teitelbaum. The lattice $\mathrm{per}(\mathrm{H}^1(\Gamma,\mathscr{A}^\times\,/\,\mathbb{C}_p^\times))$ is studied in greater depth in [Das].

3.8. **Unitary classes.** As in § 2.6, a natural $\mathbb{Q}$-vector space complement of $j(\mathrm{H}_2(\Gamma,\mathbb{Q}))$ in $\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))_\mathbb{Q}$ can now be produced, by exploiting the map

$$\eta_\Gamma := \mathrm{ord}_p\circ\mathrm{per}\circ\Pi_\Gamma : \mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p)) \longrightarrow \mathrm{H}^2(\Gamma,\mathbb{Z}),$$

and setting

(32)
$$\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))_\mathbb{Q}^\sharp := \ker(\eta_\Gamma) \subset \mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))_\mathbb{Q}.$$

Proposition 20 implies that $\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))^\sharp_{\mathbb{Q}}$ is complementary to $\delta(\mathrm{H}_2(\Gamma,\mathbb{Q}))$. Any class $\mathscr{D}\in\mathrm{H}_1(\Gamma,\mathrm{Div}(\mathfrak{H}_p))_{\mathbb{Q}}$ therefore admits a unique unitary lift, denoted

$$\mathscr{D}^\sharp\in\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))^\sharp_{\mathbb{Q}}.$$

The following analogue of Lemma 9 provides an explicit construction of unitary classes in some cases:

Lemma 22. *Let* $\mathrm{Div}_{00}(\mathfrak{H}_p^{\mathrm{nr}})$ *be defined as in Lemma 9. The natural image of the map*

$$\mathrm{H}_1(\Gamma,\mathrm{Div}_{00}(\mathfrak{H}_p^{\mathrm{nr}}))_{\mathbb{Q}}\longrightarrow\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))_{\mathbb{Q}}$$

is contained in $\mathrm{H}_1(\Gamma,\mathrm{Div}_0(\mathfrak{H}_p))^\sharp_{\mathbb{Q}}$.

Proof. This follows from the same reasoning as in the proof of Lemma 9. Namely, if $\mathscr{D}$ belongs to $\mathrm{Div}_{00}(\mathfrak{H}_p^{\mathrm{nr}})$ then the Weil symbol $[\mathscr{D};(b\gamma\xi_p)-(b\xi_p)]$ is always a p-adic unit for any $b\in\Gamma$, as can be seen by reducing to the case where $\mathscr{D}$ is supported on $\mathrm{red}^{-1}(v)$ for a single vertex v of the tree, and choosing the base point ξ_p to reduce to a vertex that is not Γ-equivalent to v. Hence the same holds for

$$(\mathrm{per}\circ\Pi_\Gamma(\mathscr{D}))(\gamma)=\prod_{b\in\Gamma}[\mathscr{D};(b\gamma\xi_p)-(b\xi_p)],$$

and the lemma follows. □

3.9. **Rigid meromorphic cocycles and principal classes.** A *rigid meromorphic cocycle* for Γ is a one-cocycle on Γ with values in $\mathscr{M}^\times$. They play the same role in the indefinite setting as rigid meromorphic functions on the Shimura curve X in the definite setting.

A *rigid meromorphic* or *analytic theta-cocycle* for Γ is a one-cocycle on Γ with values in $\mathscr{M}^\times/\mathbb{C}_p^\times$ or $\mathscr{A}^\times/\mathbb{C}_p^\times$ respectively. Recall the homomorphism

$$\mathrm{per}:\mathrm{H}^1(\Gamma,\mathscr{M}^\times/\mathbb{C}_p^\times)\longrightarrow\mathrm{H}^2(\Gamma,\mathbb{C}_p^\times)\tag{33}$$

whose restriction to $\mathrm{H}^1(\Gamma,\mathscr{A}^\times/\mathbb{C}_p^\times)$ is the period map of (31).

A class in the image of the natural map

$$\mathrm{Div}:\mathrm{H}^1(\Gamma,\mathscr{M}^\times)\longrightarrow\mathrm{H}^1(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p)).\tag{34}$$

is called a *principal divisor*, and the group of such objects is denoted $P^1(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p))$. The following lemma is the analogue in the indefinite setup of Lemma 10.

Lemma 23. *The groups* $\mathrm{H}^1(\Gamma,\mathrm{Div}_0^\dagger(\mathfrak{H}_p))$ *and* $P^1(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))$ *together generate a finite index subgroup of* $\mathrm{H}^1(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p))$.

Proof. Let Ξ denote the image of the group $P^1(\Gamma,\mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))$ in $\mathrm{H}^1(\Gamma,\mathbb{Z}^{\mathcal{V}})$ under the map induced by the degree $\mathrm{Deg}:\mathrm{Div}^\dagger(\mathfrak{H}_p)\longrightarrow\mathbb{Z}^{\mathcal{V}}$. Any prime-to-$p$ Hecke operator that annihilates $S_2^{\rm new}(pD_B)$ sends $\mathrm{H}^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\rm nr}))$ to $P^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\rm nr}))$, and therefore kills the quotient $\mathrm{H}^1(\Gamma, \mathbb{Z}^{\mathcal{V}})/\Xi$. On the other hand, the target $\mathrm{H}^1(\Gamma, \mathbb{Z}^\vee)$ of the degree map is annihilated by any Hecke operator that kills $S_2^{\rm new}(D_B)$, by Lemma 18. Since the spectra of the Hecke operators on the spaces of newforms of levels D_B and pD_B are disjoint, one can choose a Hecke operator T that annihilates $\mathrm{H}^1(\Gamma, \mathbb{Z}^{\mathcal{V}})/\Xi$ while acting invertibly on $\mathrm{H}^1(\Gamma, \mathbb{Q}^{\Gamma\backslash\mathcal{V}})$. The quotient $\mathrm{H}_1(\Gamma, \mathbb{Z}^{\mathcal{V}}))/\Xi$ is annihilated by such a T, and is therefore finite. ☐

Given a principal class in $P^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\rm nr}))$ that also lies in $\mathrm{H}^1(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p))$, the following lemma recovers a rigid meromorphic cocycle with that divisor in terms of the multiplicative Weil symbol.

Lemma 24. *Consider an unramified class*

$$\mathscr{D} \in \mathrm{H}^1(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p)) \cap P^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\rm nr}))$$

which is principal and of strong degree zero. Then the rigid meromorphic functions

$$J_{\mathscr{D}}(\gamma)(z) := [(z) - (\xi_p); \mathscr{D}(\gamma)]$$

indexed by $\gamma \in \Gamma$, define a rigid meromorphic cocycle on Γ having $\mathscr{D}$ as its divisor. The natural image of this cocycle in $\mathrm{H}^1(\Gamma, \mathscr{M}^\times)$ does not depend on the choice of $\xi_p \in \mathfrak{H}_p$.

Proof. The rigid meromorphic functions $J_{\mathscr{D}}(\gamma)$ on $\mathfrak{H}_p$ satisfy the cocycle relation for Γ up to multiplicative constants, by Corollary 4. The same reasoning as in the proof of Lemma 9 shows that the image of $J_{\mathscr{D}}$ in $\mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$ under the period map per of (33) is contained in $\mathrm{H}^2(\Gamma, \mathcal{O}_{\mathbb{C}_p}^\times)$. Let $J \in \mathrm{H}^1(\Gamma, \mathscr{M}^\times)$ be a rigid meromorphic cocycle having $\mathscr{D}$ as its divisor. Then the ratio $J_{\mathscr{D}}/J$ gives rise to a class in $\mathrm{H}^1(\Gamma, \mathscr{A}^\times / \mathbb{C}_p^\times)$ whose image in $\mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$ is contained in $\mathrm{H}^2(\Gamma, \mathcal{O}_{\mathbb{C}_p}^\times)$. Because

$$\mathrm{per}(\mathrm{H}^1(\Gamma, \mathscr{A}^\times / \mathbb{C}_p^\times)) \subset \mathrm{H}^2(\Gamma, \mathbb{C}_p^\times)$$

is a lattice, it follows that $J_{\mathscr{D}}/J$ lifts to a class in $\mathrm{H}^1(\Gamma, \mathscr{A}^\times)$, and therefore that $J_{\mathscr{D}}$ belongs to $\mathrm{H}^1(\Gamma, \mathscr{M}^\times)$, as was to be shown. ☐

3.10. p-adic Néron symbols and Green's functions for RM cycles.

Note that the extended Weil symbol gives a natural pairing

$$[\ ,\ ] : \mathrm{H}_1(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p)) \times \mathrm{H}^1(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p)) \longrightarrow \mathbb{C}_p^\times.$$

The following theorem is the indefinite counterpart of Theorem 13.

Theorem 25. *There is a unique bi-additive function*

$$[\ ,\ ]_{\rm Neron} : \mathrm{H}_1(\Gamma, \mathrm{Div}(\mathfrak{H}_p)) \times \mathrm{H}^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p)) \longrightarrow \mathbb{C}_p$$

satisfying, for all $\mathscr{D}_1 \in \mathrm{H}_1(\Gamma, \mathrm{Div}(\mathfrak{H}_p))$ and $\widehat{\mathscr{D}}_2 \in \mathrm{H}^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p))$, *(1) if $\widehat{\mathscr{D}}_2 \in \mathrm{H}^1(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p))$ is of strong degree zero,*

$$[\mathscr{D}_1, \widehat{\mathscr{D}}_2]_{\mathrm{Neron}} = \log_p([\mathscr{D}_1^\sharp; \widehat{\mathscr{D}}_2]);$$

(2) if $\widehat{\mathscr{D}}_2$ is the divisor of the rigid meromorphic cocycle J_2 and is unramified, then

$$[\mathscr{D}_1, \widehat{\mathscr{D}}_2]_{\mathrm{Neron}} = \log_p(J_2[\mathscr{D}_1]).$$

Proof. The proof proceeds along the same lines as the proof of Theorem 13. The uniqueness follows from Lemma 23 since $\mathbb{C}_p$ is uniquely divisible. To check existence, let us define $[\mathscr{D}_1, \widehat{\mathscr{D}}_2]_{\mathrm{Neron}}$ by writing $\widehat{\mathscr{D}}_2 = \mathscr{P} + \widehat{\mathscr{D}}_2^0$ with

$$\mathscr{P} := \mathrm{Div}(J_{\mathscr{P}}) \in P^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))_{\mathbb{Q}}, \qquad \widehat{\mathscr{D}}_2^0 \in \mathrm{H}^1(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p))_{\mathbb{Q}},$$

and setting

$$[\mathscr{D}_1, \widehat{\mathscr{D}}_2]_{\mathrm{Neron}} := \log_p J_{\mathscr{P}}[\mathscr{D}_1] + \log_p[\mathscr{D}_1^\sharp; \widehat{\mathscr{D}}_2^0]. \tag{35}$$

The pair $(\mathscr{P}, \widehat{\mathscr{D}}_2^0)$ is well-defined up to replacing it with $(\mathscr{P} + \delta, \widehat{\mathscr{D}}_2^0 - \delta)$, where

$$\delta \in \mathrm{H}^1(\Gamma, \mathrm{Div}_0^\dagger(\mathfrak{H}_p)) \cap P^1(\Gamma, \mathrm{Div}^\dagger(\mathfrak{H}_p^{\mathrm{nr}}))$$

is the divisor of a Γ-invariant rigid meromorphic cocycle J. The resulting expression (35) for the Néron symbol $[\mathscr{D}_1, \widehat{\mathscr{D}}_2]_{\mathrm{Neron}}$ is then changed by

$$\begin{aligned} \log_p J[\mathscr{D}_1] - \log_p[\mathscr{D}_1^\sharp; \delta] &= \log_p J[\mathscr{D}_1] - \log_p J_\delta[\mathscr{D}_1^\sharp] \\ &= \log_p J[\mathscr{D}_1] - \log_p J_\delta[\mathscr{D}_1] = 0, \end{aligned}$$

where

(1) J_δ is the rigid meromorphic cocycle constructed in Lemma 24;
(2) the second equality follows from the fact that $J_\delta[\mathscr{D}]$ depends only on the image of $\mathscr{D}$ in $\mathrm{H}_1(\Gamma, \mathrm{Div}(\mathfrak{H}_p))$;
(3) the vanishing follows since the rigid meromorphic cocycles J and J_δ have the same divisor and hence differ by an element of the finite group $\mathrm{H}^1(\Gamma, \mathscr{A}^\times)$, and therefore the logarithms of the values $J[\mathscr{D}_1]$ and $J_\delta[\mathscr{D}_1]$ are equal.

□

Let α_1 and α_2 be non Γ-conjugate embeddings of real quadratic $\mathbb{Z}[1/p]$-orders into $R[1/p]$. After possibly interchanging α_1 and α_2, it can be assumed that $p \nmid D_2$. The following defines the p-adic Green's function on the pair (α_1, α_2) of RM divisors on $\mathfrak{H}_p$:

Definition 26. *The p-adic Green's function $G_p(\alpha_1, \alpha_2)$ is equal to*

$$G_p(\alpha_1, \alpha_2) = [\mathscr{D}_{\alpha_1}, \widehat{\mathscr{D}}_{\alpha_2}]_{\mathrm{Neron}}.$$

3.11. **The local trace of $G_p(\alpha_1, \alpha_2)$.** In this section, let us assume further that

$$\alpha_1 : \mathcal{O}_1 \longrightarrow R, \qquad \alpha_2 : \mathcal{O}_2 \longrightarrow R$$

are embeddings of real quadratic orders of relatively prime discriminants D_1 and D_2 respectively, and that the prime p does not divide $D_1 D_2$, and therefore is inert in both $K_1 = \mathbb{Q}(\sqrt{D_1})$ and $K_2 = \mathbb{Q}(\sqrt{D_2})$. Recall that

$$F = \mathbb{Q}(\sqrt{D_1 D_2}), \qquad L = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}),$$

viewed as subfields of $\bar{\mathbb{Q}}_p$, and that $F_p = \mathbb{Q}_p$ and L_p denote their respective completions in this p-adic field. As an application of the general theory of the p-adic Green's function, we will give an explicit formula for the *trace to* F_p of the Green's function $G_p(\alpha_1, \alpha_2)$, in the special case where the group Γ_0 of (13) has *finite abelianisation*, and hence uniformises a curve over $\mathbb{C}$ of genus zero. This assumption is satisfied in the following two cases:

(1) The algebra B is the split quaternion algebra $M_2(\mathbb{Q})$, and hence $R = M_2(\mathbb{Z})$ and $\Gamma_0 = \mathrm{SL}_2(\mathbb{Z})$. In that case,

$$\mathrm{H}_1(\Gamma_0, \mathbb{Z}) = \mathbb{Z}/12\,\mathbb{Z}, \qquad \mathrm{H}^1(\Gamma_0, \mathbb{Z}) = 0$$

and

$$\mathrm{H}_1(\Gamma, \mathbb{Z}) = \begin{cases} \mathbb{Z}/3\,\mathbb{Z} & \text{if } p = 2; \\ \mathbb{Z}/4\,\mathbb{Z} & \text{if } p = 3; \\ \mathbb{Z}/12\,\mathbb{Z} & \text{otherwise,} \end{cases} \qquad \mathrm{H}^1(\Gamma, \mathbb{Z}) = 0,$$

cf. Serre [Se1]. This corresponds to the setting considered in [DV1].

(2) The algebra B is a non-split indefinite quaternion algebra of discriminant D_B, and there are no weight two cuspidal newforms of weight two and level D_B. This happens precisely when $D_B = 6$, 10, or 22.

Let τ_1 and τ_2 be the fixed points of $\alpha_1(K_1^\times)$ and $\alpha_2(K_2^\times)$ on $\mathfrak{H}_p$, let Γ_1 and Γ_2 be the associated stabiliser groups, and let γ_1 and γ_2 be the associated automorphs.

The class $\mathscr{D}_1 := \gamma_1 \otimes [\tau_1] \in \mathrm{H}_1(\Gamma, \mathrm{Div}(\mathfrak{H}_p))$ is the image of a class in $\mathrm{H}_1(\Gamma_0, \mathrm{Div}_0(\mathfrak{H}_p^{(0)}))$ under the natural map from the latter to the former. Let $\mathscr{D}_1^\sharp$ be the unique lift of $\mathscr{D}_1$ to the group $\mathrm{H}^1(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p))^\sharp$ of unitary one-cycles. The following lemma describes this element in terms of the cohomology of the subgroup Γ_0:

Lemma 27. *The class $\mathscr{D}_1^\sharp$ can be represented by an element in the image of the natural map*

$$\mathrm{H}_1(\Gamma_0, \mathrm{Div}_0(\mathfrak{H}_p^{(0)})) \longrightarrow \mathrm{H}_1(\Gamma, \mathrm{Div}_0(\mathfrak{H}_p)).$$

Proof. That the class $\mathscr{D}_1$ admits a representative in $\mathrm{H}_1(\Gamma_0, \mathrm{Div}_0(\mathfrak{H}_p^{(0)}))$ follows from the fact that Γ_0 has finite abelianisation and hence the natural map

$$\mathrm{H}_1(\Gamma_0, \mathrm{Div}_0(\mathfrak{H}_p^{(0)})) \longrightarrow \mathrm{H}_1(\Gamma_0, \mathrm{Div}(\mathfrak{H}_p^{(0)}))$$

has finite cokernel.This representative agrees with $\mathscr{D}_1^\sharp$, since all classes in the group $\mathrm{H}_1(\Gamma_0, \mathrm{Div}_0(\mathfrak{H}_p^{(0)}))_{\mathbb{Q}}$ belong to $\mathrm{H}_1(\Gamma_0, \mathrm{Div}_0(\mathfrak{H}_p))_{\mathbb{Q}}^\sharp$, by Lemma 22. □

Now let $\mathscr{D}_2^{(n)} \in \mathrm{H}^1(\Gamma_0, \mathrm{Div}(\mathfrak{H}_p^{(n)}))$ be the image of the one-cocycle $\widehat{\mathscr{D}}_2$ under the map

$$\mathrm{H}^1(\Gamma, \mathrm{Div}(\mathfrak{H}_p)) \longrightarrow \mathrm{H}^1(\Gamma_0, \mathrm{Div}(\mathfrak{H}_p^{(n)}))$$

obtained by restricting to Γ_0 and applying the Γ_0-equivariant map $\mathscr{D} \mapsto \mathscr{D}^{(n)}$ to the coefficients. Explicitly, having choosen a generic base point $\xi_\infty \in \mathfrak{H}$, one has

$$\mathscr{D}_2^{(n)}(\gamma) = \sum_{b \in R[p^{2n}]/\Gamma_2} ([\xi_\infty, \gamma\xi_\infty] \frown b(\alpha_2) \cdot [b\tau_2]. \tag{36}$$

It is apparent that $\mathscr{D}_2^{(n)}$ satisfies a one-cocycle relation for Γ_0. Furthermore, the triviality of $\mathrm{H}^1(\Gamma_0, \mathbb{Z})$ shows that the cocycle $\mathscr{D}_2^{(n)}$ takes values in $\mathrm{Div}_0(\mathfrak{H}_p^{(n)})$, and therefore gives rise to a class in $\mathrm{H}^1(\Gamma_0, \mathrm{Div}_0(\mathfrak{H}_p^{(n)}))$. Let $\mathcal{R}_n^\times$ denote the Γ_0-module of rational functions on $\mathbb{P}_1(\mathbb{C}_p)$ whose divisor is supported in $\mathfrak{H}_p^{(n)}$. Taking the cohomology of the short exact sequence of Γ_0-modules

$$1 \to \mathbb{C}_p^\times \longrightarrow \mathcal{R}_n^\times \longrightarrow \mathrm{Div}_0(\mathfrak{H}_p^{(n)}) \to 1,$$

and observing that $\mathrm{H}^2(\Gamma_0, \mathbb{C}_p^\times)$ is essentially trivial, one expects the classes $\mathscr{D}_2^{(n)}$ to admit lifts to $\mathrm{H}^1(\Gamma_0, \mathcal{R}_n^\times)$.

The key lemma 28 below produces a partial lift to $\mathcal{R}_n^\times/\epsilon_2^{\mathbb{Z}}$, where ϵ_2 is the fundamental unit of norm 1 of the real quadratic field K_2. To formulate it, we remark that for any $b \in R$, the function

$$\det\left(\begin{pmatrix} z \\ 1 \end{pmatrix}, b\begin{pmatrix} \tau_2 \\ 1 \end{pmatrix}\right)$$

is a linear polynomial in z with a zero at $b\tau_2$, and that its image in $\mathcal{R}_n^\times/\epsilon_2^{\mathbb{Z}}$ depends only on the class of b in $R[p^{2n}]/\gamma_2^{\mathbb{Z}}$, since replacing b by $b\gamma_2^t$ has the effect of multiplying this function by ϵ_2^t.

Lemma 28. *The $\mathcal{R}_n^\times/\epsilon_2^{\mathbb{Z}}$-valued function $J_2^{(n)}$ on Γ_0 defined by*

$$J_2^{(n)}(\gamma)(z) = \prod_{b \in R[p^{2n}]/\gamma_2^{\mathbb{Z}}} \det\left(\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right), b\left(\begin{smallmatrix} \tau_2 \\ 1 \end{smallmatrix}\right)\right)^{[\eta_\infty, \gamma\eta_\infty] \frown b(\alpha_2)} \pmod{\epsilon_2^{\mathbb{Z}}} \tag{37}$$

represents a lift of $\mathscr{D}_2^{(n)}$ to $\mathrm{H}^1(\Gamma_0, \mathcal{R}_n^\times/\epsilon_2^{\mathbb{Z}})$.

Proof. The rational function on the right hand side of (37) has divisor equal to $\mathscr{D}_2^{(n)}(\gamma)$ in (36). It therefore suffices to show that $J_2^{(n)}$ satisfies the relations of a one-cocycle on Γ_0 with values in$\mathcal{R}_n^\times/\epsilon_2^{\mathbb{Z}}$. This is readily checked by a direct calculation. □

We are now ready to state the main result. Recall that, for any two quadratic elements τ_1 and τ_2 of $\mathfrak{H}_p$, having τ_1' and τ_2' as their conjugates, we have set

$$g(\tau_1,\tau_2)=\frac{(\tau_1-\tau_2)(\tau_1'-\tau_2')}{(\tau_1-\tau_1')(\tau_2-\tau_2')},$$

and $\mathbb{G}_p(\alpha_1,\alpha_2)=\mathrm{Trace}_{F_p}^{L_p}G_p(\alpha_1,\alpha_2)$.

Theorem 29. *Assume that Γ_0 has genus zero. Then*

$$\mathbb{G}_p(\alpha_1,\alpha_2)=\lim_{n\to\infty}\log_p\Big(\prod_{b\in\Gamma_1\backslash R[p^{2n}]/\Gamma_2}g(\tau_1,b\tau_2)^{(\alpha_1)\frown b(\alpha_2)}\Big).$$

Proof. By definition,

$$G_p(\alpha_1,\alpha_2)=[\mathscr{D}_1^\sharp,\widehat{\mathscr{D}}_2]_{\mathrm{Neron}}=\lim_{n\to\infty}\log_p[\mathscr{D}_1^\sharp;\mathscr{D}_2^{(n)}]. \tag{38}$$

Because the class $\mathscr{D}_1^\sharp$ is not readily described explicitly, we shall content ourselves with the evaluation of $[\mathscr{D}_1^\sharp;\mathscr{D}_2^{(n)}]$ modulo $\epsilon_2^{\mathbb{Z}}$, invoking Lemma 28 to rewrite

$$[\mathscr{D}_1^\sharp;\mathscr{D}_2^{(n)}]=J_2^{(n)}[\mathscr{D}_1^\sharp]=J_2^{(n)}[\mathscr{D}_1]\pmod{\epsilon_2^{\mathbb{Z}}}. \tag{39}$$

But the quantity $J_2^{(n)}[\mathscr{D}_1]$ equals

$$\begin{aligned}
J_2^{(n)}(\gamma_1)(\tau_1) &= \prod_{b\in R[p^{2n}]/\Gamma_2}\det\left(\left(\begin{smallmatrix}\tau_1\\1\end{smallmatrix}\right),b\left(\begin{smallmatrix}\tau_2\\1\end{smallmatrix}\right)\right)^{[\eta_\infty,\gamma_1\eta_\infty]\frown b(\alpha_2)} && \pmod{\epsilon_2^{\mathbb{Z}}}\\
&= \prod_{b\in\Gamma_1\backslash R[p^{2n}]/\Gamma_2}\ \prod_{j=-\infty}^{\infty}\det\left(\left(\begin{smallmatrix}\tau_1\\1\end{smallmatrix}\right),\gamma_1^j b\left(\begin{smallmatrix}\tau_2\\1\end{smallmatrix}\right)\right)^{[\eta_\infty,\gamma_1\eta_\infty]\frown\gamma_1^j b(\alpha_2)} && \pmod{\epsilon_2^{\mathbb{Z}}}\\
&= \prod_{b\in\Gamma_1\backslash R[p^{2n}]/\Gamma_2}\det\left(\left(\begin{smallmatrix}\tau_1\\1\end{smallmatrix}\right),b\left(\begin{smallmatrix}\tau_2\\1\end{smallmatrix}\right)\right)^{(\alpha_1)\frown b(\alpha_2)} && \pmod{\epsilon_1^{\mathbb{Z}}\epsilon_2^{\mathbb{Z}}}.
\end{aligned}$$

Taking the norm of this identity to $F_p^\times$ has the pleasant feature that it eliminates the ambiguity by the units of K_1 and K_2. One finds

$$\mathrm{Norm}_{F_p}^{L_p}(J_2^{(n)}[\mathscr{D}_1])=\prod_{b\in\gamma_1^{\mathbb{Z}}\backslash R[p^{2n}]/\gamma_2^{\mathbb{Z}}}g(\tau_1,b\tau_2)^{(\alpha_1)\frown b(\alpha_2)}. \tag{40}$$

The theorem now follows from combining (38), (39), and (40). □

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