STARK-HEEGNER POINTS AND GENERALISED KATO CLASSES

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Abstract. Stark-Heegner points are conjectural substitutes for Heegner points when the imaginary quadratic field of the theory of complex multiplication is replaced by a real quadratic field K. They are constructed analytically as local points on elliptic curves with multiplicative reduction at a prime p that remains inert in K, but are conjectured to be rational over ring class fields of K and to satisfy a Shimura reciprocity law describing the action of G_K on them. The main conjectures of [Dar] predict that any linear combination of Stark-Heegner points weighted by the values of a ring class character ψ of K should belong to the corresponding piece of the Mordell-Weil group over the associated ring class field, and should be non-trivial when $L'(E/K,\psi,1) \neq 0$. The main result of this article is that such linear combinations arise from global classes in the idoneous pro-p Selmer group, and are non-trivial when the first derivative of a weight-variable p-adic L-function $\mathscr{L}_p(\mathbf{f}/K,\psi)$ does not vanish at the point associated to $(E/K,\psi)$. The proof rests on the construction of a three-variable family $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})$ of cohomology classes associated to a triple of Hida families and a direct comparison between Stark-Heegner points and the *generalised Kato classes* arising by specialising $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})$ at weights $(2,1,1)$ for a suitable choice of Hida families. The explicit formula that emerges from this comparison is of independent interest and supplies theoretical evidence for the elliptic Stark Conjectures of [DLR].

Contents

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Introduction

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N and let K be a quadratic field of discriminant D relatively prime to N, with associated Dirichlet character χ_K.

When $\chi_K(-N) = -1$, the Birch and Swinnerton-Dyer conjecture predicts a systematic supply of rational points on E defined over abelian extensions of K. More precisely, if H is any ring class field of K attached to an order $\mathcal{O}$ of K of conductor prime to DN, the Hasse-Weil L-function $L(E/H, s)$ factors as a product

$$L(E/H, s) = \prod_{\psi} L(E/K, \psi, s) \tag{1}$$

of twisted L-series $L(E/K, \psi, s)$ indexed by the finite order characters

$$\psi : G = \mathrm{Gal}\,(H/K) \longrightarrow L^\times,$$

taking values in some fixed finite extension L of $\mathbb{Q}$. The L-series in the right-hand side of (1) all vanish to odd order, because they arise from self-dual Galois representations and have sign $\chi_K(-N)$ in their functional equation. In particular, $L(E/K, \psi, 1) = 0$ for all ψ. An equivariant refinement of the Birch and Swinnerton-Dyer conjecture predicts that the ψ-eigenspace $E(H)^\psi \subset E(H) \otimes L$ of the Mordell-Weil group for the action of $\mathrm{Gal}\,(H/K)$ has dimension ≥ 1, and hence, that $E(H) \otimes \mathbb{Q}$ contains a copy of the regular representation of G.

When K is imaginary quadratic, this prediction is largely accounted for by the theory of Heegner points on modular or Shimura curves, which for each ψ as above produces an explicit element $P_\psi \in E(H)^\psi$. The Gross-Zagier formula implies that P_ψ is non-zero when $L'(E/K, \psi, 1) \neq 0$. Thus it follows for instance that $E(H) \otimes \mathbb{Q}$ contains a copy of the regular representation of G when $L(E/H, s)$ vanishes to order $[H : K]$ at the center.

When K is real quadratic, the construction of non-trivial algebraic points in $E(H)$ appears to lie beyond the scope of available techniques. Extending the theory of Heegner points to this setting thus represents a tantalizing challenge at the frontier of our current understanding of the Birch and Swinnerton-Dyer conjecture.

Assume from now on that $D > 0$ and there is an odd prime p satisfying

$$N = pM \text{ with } p \nmid M, \qquad \chi_K(p) = -1, \qquad \chi_K(M) = 1. \tag{2}$$

A conjectural construction of Heegner-type points, under the further restriction that $\chi_K(\ell) = 1$ for all $\ell | M$, was proposed in [Dar], and extended to the more general setting of (2) in [Gr09], [DG], [LRV], [KPM] and [Re]. It leads to a canonical collection of so-called *Stark-Heegner points*

$$P_{\mathfrak{a}} \in E(H \otimes \mathbb{Q}_p) = \prod_{\wp | p} E(H_\wp),$$

indexed by the ideal classes $\mathfrak{a}$ of $\mathrm{Pic}(\mathcal{O})$, which are regarded as here as *semi-local points*, i.e., $[H : K]$-tuples $P_{\mathfrak{a}} = \{P_{\mathfrak{a},\wp}\}_{\wp|p}$ of local points in $E(K_p)$. This construction, and its equivalence with the slightly different approach of the original one, is briefly recalled in §2.1.

As a formal consequence of the definitions (cf. Lemma 2.1), the semi-local points $P_{\mathfrak{a}}$ satisfy the Shimura reciprocity law

$$P_{\mathfrak{a}}^\sigma = P_{\mathrm{rec}(\sigma)\cdot\mathfrak{a}} \qquad \text{for all } \sigma \in G,$$

where G acts on the group $E(H \otimes \mathbb{Q}_p)$ in the natural way and $\mathrm{rec} : G \to \mathrm{Pic}(\mathcal{O})$ is the Artin map of global class field theory.

The construction of the semi-local point $P_{\mathfrak{a}} \in \prod_{\wp|p} E(H_\wp)$ is purely p-adic analytic, relying on a theory of p-adic integration of 2-forms on the product $\mathcal{H}\times\mathcal{H}_p$, where $\mathcal{H}$ denotes Poincaré's complex upper half plane and $\mathcal{H}_p$ stands for Drinfeld's rigid analytic p-adic avatar of $\mathcal{H}$, the integration being performed, metaphorically speaking, on two-dimensional regions in $\mathcal{H}_p \times \mathcal{H}$ bounded by Shintani-type cycles associated to ideal classes in K. The following statement of the Stark-Heegner conjectures of loc.cit. is equivalent to [Dar, Conj. 5.6, 5.9 and 5.15], and the main conjectures in [Gr09], [DG], [LRV], [KPM] and [Re] in the general setting of (2):

Stark-Heegner Conjecture. *The semi-local points $P_{\mathfrak{a}}$ belong to the natural image of $E(H)$ in $E(H \otimes \mathbb{Q}_p)$, and the ψ-component*

$$P_\psi := \sum_{\mathfrak{a}\in \mathrm{Pic}(\mathcal{O})} \psi^{-1}(\mathfrak{a}) P_{\mathfrak{a}} \;\in\; E(H\otimes \mathbb{Q}_p)^\psi$$

is non-trivial if and only if $L'(E/K,\psi,1) \neq 0$.

The Stark-Heegner Conjecture has been proved in many cases where ψ is a *quadratic* ring class character. When $\psi^2 = 1$, the induced representation

$$V_\psi := \mathrm{Ind}_K^{\mathbb{Q}} \psi = \chi_1 \oplus \chi_2$$

decomposes as the sum of two one-dimensional Galois representations attached to quadratic Dirichlet characters satisfying

$$\chi_1(p) = -\chi_2(p), \qquad \chi_1(M) = \chi_2(M),$$

and the pair (χ_1, χ_2) can be uniquely ordered in such a way that the L-series $L(E, \chi_1, s)$ and $L(E, \chi_2, s)$ have sign 1 and -1 respectively in their functional equations.

Define the local sign $\alpha := a_p(E)$, which is equal to either 1 or -1 according to whether E has split or non-split multiplicative reduction at p. Let $\mathfrak{p}$ be a prime of H above p, and let $\sigma_{\mathfrak{p}} \in \mathrm{Gal}\,(H/\mathbb{Q})$ denote the associated Frobenius element. Because p is inert in $K/\mathbb{Q}$, the unique prime of K above p splits completely in H/K and $\sigma_{\mathfrak{p}}$ belongs to a conjugacy class of reflections in the generalised dihedral group $\mathrm{Gal}\,(H/\mathbb{Q})$. It depends in an essential way on the choice of $\mathfrak{p}$, but, because ψ cuts out an abelian extension of $\mathbb{Q}$, the Stark-Heegner point

$$P_\psi^\alpha := P_\psi + \alpha \cdot \sigma_{\mathfrak{p}} P_\psi \tag{3}$$

does not depend on this choice. It can in fact be shown that

$$P_\psi^\alpha = \begin{cases} 2P_\psi & \text{if } \chi_2(p) = \alpha; \\ 0 & \text{if } \chi_2(p) = -\alpha. \end{cases}$$

The recent work [Mok2] of Mok and [LMY] of Longo, Martin and Yan, building on the methods introduced in [BD2, Thm. 1], [Mok1], and [LV], asserts:

Stark-Heegner theorem for quadratic characters. *Let ψ be a* quadratic *ring class character of conductor prime to $2DN$. Then the Stark-Heegner point P_ψ^α belongs to $E(H)\otimes\mathbb{Q}$ and is non-trivial if and only if*

$$L(E,\chi_1,1) \neq 0, \quad L'(E,\chi_2,1) \neq 0, \quad \textit{and} \quad \chi_2(p) = \alpha. \tag{4}$$

The principle behind the proof of this result is to compare P_ψ^α to suitable Heegner points arising from Shimura curve parametrisations, exploiting the fortuitous circumstance that the field over which P_ψ is conjecturally defined is a biquadratic extension of $\mathbb{Q}$ and is thus also contained in ring class fields of imaginary quadratic fields (in many different ways).

The present work is concerned with the less well understood *generic* case where $\psi^2 \neq 1$, when the induced representation V_ψ is irreducible. Note that ψ is either totally even or totally odd, i.e., complex conjugation acts as a scalar $\epsilon_\psi \in \{1, -1\}$ on the induced representation V_ψ.

The field which ψ cuts out cannot be embedded in any compositum of ring class fields of imaginary quadratic fields, and the Stark-Heegner Conjecture therefore seems impervious to the theory of Heegner points in this case.

The semi-local point P_ψ^α of (3), which will again play a key role in this work, now depends crucially on the choice of $\mathfrak{p}$, but it is not hard to check that its image under the localisation homomorphism

$$j_{\mathfrak{p}} : E(H \otimes \mathbb{Q}_p) \longrightarrow E(H_{\mathfrak{p}}) = E(K_p)$$

at $\mathfrak{p}$ is independent of this choice, up to scaling by $L^\times$ (cf. Lemma 2.4). It is the local point

$$P_{\psi,\mathfrak{p}}^\alpha := j_{\mathfrak{p}}(P_\psi^\alpha) \in E(H_{\mathfrak{p}}) \otimes L = E(K_p) \otimes L$$

which will play a key role in Theorems A, B and C below, which are the main results of the paper. Theorems A and B are conditional on either one of the two non-vanishing hypotheses below, which apply to a pair (E, K) and a choice of archimedean sign $\epsilon \in \{-1, 1\}$. The first hypothesis is the counterpart, in analytic rank one, of the non-vanishing for simultaneous twists of modular L-series arising as the special case of [DR2, Def. 6.8] discussed in (168) of loc.cit., where it plays a similar role in the proof of the Birch and Swinnerton–Dyer conjecture for $L(E/K, \psi, s)$ when $L(E/K, \psi, 1) \neq 0$. The main difference is that we are now concerned with quadratic ring class characters for which $L(E/K, \psi, s)$ vanishes to odd rather than to even order at the center.

Analytic non-vanishing hypothesis: *Given (E, K) as above, and a choice of a sign $\epsilon \in \{1, -1\}$, there exists a quadratic Dirichlet character χ of conductor prime to DN satisfying*

$$\chi(-1) = -\epsilon, \quad \chi\chi_K(p) = \alpha, \quad L(E, \chi, 1) \neq 0, \quad L'(E, \chi\chi_K, 1) \neq 0.$$

The second non-vanishing hypothesis applies to an arbitrary ring class character ξ of K.

Weak non-vanishing hypothesis for Stark-Heegner points: *Given (E, K) as above, and a sign $\epsilon \in \{1, -1\}$, there exists a ring class character ξ of K of conductor prime to DN with $\epsilon_\xi = -\epsilon$ for which $P_{\xi,\mathfrak{p}}^\alpha \neq 0$.*

That the former hypothesis implies the latter follows by applying the Stark-Heegner theorem for quadratic characters to the quadratic ring class character ξ of K attached to the pair $(\chi_1, \chi_2) := (\chi, \chi\chi_K)$ supplied by the analytic non-vanishing hypothesis. The stronger non-vanishing hypothesis is singled out because it has the virtue of tying in with mainstream questions in analytic number theory on which there has been recent progress [Mun]. On the other hand, the weak non-vanishing hypothesis is known to be true in the classical setting of Heegner points, when K is imaginary quadratic. In fact, for a given E and K, *all but finitely many* of the Heegner points $P_{\mathfrak{a}}$ (as $\mathfrak{a}$ ranges over all ideal classes of all possible orders in K) are of infinite order, and P_ξ and P_ξ^α are therefore non-trivial for infinitely many ring class characters ξ, and for at least one character of any given conductor, with finitely many exceptions. It seems reasonable to expect that Stark-Heegner points should exhibit a similar behaviour, and the experimental evidence bears this out. In practice, efficient algorithms for calculating Stark-Heegner points make it easy to produce a non-zero $P_{\xi,\mathfrak{p}}^\alpha$ for any given (E, K), and indeed, the extensive experiments carried out so far have failed to produce even a single example of a vanishing P_ξ^α when ξ has order ≥ 3. Thus, while these non-vanishing hypotheses are probably difficult to prove in general, they are expected to hold systematically. Moreover, they can easily be checked in practice for any specific triple (E, K, ϵ) and therefore play a somewhat ancillary role in studying the infinite collection of Stark-Heegner points attached to a fixed E and K.

Let $V_p(E) := \left(\varprojlim E[p^n]\right) \otimes \mathbb{Q}_p$ denote the Galois representation attached to E and let

$$\mathrm{Sel}_p(E/H) := H^1_{\mathrm{f}}(H, V_p(E))$$

be the pro-p Selmer group of E over H. The ψ-component of this Selmer group is an L_p-vector space, where L_p is a field containing both $\mathbb{Q}_p$ and L, by setting

$$\mathrm{Sel}_p(E/H)^{\psi} := \{\kappa \in H^1_{\mathrm{f}}(H, V_p(E)) \otimes_{\mathbb{Q}_p} L_p : \text{ such that } \sigma\kappa = \psi(\sigma)\cdot\kappa \text{ for all } \sigma \in \mathrm{Gal}\,(H/K)\}.$$

Since E is defined over $\mathbb{Q}$, the group

$$\mathrm{Sel}_p(E/H) \simeq \oplus_{\varrho} H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E) \otimes \varrho)$$

admits a natural decomposition indexed by the set of irreducible representations ϱ of $\mathrm{Gal}\,(H/\mathbb{Q})$. In this note we focus on the isotypic component singled out by ψ, namely

$$\mathrm{Sel}_p(E, \psi) := H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E) \otimes V_{\psi}) = \mathrm{Sel}_p(E/H)^{\psi} \oplus \mathrm{Sel}_p(E/H)^{\bar{\psi}} \tag{5}$$

where Shapiro's lemma combined with the inflation-restriction sequence gives the above canonical identifications.

It will be convenient to assume from now on that $E[p]$ is irreducible as a $G_{\mathbb{Q}}$-module. (This hypothesis could certainly be relaxed at the cost of some simplicity and transparency in some of the arguments.)

The first main result of this article is:

Theorem A. *Assume that the (analytic or weak) non-vanishing hypothesis holds for (E, K, ϵ). Let ψ be any non-quadratic ring class character of K of conductor prime to DN, for which $\epsilon_{\psi} = \epsilon$. Then there is a global Selmer class*

$$\kappa_{\psi} \in \mathrm{Sel}_p(E, \psi)$$

whose natural image in the group $E(H_{\mathfrak{p}}) \otimes L_p$ of local points agrees with $P^{\alpha}_{\psi,\mathfrak{p}}$.

In particular, it follows that

$$P^{\alpha}_{\psi,\mathfrak{p}} \neq 0 \quad \Rightarrow \quad \dim_{L_p} \mathrm{Sel}_p(E/H)^{\psi} \geq 1. \tag{6}$$

As a corollary, we obtain a criterion for the infinitude of $\mathrm{Sel}_p(E/H)^{\psi}$ in terms of the p-adic L-function $\mathscr{L}_p(\mathbf{f}/K, \psi)$ constructed in [BD2, §3], interpolating the square roots of the central critical values $L(f_k/K, \psi, k/2)$, as f_k ranges over the weight $k \geq 2$ classical specializations of the Hida family passing through the weight two eigenform f associated to E. The interpolation property implies that $\mathscr{L}_p(\mathbf{f}/K, \psi)$ vanishes at $k = 2$, and its first derivative $\mathscr{L}_p{}'(\mathbf{f}/K, \psi)(2)$ is a natural p-adic analogue of the derivative at $s = 1$ of the classical complex L-function $L(f/K, \psi, s)$. The following result can thus be viewed as a p-adic variant of the Birch and Swinnerton-Dyer Conjecture in this setting.

Theorem B. *If $\mathscr{L}_p{}'(\mathbf{f}/K, \psi)(2) \neq 0$, then* $\dim_{L_p} \mathrm{Sel}_p(E/H)^{\psi} \geq 1$.

Theorem B is a direct corollary of (6) in light of the main result of [BD2], recalled in Theorem 2.9 below, which asserts that $P^{\alpha}_{\psi,\mathfrak{p}}$ is non-trivial when $\mathscr{L}_p{}'(\mathbf{f}/K, \psi)(2) \neq 0$.

Remark 1. Assume the p-primary part of (the ψ-isotypic component of) the Tate-Shafarevich group of E/H is finite. Then Theorem A shows that $P^{\alpha}_{\psi,\mathfrak{p}}$ arises from a *global* point in $E(H) \otimes L_p$, as predicted by the Stark-Heegner conjecture. Moreover, Theorem B implies that $\dim_L E(H)^{\psi} \geq 1$ if $\mathscr{L}_p{}'(\mathbf{f}/K, \psi)(2) \neq 0$.

Remark 2. The irreducibility of V_{ψ} when ψ is non-quadratic shows that P^{α}_{ψ} is non-trivial if and only if the same is true for P_{ψ}. The Stark-Heegner Conjecture combined with the injectivity of the map from $E(H) \otimes L$ to $E(H_{\mathfrak{p}}) \otimes L$ suggests that $P^{\alpha}_{\psi,\mathfrak{p}}$ never vanishes when

$P_\psi \neq 0$, but the scenario where P_ψ^α is a non-trivial element of the kernel of $j_{\mathfrak{p}}$ seems hard to rule out unconditionally, without assuming the Stark-Heegner conjecture a priori.

Remark 3. Section 2 is devoted to review the theory of Stark-Heegner points. For notational simplicity, §2 has been written under the stronger Heegner hypothesis

$$\chi_K(p) = -1, \qquad \chi_K(\ell) = 1 \text{ for all } \ell | M$$

of [Dar]. This section contains no new results and merely collects together the basic notations and principal results of [Dar], [BD2], [Mok2] and [LMY]. Exact references for the analogous results needed to cover the more general setting of (2) are given along the way. The remaining sections §3 and §4, which form the main body of the article, adapt without change to proving Theorems A and B under the general assumption (2). In particular, while *quaternionic* modular forms need to be invoked in the general construction of Stark-Heegner points of [Gr09], [DG] and [LRV], the arguments in §3 and §4 only employ *classical elliptic modular forms* in order to deal with the general setting. The method described in this work also adapts, multatis mutandis, to proving the main conjecture of [Das] for abelian varieties of GL_2-type, and the main conjecture of [RS] on "Stark-Heegner cycles" associated to higher weight modular forms under a similar analytic or weak non-vanishing hypothesis: it suffices for this to invoke the main theorem of [Se] in place of the *Stark-Heegner theorem for quadratic characters.*

We now describe the main steps in the proof of Theorem A, which rests on a comparison between Stark-Heegner points and the *generalised Kato classes* introduced in [DR2].

• *Step 1. An auxiliary Stark-Heegner point.* Invoking the weak non-vanishing Hypothesis for Stark-Heegner points or its stronger analytic variant, let ξ be an auxiliary ring class character of K having parity opposite to that of ψ, and for which the Stark-Heegner point $P^\alpha_{\xi,\mathfrak{p}}$ is non-zero.

• *Step 2*: *Theta series of weight one attached to K.* A lemma of Tate on lifting projective Galois representations from $\mathrm{PGL}_2(\mathbb{C})$ to $\mathrm{GL}_2(\mathbb{C})$ can be used, as in the statement of [DR2, Lemma 6.9] and the discussion following it, to exhibit two ray class characters ψ_g and ψ_h of K of conductor prime to N satisfying

$$\psi_g\psi_h = \psi, \qquad \psi_g\psi_h' = \xi,$$

where ψ_h' is the composition of ψ_h with the involution in $\mathrm{Gal}\,(K/\mathbb{Q})$. Letting V_g and V_h denote the two-dimensional Artin representations induced from ψ_g and ψ_h respectively, it is easy to check that

$$V_{gh} := V_g \otimes V_h = V_\psi \oplus V_\xi.$$

The fact that V_ψ and V_ξ have opposite parity implies that the characters ψ_g and ψ_h both have *mixed signature* at infinity, and hence, that V_g and V_h are odd two-dimensional Artin representations of $G_{\mathbb{Q}}$. A theorem of Hecke shows that the theta series g and h associated to ψ_g and ψ_h are holomorphic modular forms of weight one, having V_g and V_h as associated Galois representations. The eigenvalues of the frobenius element $\sigma_{\mathfrak{p}}$, which acts on V_g and V_h as a reflection modulo the center, can be ordered so that they are of the form

$$(\alpha_g, \beta_g) = (\iota, -\iota), \qquad (\alpha_h, \beta_h) = (\alpha\iota^{-1}, -\alpha\iota^{-1})$$

for a suitable $\iota \in L^\times$, where we recall that $\alpha \in \{1, -1\}$ is the local sign at p determined by E. Let g_α and h_α be the p-stabilisations of g and h satisfying

$$U_p g_\alpha = \alpha_g g_\alpha, \qquad U_p h_\alpha = \alpha_h h_\alpha.$$

• *Step 3*: *Generalised Kato classes.* In addition to the Hida family $\mathbf{f}$ passing through f, a theorem of Wiles ensures the existence of two Hida families $\mathbf{g}$ and $\mathbf{h}$ specialising to g_α and h_α

at suitable weight one points. The main object of §3 is the construction of a three-variable Euler system $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})$ of diagonal cycles associated to $(\mathbf{f},\mathbf{g},\mathbf{h})$; cf. (104). This extends in new directions (notably along $\mathbf{f}$) the results obtained in [DR2], which were insufficient for the purposes of this note, and required a different aproach. We do so by introducing a family of cycles on the cube of the modular curve $X(N)$ of full level structure, which may be of independent interest and deserves to be explored in more detail. As we prove in Theorem 3.29, the class $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})$ recovers the three different unbalanced triple-product p-adic Garrett-Hida L-functions under suitable Perrin-Riou Λ-adic regulators.

The specialisation of $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})$ in weight $(2,1,1)$ gives rise to the *generalised Kato class*

$$\kappa(f,g_\alpha,h_\alpha)\in H^1(\mathbb{Q},V_p(E)\otimes V_{gh})=H^1(\mathbb{Q},V_p(E)\otimes V_\psi)\oplus H^1(\mathbb{Q},V_p(E)\otimes V_\xi),$$

obtained, roughly speaking, as a p-adic limit of the p-adic Abel Jacobi images of Gross-Kudla Schoen cycles attached to the triple (f,g_x,h_x) where (g_x,h_x) ranges over pairs of classical specializations of $(\mathbf{g},\mathbf{h})$ at points of weight $k\geq 2$. (More precisely, the class $\kappa(f,g_\alpha,h_\alpha)$ is the image under a non-canonical projection of a class taking values in several copies of $V_p(E)\otimes V_{gh}$, arising from the cohomology of modular varieties in level $N=\mathrm{lcm}(N_f,N_g,N_h)$. This technical issue, which is suppressed in the introduction to lighten the exposition, is dealt with in the main body of the article.) The *generalised Kato reciprocity law* proved in [DR2] parlays the vanishing of $L(f,V_{gh},1)$ into the conclusion that $\kappa(f,g_\alpha,h_\alpha)$ is cristalline at p and hence belongs to the Selmer group $H^1_{\mathrm{f}}(\mathbb{Q},V_p(E)\otimes V_{gh})$. Let

$$\kappa_\psi(f,g_\alpha,h_\alpha)\in H^1_{\mathrm{f}}(\mathbb{Q},V_p(E)\otimes V_\psi),\qquad \kappa_\xi(f,g_\alpha,h_\alpha)\in H^1_{\mathrm{f}}(\mathbb{Q},V_p(E)\otimes V_\xi)$$

denote the two components of the global cohomology class $\kappa(f,g_\alpha,h_\alpha)$.

Just as in (5), there are canonical identifications at the local level

$$\begin{aligned} H^1_{\mathrm{f}}(\mathbb{Q}_p,V_p(E)\otimes V_\psi) &= H^1_{\mathrm{f}}(H_p,V_p(E))^{\psi}\oplus H^1_{\mathrm{f}}(H_p,V_p(E))^{\bar\psi} \\ &= E(H\otimes\mathbb{Q}_p)^{\psi}\oplus E(H\otimes\mathbb{Q}_p)^{\bar\psi}, \end{aligned}\tag{7}$$

where

$$H^1_{\mathrm{f}}(H_p,V_p(E)):=\oplus_{\mathfrak{p}|p}H^1_{\mathrm{f}}(H_{\mathfrak{p}},V_p(E)).$$

Similar remarks apply equally of course when ψ is replaced by ξ.

Given a prime $\mathfrak{p}$ of H above p, let

$$\log_{E,\mathfrak{p}}:E(H_{\mathfrak{p}})\longrightarrow H_{\mathfrak{p}}=K_p$$

denote the formal group logarithm map, which is obtained by composing the inverse of the Tate uniformization of E with the branch of the p-adic logarithm map that vanishes at the Tate period of E. By an abuse of notation, we will identity $\log_{E,\mathfrak{p}}$ with the homomorphisms

$$\log_{E,\mathfrak{p}}:E(H\otimes\mathbb{Q}_p)\longrightarrow K_p,\qquad H^1_{\mathrm{f}}(\mathbb{Q}_p,V_p(E)\otimes V_\psi)\longrightarrow K_p\otimes L_p$$

that it induces via composition with the localisation map $j_{\mathfrak{p}}$ and the identifications in (7).

Let

$$\kappa^\alpha_\psi(f,g_\alpha,h_\alpha)=(1+\alpha\sigma_{\mathfrak{p}})\kappa_\psi(f,g_\alpha,h_\alpha)\in \mathrm{Sel}_p(E,\psi):=H^1_{\mathrm{f}}(H,V_p(E))^{\psi\oplus\bar\psi}$$

denote the component of $\kappa_\psi(f,g_\alpha,h_\alpha)$ on which $\sigma_{\mathfrak{p}}$ acts with eigenvalue α, and likewise with ψ replaced by the auxiliary character ξ.

• *Step 4*: *Generalised Kato classes and Stark-Heegner points.* Properties of generalised Kato classes already established in [DR2] and [DR3] can be used to show that

$$\log_{E,\mathfrak{p}}(\kappa^\alpha_\psi(f,g_\alpha,h_\alpha))=\log_{E,\mathfrak{p}}(\kappa^\alpha_\xi(f,g_\alpha,h_\alpha)).$$

Theorem A is now a consequence of the following theorem after setting

$$\kappa_\psi:=\log_{E,\mathfrak{p}}(P^\alpha_\xi)^{-1}\times\kappa^\alpha_\psi(f,g_\alpha,h_\alpha).$$

Theorem C. *For all g and h as above,*

$$\log_{E,\mathfrak{p}}(\kappa^{\alpha}_{\psi}(f,g_{\alpha},h_{\alpha})) = \log_{E,\mathfrak{p}}(P^{\alpha}_{\psi}) \times \log_{E,\mathfrak{p}}(P^{\alpha}_{\xi}) \pmod{L^{\times}}.$$

Theorem C, which makes an explicit comparison between generalised Kato classes and Stark-Heegner points, is the third main theorem in this paper. It is consistent with the conjectures of [DR3] on the position of the generalised Kato class $\kappa(f,g_{\alpha},h_{\alpha})$ in the V_{gh}-isotypic component of the Mordell-Weil group of E, in light of the expected algebraicity of P_{ψ} and P_{ξ}, and shows that the Stark-Heegner conjectures are compatible with the elliptic Stark conjectures of [DLR] in the special case where g and h are theta series attached to characters of a common real quadratic field.

The key ingredients in the proof of Theorem C are

• A Perrin-Riou-style "explicit reciprocity law" in the exceptional zero setting that follows from the work of R. Venerucci [Ve]. Let $\mathbf{f}$ be a Hida family specializing to f in weight two, restricted to a neighbourhood of f in the eigencurve which maps isomorphically to the appropriate neighbourhood U of 2 in weight space, and hence admitting at most one weight k specialisation f_k for any given weight k. Venerucci's reciprocity law involves the Garrett-Rankin triple product p-adic L-function $\mathscr{L}_p{}^f(\mathbf{f},g,h)$ interpolating the *square roots* of the central critical values $L(f_k,V_{gh},k/2)$ as f_k runs over the classical specializations with $k\in U\cap\mathbb{Z}^{>2}$, and asserts that

$$\log_{E,\mathfrak{p}}(\kappa^{\alpha}_{\psi}(f,g_{\alpha},h_{\alpha})) = \frac{d^2}{dk^2}\mathscr{L}_p{}^f(\mathbf{f},g,h)_{k=2} \pmod{L^{\times}}. \tag{8}$$

The above p-adic L-function in fact depends on a choice of test vectors $(\breve{f},\breve{g},\breve{h})$in level $N=\mathrm{lcm}(N_f,N_g,N_h)$. This technical issue, which is also is suppressed in the introduction to lighten the exposition, is dealt with in the main body of the article.

• A factorisation formula of the form

$$\mathscr{L}_p{}^f(\mathbf{f},g,h) = \mathscr{L}_p(\mathbf{f}/K,\psi)\times\mathscr{L}_p(\mathbf{f}/K,\xi) \pmod{L^{\times}}, \tag{9}$$

where $\mathscr{L}_p(\mathbf{f}/K,\psi)$ is defined using a formula of Waldspurger for the square roots of the central critical values $L(f_k/K,\psi,k/2)$, in the form made explicit by A. Popa in his Harvard PhD thesis. This factorisation is a manifestation of the Artin formalism for p-adic L-series, reflecting the fact that $V_g\otimes V_h = V_{\psi}\oplus V_{\xi}$ together with the fact that the Garrett-Rankin and Waldspurger type "square root p-adic L-series" have a common domain of classical interpolation.

• A formula already established in [BD2] showing that

$$\log_{E,\mathfrak{p}}(P^{\alpha}_{\psi}) = \frac{d}{dk}\mathscr{L}_p(\mathbf{f}/K,\psi)_{k=2} \pmod{L^{\times}} \tag{10}$$

Theorem C follows by combining (8), (9) and (10).

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1. Background

1.1. **Basic notations.** Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. All the number fields that arise will be viewed as embedded in this algebraic closure. For each such K, let $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ denote its absolute Galois group. For each prime p, an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ is also assumed to be fixed, and ord_p denotes the resulting p-adic valuation on $\bar{\mathbb{Q}}^\times$, normalized in such a way that $\mathrm{ord}_p(p) = 1$.

For a variety V defined over $K \subset \bar{\mathbb{Q}}$, let $\bar{V}$ denote the base change of V to $\bar{\mathbb{Q}}$. If $\mathcal{F}$ is an étale sheaf on V, write $H^i_{\mathrm{et}}(\bar{V}, \mathcal{F})$ for the ith étale cohomology group of $\bar{V}$ with values in $\mathcal{F}$, equipped with its continuous action by G_K.

Given a prime p, let $\mathbb{Q}(\mu_{p^\infty}) = \cup_{r\geq 1}\mathbb{Q}(\zeta_r)$ be the cyclotomic extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ a primitive p^r-th root of unity ζ_r. Let

$$\varepsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Z}_p^\times$$

denote the p-adic cyclotomic character. It can be factored as $\varepsilon_{\mathrm{cyc}} = \omega\langle\varepsilon_{\mathrm{cyc}}\rangle$, where

$$\omega : G_{\mathbb{Q}} \longrightarrow \mu_{p-1} \qquad \langle\varepsilon_{\mathrm{cyc}}\rangle : G_{\mathbb{Q}} \longrightarrow 1 + p\mathbb{Z}_p$$

are obtained by composing $\varepsilon_{\mathrm{cyc}}$ with the projection onto the first and second factors in the canonical decomposition $\mathbb{Z}_p^\times \simeq \mu_{p-1} \times (1+p\mathbb{Z}_p)$. If $\mathcal{M}$ is a $\mathbb{Z}_p[G_{\mathbb{Q}}]$-module and j is an integer, write $\mathcal{M}(j) = \mathcal{M} \otimes \varepsilon_{\mathrm{cyc}}^j$ for the j-th Tate twist of $\mathcal{M}$.

Let

$$\mathring{\Lambda}_r := \mathbb{Z}_p[(\mathbb{Z}/p^r\mathbb{Z})^\times], \qquad \mathring{\Lambda} := \mathbb{Z}_p[[\mathbb{Z}_p^\times]] := \varprojlim_r \mathring{\Lambda}_r$$

denote the group ring and completed group ring attached to the profinite group $\mathbb{Z}_p^\times$. The ring $\mathring{\Lambda}$ is equipped with $p-1$ distinct algebra homomorphisms $\omega^i : \mathring{\Lambda} \to \Lambda$ (for $0 \leq i \leq p-2$) to the local ring

$$\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]] = \varprojlim \mathbb{Z}_p[1+p\mathbb{Z}/p^r\mathbb{Z}] \simeq \mathbb{Z}_p[[T]],$$

where ω^i sends a group-like element $a \in \mathbb{Z}_p^\times$ to $\omega^i(a)\langle a\rangle \in \Lambda$. These homomorphisms identify $\mathring{\Lambda}$ with the direct sum

$$\mathring{\Lambda} = \bigoplus_{i=0}^{p-2} \Lambda.$$

The local ring Λ is called the one variable Iwasawa algebra. More generally, for any integer $t \geq 1$, let

$$\mathring{\Lambda}^{\otimes t} := \mathring{\Lambda}\hat{\otimes}_{\mathbb{Z}_p} \overset{t}{\cdots} \hat{\otimes}_{\mathbb{Z}_p}\mathring{\Lambda}, \qquad \Lambda^{\otimes t} = \Lambda\hat{\otimes}_{\mathbb{Z}_p} \overset{t}{\cdots} \hat{\otimes}_{\mathbb{Z}_p}\Lambda \simeq \mathbb{Z}_p[[T_1, \dots T_t]].$$

The later ring is called the Iwasawa algebra in t variables, and is isomorphic to the power series ring in t variables over $\mathbb{Z}_p$, while

$$\mathring{\Lambda}^{\otimes t} = \bigoplus_{\alpha} \Lambda^{\otimes t},$$

the sum running over the $(p-1)^t$ distinct $\mathbb{Z}_p^\times$ valued characters of $(\mathbb{Z}/p\mathbb{Z})^{\times t}$.

1.2. **Modular forms and Galois representations.** Let

$$\phi = q + \sum_{n\geq 2} a_n(\phi)q^n \in S_k(M, \chi)$$

be a cuspidal modular form of weight $k \geq 1$, level M and character $\chi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$, and assume that ϕ is an eigenform with respect to all good Hecke operators T_ℓ, $\ell \nmid M$.

Fix an odd prime number p (that in this section may or may not divide M). Let $\mathcal{O}_\phi$ denote the valuation ring of the finite extension of $\mathbb{Q}_p$ generated by the fourier coefficients of ϕ, and let $\mathbb{T}$ denote the Hecke algebra generated over $\mathbb{Z}_p$ by the good Hecke operators T_ℓ with $\ell \nmid M$

and by the diamond operators acting on $S_k(M,\chi)$. The eigenform ϕ gives rise to an algebra homomorphism

$$\xi_\phi : \mathbb{T} \longrightarrow \mathcal{O}_\phi$$

sending T_ℓ to $a_\ell(\phi)$ and the diamond operator $\langle \ell \rangle$ to $\chi(\ell)$.

A fundamental construction of Shimura, Deligne, and Serre-Deligne attaches to ϕ an irreducible two-dimensional Galois representation

$$\varrho_\phi : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V_\phi) \simeq \mathrm{GL}_2(\mathcal{O}_\phi),$$

which is unramified at all primes $\ell \nmid Mp$, and for which

$$\det(1 - \varrho_\phi(\mathrm{Fr}_\ell)x) = 1 - a_\ell(\phi)x + \chi(\ell)\ell^{k-1}x^2, \tag{11}$$

where Fr_ℓ denotes the arithmetic Frobenius element at ℓ. This property characterises the semisimple representation ϱ_ϕ up to isomorphism.

When $k := k_\circ + 2 \geq 2$, the representation V_ϕ can be realised in the p-adic étale cohomology of an appropriate Kuga-Sato variety. Since this realisation is important for the construction of generalised Kato classes, we now briefly recall its salient features. Let $Y = Y_1(M)$ and $X = X_1(M)$ denote the open and closed modular curve representing the fine moduli functor of isomorphism classes of pairs (A,P) formed by a (generalised) elliptic curve A together with a torsion point P on A of exact order M. Let

$$\pi : \mathcal{A}_\circ \longrightarrow Y \tag{12}$$

denote the universal elliptic curve over Y.

The $k_\circ$-th open Kuga-Sato variety over Y is the $k_\circ$-fold fiber product

$$\mathcal{A}_\circ^{k_\circ} := \mathcal{A}_\circ \times_Y \overset{(k_\circ)}{\cdots} \times_Y \mathcal{A}_\circ \tag{13}$$

of $\mathcal{A}_\circ$ over Y. The variety $\mathcal{A}_\circ^{k_\circ}$ admits a smooth compactification $\mathcal{A}^{k_\circ}$ which is fibered over X and is called the $k_\circ$-th Kuga-Sato variety over X; we refer to Conrad's appendix in [BDP1] for more details. The geometric points in $\mathcal{A}^{k_\circ}$ that lie above Y are in bijection with isomorphism classes of tuples $[(A,P), P_1, \ldots, P_{k_\circ}]$, where (A,P) is associated to a point of Y as in the previous paragraph and $P_1, ..., P_{k_\circ}$ are points on A.

The representation V_ϕ is realised (up to a suitable Tate twist) in the middle degree étale cohomology $H^{k_\circ+1}_{\mathrm{et}}(\bar{\mathcal{A}}^{k_\circ}, \mathbb{Z}_p)$. More precisely, let

$$\mathcal{H}_r := R^1\pi_*\,\mathbb{Z}/p^r\mathbb{Z}(1), \qquad \mathcal{H} := R^1\pi_*\,\mathbb{Z}_p(1),$$

and for any $k_\circ \geq 0$, define

$$\mathcal{H}_r^{k_\circ} := \mathrm{TSym}^{k_\circ}(\mathcal{H}_r), \qquad \mathcal{H}^{k_\circ} := \mathrm{TSym}^{k_\circ}(\mathcal{H}) \tag{14}$$

to be the sheaves of symmetric $k_\circ$-tensors of $\mathcal{H}_r$ and $\mathcal{H}$, respectively. As defined in e.g. [BDP1, (2.1.2)], there is an idempotent $\epsilon_{k_\circ}$ in the ring of rational correspondences of $\mathcal{A}^{k_\circ}$ whose induced projector on the étale cohomology groups of this variety satisfy:

$$\epsilon_{k_\circ}\left(H^{k_\circ+1}_{\mathrm{et}}(\bar{\mathcal{A}}^{k_\circ}, \mathbb{Z}_p(k_\circ))\right) = H^1_{\mathrm{et}}(\bar{X}, \mathcal{H}^{k_\circ}). \tag{15}$$

Define the $\mathcal{O}_\phi$-module

$$V_\phi(M) := H^1_{\mathrm{et}}(\bar{X}, \mathcal{H}^{k_\circ}(1)) \otimes_{\mathbb{T},\xi_\phi} \mathcal{O}_\phi, \tag{16}$$

and write

$$\varpi_\phi : H^1_{\mathrm{et}}(\bar{X}, \mathcal{H}^{k_\circ}(1)) \longrightarrow V_\phi(M) \tag{17}$$

for the canonical projection of $\mathbb{T}[G_{\mathbb{Q}}]$-modules arising from (16). Deligne's results and the theory of newforms show that the module $V_\phi(M)$ is the direct sum of several copies of a locally free module V_ϕ of rank 2 over $\mathcal{O}_\phi$ that satisfies (11).

Let α_ϕ and β_ϕ the two roots of the p-th Hecke polynomial $T^2 - a_p(\phi)T + \chi(p)p^{k-1}$, ordered in such a way that $\mathrm{ord}_p(\alpha_\phi) \le \mathrm{ord}_p(\beta_\phi)$. (If α_ϕ and β_ϕ have the same p-adic valuation, simply fix an arbitrary ordering of the two roots.) We set $\chi(p) = 0$ whenever p divides the primitive level of ϕ and thus $\alpha_\phi = a_p(\phi)$ and $\beta_\phi = 0$ in this case. The eigenform ϕ is said to be *ordinary* at p when $\mathrm{ord}_p(\alpha_\phi) = 0$. In that case, there is an exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0 \to V_\phi^+ \longrightarrow V_\phi \longrightarrow V_\phi^- \to 0, \qquad V_\phi^+ \simeq \mathcal{O}_\phi(\varepsilon_{\mathrm{cyc}}^{k-1}\chi\psi_\phi^{-1}), \quad V_\phi^- \simeq \mathcal{O}_\phi(\psi_\phi), \tag{18}$$

where ψ_ϕ is the unramified character of $G_{\mathbb{Q}_p}$ sending Fr_p to α_ϕ.

1.3. **Hida families and Λ-adic Galois representations.** Fix a prime $p \ge 3$. The formal spectrum

$$\mathcal{W} := \mathrm{Spf}(\Lambda)$$

of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$ is called the *weight space* attached to Λ. The A-valued points of $\mathcal{W}$ over a p-adic ring A are given by

$$\mathcal{W}(A) = \mathrm{Hom}_{\mathrm{alg}}(\Lambda, A) = \mathrm{Hom}_{\mathrm{grp}}(1 + p\mathbb{Z}_p, A^\times),$$

where the Hom's in this definition denote continuous homomorphisms of p-adic rings and profinite groups respectively. Weight space is equipped with a distinguished collection of *arithmetic points* $\nu_{k_\circ,\varepsilon}$, indexed by integers $k_\circ \ge 0$ and Dirichlet characters $\varepsilon : (1+p\mathbb{Z}/p^r\mathbb{Z}) \to \mathbb{Q}_p(\zeta_{r-1})^\times$ of p-power conductor. The point $\nu_{k_\circ,\varepsilon} \in \mathcal{W}(\mathbb{Z}_p[\zeta_r])$ is defined by

$$\nu_{k_\circ,\varepsilon}(n) = \varepsilon(n)n^{k_\circ},$$

and the notational shorthand $\nu_{k_\circ} := \nu_{k_\circ,1}$ is adopted throughout. More generally, if $\tilde{\Lambda}$ is any finite flat Λ-algebra, a point $x \in \tilde{\mathcal{W}} := \mathrm{Spf}(\tilde{\Lambda})$ is said to be arithmetic if its restriction to Λ agrees with $\nu_{k_\circ,\varepsilon}$ for some $k_\circ$ and ϵ. The integer $k = k_\circ + 2$ is called the *weight* of x and denoted $\mathrm{wt}(x)$.

Let

$$\underline{\varepsilon}_{\mathrm{cyc}} : G_{\mathbb{Q}} \longrightarrow \Lambda^\times \tag{19}$$

denote the Λ-adic cyclotomic character which sends a Galois element σ to the group-like element $[\langle\varepsilon_{\mathrm{cyc}}(\sigma)\rangle]$. This character interpolates the powers of the cyclotomic character, in the sense that

$$\nu_{k_\circ,\varepsilon} \circ \underline{\varepsilon}_{\mathrm{cyc}} = \varepsilon \cdot \langle\varepsilon_{\mathrm{cyc}}\rangle^{k_\circ} = \varepsilon \cdot \varepsilon_{\mathrm{cyc}}^{k_\circ} \cdot \omega^{-k_\circ}. \tag{20}$$

Let $M \ge 1$ be an integer not divisible by p.

Definition 1.1. A *Hida family of tame level M and tame character* $\chi : (\mathbb{Z}/M\mathbb{Z})^\times \to \bar{\mathbb{Q}}_p^\times$ is a formal q-expansion

$$\boldsymbol{\phi} = \sum_{n\ge1} a_n(\boldsymbol{\phi})q^n \in \Lambda_{\boldsymbol{\phi}}[[q]]$$

with coefficients in a finite flat Λ-algebra $\Lambda_{\boldsymbol{\phi}}$, such that for any arithmetic point $x \in \mathcal{W}_{\boldsymbol{\phi}} := \mathrm{Spf}(\Lambda_{\boldsymbol{\phi}})$ above $\nu_{k_\circ,\epsilon}$, where $k_\circ \ge 0$ and ε is a character of conductor p^r, the series

$$\boldsymbol{\phi}_x := \sum_{n\ge1} x(a_n(\boldsymbol{\phi}))q^n \in \bar{\mathbb{Q}}_p[[q]]$$

is the q-expansion of a classical p-ordinary eigenform in the space $S_k(Mp^r, \chi\varepsilon\omega^{-k_\circ})$ of cusp forms of weight $k = k_\circ + 2$, level Mp^r and nebentype $\chi\varepsilon\omega^{-k_\circ}$.

By enlarging $\Lambda_{\boldsymbol{\phi}}$ if necessary, we shall assume throughout that $\Lambda_{\boldsymbol{\phi}}$ contains the M-th roots of unity.

Definition 1.2. Let $x \in \mathcal{W}_{\boldsymbol{\phi}}$ be an arithmetic point lying above the point $\nu_{k_\circ,\epsilon}$ of weight space. The point x is said to be

- *tame* if the character ϵ is tamely ramified, i.e., factors through $(\mathbb{Z}/p\mathbb{Z})^\times$.
- *cristalline* if $\epsilon\omega^{-k_\circ} = 1$, i.e., if the weight k specialisation of $\boldsymbol{\phi}$ at x has trivial nebentypus character at p.

We let $\mathcal{W}_{\boldsymbol{\phi}}^\circ$ denote the set of cristalline arithmetic points of $\mathcal{W}_{\boldsymbol{\phi}}$.

Note that a cristalline point is necessarily tame but of course there are tame points that are not cristalline. The justification for this terminology is that the Galois representation V_{ϕ_x} is cristalline at p when x is a cristalline.

If x is a cristalline point, then the classical form $\boldsymbol{\phi}_x$ is always old at p if $k > 2$. In that case there exists an eigenform ϕ_x° of level M such that $\boldsymbol{\phi}_x$ is the ordinary p-stabiization of $\boldsymbol{\phi}_x^\circ$. If the weight is $k = 1$ or 2, $\boldsymbol{\phi}_x$ may be either old or new at p; if it is new at p then we set $\boldsymbol{\phi}_x^\circ = \boldsymbol{\phi}_x$ in order to have uniform notations.

We say $\boldsymbol{\phi}$ is residually irreducible if the mod p Galois representation associated to the Deligne representations associated to ϕ_x° for any cristalline classical point is irreducible.

Finally, the Hida family $\boldsymbol{\phi}$ is said to be *primitive* of tame level $M_{\boldsymbol{\phi}} \mid M$ if for all but finitely many arithmetic points $x \in \mathcal{W}_{\boldsymbol{\phi}}$ of weight $k \geq 2$, the modular form $\boldsymbol{\phi}_x$ arises from a newform of level $M_{\boldsymbol{\phi}}$.

The following theorem of Hida and Wiles associates a two-dimensional Galois representation to a Hida family $\boldsymbol{\phi}$ (cf. e.g. [MT, Théorème 7]).

Theorem 1.3. *Assume $\boldsymbol{\phi}$ is residually irreducible. Then there is a rank two $\Lambda_{\boldsymbol{\phi}}$-module $\mathbb{V}_{\boldsymbol{\phi}}$ equipped with a Galois action*

$$\varrho_{\boldsymbol{\phi}} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\Lambda_{\boldsymbol{\phi}}}(\mathbb{V}_{\boldsymbol{\phi}}) \simeq \mathrm{GL}_2(\Lambda_{\boldsymbol{\phi}}), \tag{21}$$

such that, for all arithmetic points $x : \Lambda_{\boldsymbol{\phi}} \longrightarrow \bar{\mathbb{Q}}_p$,

$$\mathbb{V}_{\boldsymbol{\phi}} \otimes_{x,\Lambda_{\boldsymbol{\phi}}} \bar{\mathbb{Q}}_p \simeq V_{\boldsymbol{\phi}_x}.$$

Let

$$\psi_{\boldsymbol{\phi}} : G_{\mathbb{Q}_p} \longrightarrow \Lambda_{\boldsymbol{\phi}}^\times$$

denote the unramified character sending a Frobenius element Fr_p *to* $\mathbf{a}_p(\boldsymbol{\phi})$*. The restriction of* $\mathbb{V}_{\boldsymbol{\phi}}$ *to* $G_{\mathbb{Q}_p}$ *admits a filtration*

$$0 \to \mathbb{V}_{\boldsymbol{\phi}}^+ \to \mathbb{V}_{\boldsymbol{\phi}} \to \mathbb{V}_{\boldsymbol{\phi}}^- \to 0 \quad \textit{where } \mathbb{V}_{\boldsymbol{\phi}}^+ \simeq \Lambda_{\boldsymbol{\phi}}(\psi_{\boldsymbol{\phi}}^{-1}\chi\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathrm{cyc}}) \ \textit{ and } \ \mathbb{V}_{\boldsymbol{\phi}}^- \simeq \Lambda_{\boldsymbol{\phi}}(\psi_{\boldsymbol{\phi}}). \tag{22}$$

The explicit construction of the Galois representation $\mathbb{V}_{\boldsymbol{\phi}}$ plays an important role in defining the generalised Kato classes, and we now recall its main features.

For all $0 \leq r < s$, let

$$X_r := X_1(Mp^r), \qquad X_{r,s} := X_1(Mp^r) \times_{X_0(Mp^r)} X_0(Mp^s),$$

where the fiber product is taken relative to the natural projection maps. In particular,

- the curve $X := X_0 := X_1(M)$ represents the functor of elliptic curves A with $\Gamma_1(M)$-level structure, i.e., with a marked point of order M;
- the curve X_r represents the functor classifying pairs (A, P) consisting of a generalized elliptic curve A with $\Gamma_1(M)$-level structure and a point P of order p^r on A;
- the curve $X_{0,s} = X_1(M) \times_{X_0(M)} X_0(Mp^s)$ classifies pairs (A, C) consisting of a generalized elliptic curve A with $\Gamma_1(M)$ structure and a cyclic subgroup scheme C of order p^s on A;
- the curve $X_{r,s}$ classifies pairs (A, P, C) consisting of a generalized elliptic curve A with $\Gamma_1(M)$ structure, a point P of order r on A and and a cyclic subgroup scheme C of order p^s on A containing P.

The curves X_r and $X_{0,r}$ are smooth geometrically connected curves over $\mathbb{Q}$. The natural covering map $X_r \longrightarrow X_{0,r}$ is Galois with Galois group $(\mathbb{Z}/p^r\mathbb{Z})^\times$ acting on the left via the diamond operators defined by

$$\langle a\rangle(A,P) = (A,aP). \tag{23}$$

Let

$$\varpi_1 : X_{r+1} \longrightarrow X_r \tag{24}$$

denote the natural projection from level $r+1$ to level r which corresponds to the map $(A,P) \mapsto (A,pP)$, and to the map $\tau \mapsto \tau$ on upper half planes. Let

$$\varpi_2 : X_{r+1} \longrightarrow X_r$$

denote the other projection, corresponding to the map $(A,P) \mapsto (A/\langle p^r P\rangle, P + \langle p^r P\rangle)$, which on the upper half plane sends τ to $p\tau$. These maps can be factored as

$$\begin{tikzcd} X_{r+1} \arrow[d, "\mu"] \arrow[dr, "\varpi_1"] & \\ X_{r,r+1} \arrow[r, "\pi_1"'] & X_r, \end{tikzcd} \qquad \begin{tikzcd} X_{r+1} \arrow[d, "\mu"] \arrow[dr, "\varpi_2"] & \\ X_{r,r+1} \arrow[r, "\pi_2"'] & X_r. \end{tikzcd} \tag{25}$$

For all $r \geq 1$, the vertical map μ is a cyclic Galois covering of degree p, with Galois group canonically isomorphic to $(1 + p^r(\mathbb{Z}/p^{r+1}\mathbb{Z}))$, while the horizontal maps π_1 and π_2 are non-Galois coverings of degree p. When $r = 0$, the map μ is a cyclic Galois covering of degree $p-1$, with Galois groups canonically isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\times$, while π_1 and π_2 are non-Galois coverings of degree $p+1$.

The Λ-adic representation $\mathbb{V}_\phi$ shall be realised (up to twists) in quotients of the inverse limit of étale cohomology groups arising from the tower

$$X_\infty^* \quad : \quad \cdots \xrightarrow{\varpi_1} X_{r+1} \xrightarrow{\varpi_1} X_r \xrightarrow{\varpi_1} \cdots \xrightarrow{\varpi_1} X_1 \xrightarrow{\varpi_1} X_0$$

of modular curves. Define the inverse limit

$$H^1_{\mathrm{et}}(\bar{X}_\infty^*, \mathbb{Z}_p) := \varprojlim_{\varpi_{1*}} H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p) \tag{26}$$

where the transition maps arise from the pushforward induced by the morphism ϖ_1. This inverse limit is a module over the completed group rings $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ arising from the action of the diamond operators, and is endowed with a plethora of extra structures that we now describe.

Hecke operators. The transition maps in (26) are compatible with the action of the Hecke operators T_n for all n that are not divisible by p. Of crucial importance for us in this article is Atkin's operator U_p^*, which operates on $H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)$ via the composition

$$U_p^* := \pi_{1*}\pi_2^*$$

arising from the maps in (25).

The operator U_p^* is compatible with the transition maps defining $H^1_{\mathrm{et}}(\bar{X}_\infty^*, \mathbb{Z}_p)$,

Inverse systems of étale sheaves. The cohomology group $H^1_{\mathrm{et}}(\bar{X}_\infty^*, \mathbb{Z}_p)$ can be identified with the first cohomology group of the base curve X_1 with values in a certain inverse systems of étale sheaves.

For each $r \geq 1$, let

$$\mathcal{L}_r^* := \varpi_{1*}^{r-1}\mathbb{Z}_p \tag{27}$$

be the pushforward of the constant sheaf on X_r via the map

$$\varpi_1^{r-1} : X_r \longrightarrow X_1$$

The stalk of $\mathcal{L}_r^*$ at a geometric point $x = (A, P)$ on X_1 is given by

$$\mathcal{L}_{r,x}^* = \mathbb{Z}_p[A[p^r]\langle P\rangle],$$

where

$$A[p^r]\langle P\rangle := \{Q \in A[p^r] \text{ such that } p^{r-1}Q = P\}.$$

The multiplication by p map on the fibers gives rise to natural homomorphisms of sheaves

$$[p] : \mathcal{L}_{r+1}^* \longrightarrow \mathcal{L}_r^*, \tag{28}$$

and Shapiro's lemma gives canonical identifications

$$H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p) = H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{L}_r^*),$$

for which the following diagram commutes:

$$\begin{tikzcd} H^1_{\mathrm{et}}(\bar{X}_{r+1}, \mathbb{Z}_p) \arrow[r, "\varpi_{1*}"] \arrow[d, equal] & H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p) \arrow[d, equal] \\ H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{L}_{r+1}^*) \arrow[r, "{[p]}"] & H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{L}_r^*). \end{tikzcd}$$

Let $\mathcal{L}_\infty^* := \varprojlim_r \mathcal{L}_r^*$ denote the inverse system of étale sheaves relative to the maps $[p]$ arising in (28). By passing to the limit, we obtain an identification

$$H^1_{\mathrm{et}}(\bar{X}_\infty^*, \mathbb{Z}_p) = \varprojlim_{r\geq 1} H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{L}_r^*) = H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{L}_\infty^*). \tag{29}$$

Weight k specialisation maps. Recall the p-adic étale sheaves $\mathcal{H}^{k_\circ}$ introduced in (14), whose cohomology gave rise to the Deligne representations attached to modular forms of weight $k = k_\circ + 2$ via (16). The natural $k_\circ$-th power symmetrisation function

$$A[p^r] \longrightarrow \mathcal{H}_r^{k_\circ}, \qquad Q \mapsto Q^{k_\circ},$$

restricted to $A[p^r]\langle P\rangle$ and extended to $\mathcal{L}_{r,x}^*$ by $\mathbb{Z}_p$-linearity, induces morphisms

$$\mathrm{sp}_{k,r}^* : \mathcal{L}_r^* \longrightarrow \mathcal{H}_r^{k_\circ} \tag{30}$$

of sheaves over X_1 (which are thus compatible with the action of $G_{\mathbb{Q}}$ on the fibers). These specialisation morphisms are compatible with the transition maps $[p]$ in the sense that the diagram

$$\begin{tikzcd} \mathcal{L}_{r+1}^* \arrow[r, "{[p]}"] \arrow[d, "\mathrm{sp}_{k,r+1}^*"] & \mathcal{L}_r^* \arrow[d, "\mathrm{sp}_{k,r}^*"] \\ \mathcal{H}_{r+1}^{k_\circ} \arrow[r] & \mathcal{H}_r^{k_\circ} \end{tikzcd}$$

commutes, where the bottom horizontal arrow denotes the natural reduction map. The maps $\mathrm{sp}_{k,r}^*$ can thus be pieced together into morphisms

$$\mathrm{sp}_k^* : \mathcal{L}_\infty^* \longrightarrow \mathcal{H}^{k_\circ}. \tag{31}$$

The induced morphism

$$\mathrm{sp}_k^* : H^1_{\mathrm{et}}(\bar{X}_\infty^*, \mathbb{Z}_p) \longrightarrow H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{H}^{k_\circ}), \tag{32}$$

arising from those on $H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{L}_\infty^*)$ via (29) will be denoted by the same symbol by abuse of notation, and is referred to as the *weight $k = k_\circ + 2$ specialisation map.* The existence of such maps having finite cokernel reveals that the Λ-adic Galois representation $H^1_{\mathrm{et}}(\bar{X}_\infty^*, \mathbb{Z}_p)$ is rich enough to capture the Deligne representations attached to modular forms on X_1 of *arbitrary* weight $k \geq 2$.

For each $a \in 1 + p\mathbb{Z}_p$, the diamond operator $\langle a \rangle$ acts trivially on X_1 and as multiplication by $a^{k_\circ}$ on the stalks of the sheaves $\mathcal{H}_r^{k_\circ}$. It follows that the weight k specialisation map sp_k^* factors through the quotient $H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p) \otimes_{\Lambda,\nu_{k_\circ}} \mathbb{Z}_p$, i.e., one obtains a map

$$\mathrm{sp}_k^* : H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p) \otimes_{\Lambda,\nu_{k_\circ}} \mathbb{Z}_p \quad \longrightarrow \quad H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{H}^{k_\circ}).$$

Remark 1.4. The inverse limit $\mathcal{L}^*_\infty$ of the sheaves $\mathcal{L}^*_r$ on X_1 has been systematically studied by G. Kings in [Ki, §2.3-2.4], and is referred to as a *sheaf of Iwasawa modules.* Jannsen introduced in [Ja] the étale cohomology groups of such inverse systems of sheaves, and proved the existence of a Hoschild-Serre spectral sequence, Gysin excision exact sequences and cycle map in this context.

Ordinary projections. Let

$$e^* := \lim_{n\to\infty} U_p^{*n!} \tag{33}$$

denote Hida's ordinary projector. Since U_p^* commutes with the push-forward maps ϖ_{1*}, this idempotent operates on $H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p)$. While the structure of the Λ-module $H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p)$ seems rather complicated, a dramatic simplification occurs after passing to their ordinary part $e^* H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p)$, as the following classical theorem of Hida shows.

Theorem 1.5. [Hi2, Corollaries 3.3 and 3.7] *The Galois representation $e^* H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p(1))$ is a free Λ-module. For each $\nu_{k_\circ} \in \mathcal{W}$ with $k_\circ \geq 0$, the weight $k = k_\circ + 2$ specialisation map induces maps with bounded cokernel (independent of k)*

$$\mathrm{sp}_k^* \quad : \quad e^* H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p(1)) \otimes_{\nu_{k_\circ}} \mathbb{Z}_p \longrightarrow e^* H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{H}^{k_\circ}(1)).$$

Galois representations attached to Hida families. The Galois representation $\mathbb{V}_{\boldsymbol{\phi}}$ of Theorem 1.3 associated by Hida and Wiles to a Hida family $\boldsymbol{\phi}$ of tame level M and character χ can be realised as a quotient of the Λ-module $e^* H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p(1))$. More precisely, let

$$\xi_{\boldsymbol{\phi}} : \mathbb{T}_\Lambda \longrightarrow \Lambda_{\boldsymbol{\phi}}$$

be the Λ-algebra homomorphism from the Λ-adic Hecke algebra $\mathbb{T}_\Lambda$ to the Λ-algebra $\Lambda_{\boldsymbol{\phi}}$ generated by the fourier coefficients of $\boldsymbol{\phi}$ sending T_ℓ to $a_\ell(\boldsymbol{\phi})$.

Then we have, much as in (16), a quotient map of Λ-adic Galois representations

$$\varpi_{\boldsymbol{\phi}} : e^* H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p(1)) \longrightarrow e^* H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}_\Lambda, \xi_{\boldsymbol{\phi}}} \Lambda_\phi =: \mathbb{V}_{\boldsymbol{\phi}}(M), \tag{34}$$

for which the following diagram of $\mathbb{T}_\Lambda[G_\mathbb{Q}]$-modules is commutative:

$$\begin{tikzcd} e^* H^1_{\mathrm{et}}(\bar{X}^*_\infty, \mathbb{Z}_p(1)) \arrow[r, "\varpi_{\boldsymbol{\phi}}"] \arrow[d, "\mathrm{sp}_k^*"] & \mathbb{V}_{\boldsymbol{\phi}}(M) \arrow[d, "x"] \\ H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{H}^{k_\circ}(1)) \arrow[r, "\varpi_{\boldsymbol{\phi}_x}"] & V_{\boldsymbol{\phi}_x}(Mp), \end{tikzcd} \tag{35}$$

for all arithmetic points x of $\mathcal{W}_{\boldsymbol{\phi}}$ of weight $k = k_\circ + 2$ and trivial character.

As in (17), $\mathbb{V}_{\boldsymbol{\phi}}(M)$ is non-canonically isomorphic to a finite direct sum of copies of a $\Lambda_{\boldsymbol{\phi}}[G_\mathbb{Q}]$-module $\mathbb{V}_{\boldsymbol{\phi}}$ of rank 2 over $\Lambda_{\boldsymbol{\phi}}$, satisfying the properties stated in Theorem 1.3.

1.4. **Families of Dieudonné modules.** Let $\mathbf{B}_{\rm dR}$ denote Fontaine's field of de Rham periods, $\mathbf{B}_{\rm dR}^+$ be its ring of integers and $\log[\zeta_{p^\infty}]$ denote the uniformizer of $\mathbf{B}_{\rm dR}^+$ associated to a norm-compatible system $\zeta_{p^\infty} = \{\zeta_{p^n}\}_{n\geq 0}$ of p^n-th roots of unity. (cf. e.g.[BK93, §1]). For any finite-dimensional de Rham Galois representation V of $G_{\mathbb{Q}_p}$ with coefficients in a finite extension $L_p/\mathbb{Q}_p$, define the de Rham Dieudonné module

$$D(V) = (V \otimes \mathbf{B}_{\rm dR})^{G_{\mathbb{Q}_p}}.$$

It is an L_p-vector space of the same dimension as V, equipped with a descending exhaustive filtration $\mathrm{Fil}^j D(V) = (V \otimes \log^j[\zeta_{p^\infty}]\mathbf{B}_{\rm dR}^+)^{G_{\mathbb{Q}_p}}$ by L_p-vector subspaces.

Let $\mathbf{B}_{\rm cris} \subset \mathbf{B}_{\rm dR}$ denote Fontaine's ring of crystalline p-adic periods. If V is crystalline (which is always the case if it arises as a subquotient of the étale cohomology of an algebraic variety with good reduction at p), then there is a canonical isomorphism

$$D(V) \simeq (V \otimes \mathbf{B}_{\rm cris})^{G_{\mathbb{Q}_p}},$$

which furnishes $D(V)$ with a linear action of a Frobenius endomorphism Φ.

In [BK93] Bloch and Kato introduced a collection of subspaces of the local Galois cohomology group $H^1(\mathbb{Q}_p, V)$, denoted respectively

$$H^1_{\rm e}(\mathbb{Q}_p, V) \subseteq H^1_{\rm f}(\mathbb{Q}_p, V) \subseteq H^1_{\rm g}(\mathbb{Q}_p, V) \subseteq H^1(\mathbb{Q}_p, V),$$

and constructed homomorphisms

$$\log_{\rm BK} : H^1_{\rm e}(\mathbb{Q}_p, V) \xrightarrow{\sim} D(V)/\big(\mathrm{Fil}^0 D(V) + D(V)^{\Phi=1}\big) \tag{36}$$

and

$$\exp^*_{\rm BK} : H^1(\mathbb{Q}_p, V)/H^1_{\rm g}(\mathbb{Q}_p, V) \xrightarrow{\sim} \mathrm{Fil}^0 D(V) \tag{37}$$

that are usually referred to as the Bloch-Kato logarithm and dual exponential map.

We illustrate the above Bloch-Kato homomorphisms with a few basic examples that shall be used several times in the remainder of this article.

Example 1.6. As shown e.g. in [BK93], [Bel, §2.2], for any unramified character ψ of $G_{\mathbb{Q}_p}$ and all $n \in \mathbb{Z}$ we have:

(a) If $n \geq 2$, or $n = 1$ and $\psi \neq 1$, then $H^1_{\rm e}(\mathbb{Q}_p, L_p(\psi\varepsilon^n_{\rm cyc})) = H^1(\mathbb{Q}_p, L_p(\psi\varepsilon^n_{\rm cyc}))$ is one-dimensional over L_p and the Bloch-Kato logarithm induces an isomorphism

$$\log_{\rm BK} : H^1(\mathbb{Q}_p, L_p(\psi\varepsilon^n_{\rm cyc})) \xrightarrow{\sim} D(L_p(\psi\varepsilon^n_{\rm cyc})).$$

(b) If $n < 0$, or $n = 0$ and $\psi \neq 1$, then $H^1_{\rm g}(\mathbb{Q}_p, L_p(\psi\varepsilon^n_{\rm cyc})) = 0$ and $H^1(\mathbb{Q}_p, L_p(\psi\varepsilon^n_{\rm cyc}))$ is one-dimensional. The dual exponential gives rise to an isomorphism

$$\exp^*_{\rm BK} : H^1(\mathbb{Q}_p, L_p(\psi\varepsilon^n_{\rm cyc})) \xrightarrow{\sim} \mathrm{Fil}^0 D(L_p(\psi\varepsilon^n_{\rm cyc})) = D(L_p(\psi\varepsilon^n_{\rm cyc})).$$

(c) Assume $\psi = 1$. If $n = 0$, then $H^1(\mathbb{Q}_p, L_p)$ has dimension 2, $H^1_{\rm f}(\mathbb{Q}_p, L_p) = H^1_{\rm g}(\mathbb{Q}_p, L_p)$ has dimension 1 and $H^1_{\rm e}(\mathbb{Q}_p, L_p)$ has dimension 0 over L_p. The Bloch-Kato dual exponential map induces an isomorphism

$$\exp^*_{\rm BK} : H^1(\mathbb{Q}_p, L_p)/H^1_{\rm f}(\mathbb{Q}_p, L_p) \xrightarrow{\sim} \mathrm{Fil}^0 D(L_p) = D(L_p) = L_p.$$

Class field theory identifies $H^1(\mathbb{Q}_p, L_p)$ with $\mathrm{Hom}_{\rm cont}(\mathbb{Q}_p^\times, \mathbb{Q}_p) \otimes L_p$, which is spanned by the homomorphisms ord_p and $\log_p$.

If $n = 1$, then $H^1(\mathbb{Q}_p, L_p(1)) = H^1_{\rm g}(\mathbb{Q}_p, L_p(1))$ is 2-dimensional and $H^1_{\rm f}(\mathbb{Q}_p, L_p(1)) = H^1_{\rm e}(\mathbb{Q}_p, L_p(1))$ has dimension 1 over L_p. As proved e.g. in [Bel, Prop. 2.9], Kummer theory identifies the spaces $H^1_{\rm f}(\mathbb{Q}_p, L_p(1)) \subset H^1(\mathbb{Q}_p, L_p(1))$ with $\mathbb{Z}_p^\times \hat{\otimes} L_p$ sitting inside $\mathbb{Q}_p^\times \hat{\otimes} L_p$. Under this identification, the Bloch-Kato logarithm is the usual p-adic logarithm on $\mathbb{Z}_p^\times$.

Let $\hat{\mathbb{Z}}_p^{\mathrm{nr}}$ denote the ring of integers of the completion of the maximal unramified extension of $\mathbb{Q}_p$. If V is unramified then there is a further canonical isomorphism

$$D(V) \simeq (V \otimes \hat{\mathbb{Z}}_p^{\mathrm{nr}})^{G_{\mathbb{Q}_p}}. \tag{38}$$

Let ϕ be an eigenform (with respect to the good Hecke operators) of weight $k = k_\circ + 2 \geq 2$, level M and character χ, with fourier coefficients in a finite extension L_p of $\mathbb{Q}_p$. The comparison theorem [Fa] of Faltings-Tsuji combined with (16) asserts that there is a natural isomorphism

$$D(V_\phi(M)) \simeq H^1_{\mathrm{dR}}(X_1(M), \mathcal{H}^{k_\circ}(1))[\phi]$$

of Dieudonné modules over L_p. Note that $D(V_\phi(M))$ is the direct sum of several copies of the two-dimensional Dieudonné module $D(V_\phi)$.

Assume that $p \nmid M$ and ϕ is ordinary at p. Then $V_\phi(M)$ is crystalline and Φ acts on $D(V_\phi(M))$ as

$$\Phi = \chi(p) p^{k_\circ + 1} U_p^{-1}. \tag{39}$$

In particular the eigenvalues of Φ on $D(V_\phi(M))$ are $\chi(p)p^{k_\circ+1}\alpha_\phi^{-1} = \beta_\phi$ and $\chi(p)p^{k_\circ+1}\beta_\phi^{-1} = \alpha_\phi$, the two roots of the Hecke polynomial of ϕ at p.

Let $\phi^* = \phi \otimes \bar{\chi} \in S_k(M, \bar{\chi})$ denote the twist of ϕ by the inverse of its nebentype character. Poincaré duality induces a perfect pairing

$$\langle\, , \rangle : D(V_\phi(M)) \times D(V_{\phi^*}(M)) \longrightarrow D(L_p) = L_p.$$

The exact sequence (18) induces in this setting an exact sequence of Dieudonné modules

$$0 \longrightarrow D(V_\phi^+(M)) \xrightarrow{i} D(V_\phi(M)) \xrightarrow{\pi} D(V_\phi^-(M)) \longrightarrow 0. \tag{40}$$

Since $V_\phi^-(M$ is unramified, we have $D(V_\phi^-(M)) \simeq (V_\phi^-(M) \otimes \hat{\mathbb{Z}}_p^{\mathrm{nr}})^{G_{\mathbb{Q}_p}}$. This submodule may also be characterized as the eigenspace $D(V_\phi^-(M)) = D(V_\phi(M))^{\Phi = \alpha_\phi}$ of eigenvalue α_ϕ for the action of frobenius.

The rule $\breve{\phi} \mapsto \omega_{\breve{\phi}}$ that attaches to any modular form its associated differential form gives rise to an isomorphism $S_k(M,\chi)_{L_p}[\phi] \xrightarrow{\sim} \mathrm{Fil}^0(D(V_\phi(M))) \subset D(V_\phi(M))$. Moreover, the map π of (40) induces an isomorphism

$$S_k(M,\chi)_{L_p}[\phi] \xrightarrow{\sim} \mathrm{Fil}^0(D(V_\phi(M))) \xrightarrow{\pi} D(V_\phi^-(M)). \tag{41}$$

Any element $\omega \in D(V_{\phi^*}^-(M))$ gives rise to a linear map

$$\omega : D(V_\phi^+(M)) \longrightarrow L_p, \quad \eta \mapsto \langle \eta, \pi^{-1}(\omega) \rangle.$$

Similarly, any $\eta \in D(V_{\phi^*}^+(M))$ also gives rise to a linear functional

$$\eta : D(V_\phi^-(M)) \longrightarrow L_p, \quad \omega \mapsto \langle \pi^{-1}(\omega), \eta \rangle.$$

Let now $\tilde{\Lambda}$ be a finite flat extension of the Iwasawa algebra Λ and let $\mathbb{U}$ denote a free $\tilde{\Lambda}$-module of finite rank equipped with an *unramified* $\tilde{\Lambda}$-linear action of $G_{\mathbb{Q}_p}$. Define the Λ-adic Dieudonné module

$$\mathbb{D}(\mathbb{U}) := (\mathbb{U} \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{nr}})^{G_{\mathbb{Q}_p}}.$$

As shown in e.g. [Och03, Lemma 3.3], $\mathbb{D}(\mathbb{U})$ is a free module over $\tilde{\Lambda}$ of the same rank as $\mathbb{U}$.

Examples of such Λ-adic Dieudonné modules arise naturally in the context of families of modular forms thanks to Theorem 1.3. Indeed, let $\boldsymbol{\phi}$ be a Hida family of tame level M and character χ, and let $\boldsymbol{\phi}^*$ denote the Λ-adic modular form obtained by twisting $\boldsymbol{\phi}$ by $\bar{\chi}$.

Let $\mathbb{V}_{\boldsymbol{\phi}}$ and $\mathbb{V}_{\boldsymbol{\phi}}(M)$ denote the global Λ-adic Galois representations described in (34). It follows from (22) that to the restriction of $\mathbb{V}_{\boldsymbol{\phi}}$ to $G_{\mathbb{Q}_p}$ one might associate two natural unramified $\Lambda[G_{\mathbb{Q}_p}]$-modules of rank one, namely

$$\mathbb{V}_{\boldsymbol{\phi}}^{-} \simeq \Lambda_{\boldsymbol{\phi}}(\psi_{\boldsymbol{\phi}}) \quad \text{and} \quad \mathbb{U}_{\boldsymbol{\phi}}^{+} = \mathbb{V}_{\boldsymbol{\phi}}^{+}(\chi^{-1}\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}).$$

Define similarly the unramified modules $\mathbb{V}_{\boldsymbol{\phi}}^{-}(M)$ and $\mathbb{U}_{\boldsymbol{\phi}}^{+}(M)$.

Let

$$S_{\Lambda}^{\mathrm{ord}}(M,\chi)[\boldsymbol{\phi}] := \left\{ \breve{\boldsymbol{\phi}} \in S_{\Lambda}^{\mathrm{ord}}(M,\chi) \quad \text{s.t.} \quad \left| \begin{array}{l} T_\ell \breve{\boldsymbol{\phi}} = a_\ell(\boldsymbol{\phi})\breve{\boldsymbol{\phi}}, \quad \forall \ell \nmid Mp, \\ U_p \breve{\boldsymbol{\phi}} = a_p(\boldsymbol{\phi})\breve{\boldsymbol{\phi}} \end{array} \right. \right\}, \tag{42}$$

For any cristalline arithmetic point $x \in \mathcal{W}_{\boldsymbol{\phi}}^{\circ}$ of weight k, the specialization of a Λ-adic test vector $\breve{\phi} \in S_{\Lambda}^{\mathrm{ord}}(M,\chi)[\boldsymbol{\phi}]$ at x is a classical eigenform $\breve{\phi}_x \in S_k(Mp,\chi)$ with coefficients in $L_p = x(\Lambda_{\boldsymbol{\phi}}) \otimes \mathbb{Q}_p$ and the same eigenvalues as ϕ_x for the good Hecke operators.

Likewise, define

$$S_{\Lambda}^{\mathrm{ord}}(M,\bar{\chi})^{\vee}[\boldsymbol{\phi}] = \left\{ \eta : S_{\Lambda}^{\mathrm{ord}}(M,\bar{\chi}) \to \Lambda_{\boldsymbol{\phi}} \left| \begin{array}{l} \eta \circ T_\ell^* = a_\ell(\boldsymbol{\phi})\eta, \quad \forall \ell \nmid Mp, \\ \eta \circ U_p^* = a_p(\boldsymbol{\phi})\eta \end{array} \right. \right\}$$

Let $\mathcal{Q}_{\boldsymbol{\phi}}$ denote the field of fractions of $\Lambda_{\boldsymbol{\phi}}$. Associated to any test vector $\breve{\boldsymbol{\phi}} \in S_{\Lambda}^{\mathrm{ord}}(M,\chi)[\boldsymbol{\phi}]$, [DR1, Lemma 2.19] describes a $\mathcal{Q}_{\boldsymbol{\phi}}$-linear dual test vector

$$\breve{\boldsymbol{\phi}}^{\vee} \in S_{\Lambda}^{\mathrm{ord}}(M,\bar{\chi})^{\vee}[\boldsymbol{\phi}] \hat{\otimes} \mathcal{Q}_{\boldsymbol{\phi}} \tag{43}$$

such that for any $\boldsymbol{\varphi} \in S_{\Lambda}^{\mathrm{ord}}(M,\bar{\chi})$ and any point $x \in \mathcal{W}_{\mathbf{f}}^{\circ}$,

$$x(\breve{\boldsymbol{\phi}}^{\vee}(\boldsymbol{\varphi})) = \frac{\langle \breve{\boldsymbol{\phi}}_x, \boldsymbol{\varphi}_x \rangle}{\langle \breve{\boldsymbol{\phi}}_x, \breve{\boldsymbol{\phi}}_x^{*} \rangle}$$

where $\langle\, , \rangle$ denotes the pairing induced by Poincaré duality on the modular curve associated to the congruence subgroup $\Gamma_1(M) \cap \Gamma_0(p)$. This way, the specialization of a Λ-adic dual test vector $\breve{\boldsymbol{\phi}}^{\vee} \in S_{\Lambda}^{\mathrm{ord}}(M,\bar{\chi})^{\vee}[\boldsymbol{\phi}]$ at x gives rise to a linear functional

$$\breve{\boldsymbol{\phi}}_x^{\vee} : S_k(Mp,\bar{\chi})[\phi_x^*] \longrightarrow L_p,$$

which in view of the above isomorphisms we may identify with an element $\eta_{\breve{\boldsymbol{\phi}}_x} \in D(V_{\boldsymbol{\phi}_x}^{+}(Mp))$.

A natural $\mathcal{Q}_{\mathbf{f}}$-basis of $S_{\Lambda}^{\mathrm{ord}}(M,\chi)[\boldsymbol{\phi}] \hat{\otimes} \mathcal{Q}_{\boldsymbol{\phi}}$ is given by the Λ-adic modular forms $\boldsymbol{\phi}(q^d)$ as d ranges over the positive divisors of $M/M_{\boldsymbol{\phi}}$ and it is also obvious that $\{\boldsymbol{\phi}(q^d)^{\vee} : d \mid \frac{M}{M_{\boldsymbol{\phi}}}\}$ provides a $\mathcal{Q}_{\boldsymbol{\phi}}$-basis of $S_{\Lambda}^{\mathrm{ord}}(M,\bar{\chi})^{\vee}[\boldsymbol{\phi}] \hat{\otimes} \mathcal{Q}_{\boldsymbol{\phi}}$.

The following statement shows that the linear maps described above can be made to vary in families.

Proposition 1.7. *For any Λ-adic test vector $\breve{\boldsymbol{\phi}} \in S_{\Lambda}^{\mathrm{ord}}(M,\chi)[\boldsymbol{\phi}]$ there exist*

(1) *a homomorphism of $\Lambda_{\boldsymbol{\phi}}$-modules*

$$\langle\, , \omega_{\breve{\boldsymbol{\phi}}} \rangle : D(\mathbb{U}_{\boldsymbol{\phi}^*}^{+}(M)) \longrightarrow \Lambda_{\boldsymbol{\phi}}$$

such that for every $x \in \mathcal{W}_{\boldsymbol{\phi}}^{\circ}$, the specialization of $\omega_{\breve{\boldsymbol{\phi}}}$ at x is the linear form

$$x \circ \langle\, , \omega_{\breve{\boldsymbol{\phi}}} \rangle = \langle\, , \omega_{\breve{\boldsymbol{\phi}}_x} \rangle : D(U_{\boldsymbol{\phi}_x^*}^{+}(Mp)) \longrightarrow L_p.$$

(2) *and a homomorphism of $\Lambda_{\boldsymbol{\phi}}$-modules*

$$\langle\, , \eta_{\breve{\boldsymbol{\phi}}} \rangle : D(\mathbb{V}_{\boldsymbol{\phi}^*}^{-}(M)) \longrightarrow \mathcal{Q}_{\boldsymbol{\phi}},$$

whose specialization at a classical point $x \in \mathcal{W}_{\phi}^{\circ}$ such that ϕ_x is the ordinary stabilization of an eigenform ϕ_x° of level M, agrees with the functional

$$x \circ \langle\, , \eta_{\breve{\phi}}\rangle = \frac{\breve{\phi}_x^{\vee}}{\mathcal{E}_0(\phi_x^{\circ})\mathcal{E}_1(\phi_x^{\circ})} : D(V_{\phi_x^*}^-(Mp)) \longrightarrow L_p.$$

Here

$$\mathcal{E}_0(\phi_x^{\circ}) = 1 - \chi^{-1}(p)\beta_{\phi_x^{\circ}}^2 p^{1-k}, \quad \mathcal{E}_1(\phi_x^{\circ}) = 1 - \chi(p)\alpha_{\phi_x^{\circ}}^{-2} p^{k-2} \tag{44}$$

are the Euler factors appearing in [DR1, Theorem 1.3].

Proof. This is a reformulation of [KLZ, Proposition 10.1.1 and 10.1.2], which in turn builds on [Oh2]. Namely, Prop. 10.1.1 of loc. cit. proves the statement, except that the interpolation property in the second claim reads as

$$x \circ \langle\, , \eta_{\breve{\phi}}\rangle = \frac{\breve{\phi}_x^{\vee}}{\lambda(\phi_x^{\circ})\mathcal{E}_0(\phi_x^{\circ})\mathcal{E}_1(\phi_x^{\circ})} : D(V_{\phi_x^*}^-(Mp)) \longrightarrow L_p.$$

where $\lambda(\phi_x^{\circ}) \in \bar{\mathbb{Q}}^{\times}$ denotes the pseudo-eigenvalue of ϕ_x°, which we recall it is the scalar given by

$$W_M(\phi_x^{\circ}) = \lambda(\phi_x^{\circ}) \cdot \phi_x^{\circ *}, \tag{45}$$

where $W_M : S_k(M,\chi) \to S_k(M,\chi^{-1})$ stands for the Atkin-Lehner operator. Since we are assuming that Λ_{ϕ} contains the M-th roots of unity (cf. the remark right after Definition 1.1), Prop. 10.1.2 of loc. cit. shows that there exists an element $\lambda(\phi) \in \Lambda_{\phi}$ interpolating the pseudo-eigenvalues of the classical p-stabilized specializations of ϕ. The claim follows, as the functional $\langle\, , \eta_{\breve{\phi}}\rangle$ above is obtained as the product of that of [KLZ] and $\lambda(\phi)$. □

Remark 1.8. If $\breve{\phi}_x$ is the p-stabilization of an eigenform $\breve{\phi}_x^{\circ}$ of level M, then

$$\omega_{\breve{\phi}_x} = (1 - \frac{\beta_{\phi_x^{\circ}}}{\alpha_{\phi_x^{\circ}}})\varpi_1^* \omega_{\phi_x^{\circ}} \quad \text{and} \quad \breve{\phi}_x^{\vee} = (1 - \frac{\beta_{\phi_x^{\circ}}}{\alpha_{\phi_x^{\circ}}})\varpi_1^* \breve{\phi}_x^{\circ\vee}.$$

2. Stark-Heegner points

2.1. **Review of Stark-Heegner points.** This section recalls briefly the construction of Stark-Heegner points originally proposed in [Dar] and compares it with the equivalent but slightly different presentation given in the introduction. As explained in Remark 3, we provide the details under the running assumptions of loc. cit., and we refer to the references quoted in the introduction for the analogous story under the more general hypothesis (2).

Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N := pM$ with $p \nmid M$. Since E has multiplicative reduction at p, the group $E(\mathbb{Q}_{p^2})$ of local points over the quadratic unramified extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$ is equipped with Tate's p-adic uniformisation

$$\Phi_{\mathrm{Tate}} : \mathbb{Q}_{p^2}^{\times}/q^{\mathbb{Z}} \longrightarrow E(\mathbb{Q}_{p^2}).$$

Let f be the weight two newform attached to E via Wiles' modularity theorem, which satisfies the usual invariance properties under Hecke's congruence group $\Gamma_0(N)$, and let

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/p]), \qquad c \equiv 0 \pmod{M} \right\}$$

denote the associated p-arithmetic group, which acts by Möbius transformations both on the complex upper-half plane $\mathcal{H}$ and on Drinfeld's p-adic analogue $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$. The main construction of Sections 1-3 of [Dar] attaches to f a non-trivial *indefinite multiplicative integral*

$$\mathcal{H}_p \times \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \mathbb{C}_p^{\times}/q^{\mathbb{Z}}, \qquad (\tau, x, y) \mapsto \times\!\!\!\!\!\!\int^{\tau}\!\!\int_x^y \omega_f$$

satisfying

$$\text{(46)} \qquad \times\!\!\!\!\!\!\int^{\gamma\tau}\!\!\int_{\gamma x}^{\gamma y} \omega_f = \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{x}^{y} \omega_f, \qquad \text{for all } \gamma \in \Gamma,$$

along with the requirement that

$$\text{(47)} \qquad \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{x}^{y} \omega_f = \left(\times\!\!\!\!\!\!\int^{\tau}\!\!\int_{y}^{x} \omega_f\right)^{-1}, \qquad \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{x}^{y} \omega_f \times \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{y}^{z} \omega_f = \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{x}^{z} \omega_f.$$

This function is obtained, roughly speaking, by applying the Schneider-Teitelbaum p-adic Poisson transform to a suitable harmonic cocycle constructed from the modular symbol attached to f. It is important to note that there are in fact *two distinct* such modular symbols, which depend on a choice of a sign $w_\infty = \pm 1$ at ∞ and are referred to as the plus and the minus modular symbols, and therefore two distinct multiplicative integral functions, with different transformation properties under matrices of determinant -1 in $\mathrm{GL}_2(\mathbb{Z}[1/p])$. More precisely, the multiplicative integral associated to w_∞ satisfies the further invariance property

$$\times\!\!\!\!\!\!\int^{-\tau}\!\!\int_{-x}^{-y} \omega_f = \left(\times\!\!\!\!\!\!\int^{\tau}\!\!\int_{x}^{y} \omega_f\right)^{w_\infty}.$$

See sections 1-3 of loc. cit., and §3.3. in particular, for further details.

Let K be a real quadratic field of discriminant $D > 0$, whose associated Dirichlet character χ_K satisfies the *Heegner hypothesis*

$$\chi_K(p) = -1, \qquad \chi_K(\ell) = 1 \text{ for all } \ell | M.$$

It follows that D is a quadratic residue modulo M, and we may fix a $\delta \in (\mathbb{Z}/M\mathbb{Z})^\times$ satisfying $\delta^2 = D \pmod{M}$. Let $K_p \simeq \mathbb{Q}_{p^2}$ denote the completion of K at p, and let $\sqrt{D}$ denote a chosen square root of D in K_p.

Fix an order $\mathcal{O}$ of K, of conductor c relatively prime to DN. The narrow Picard group $G_{\mathcal{O}} := \mathrm{Pic}(\mathcal{O})$ is in bijection with the set of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of binary quadratic forms of discriminant Dc^2. A binary quadratic form $F = Ax^2+Bxy+Cy^2$ of this discriminant is said to be a *Heegner form* relative to the pair (M, δ) if M divides A and $B \equiv \delta c \pmod{M}$. Every class in $G_{\mathcal{O}}$ admits a representative which is a Heegner form, and all such representatives are equivalent under the natural action of the group $\Gamma_0(M)$. In particular, we can write

$$G_{\mathcal{O}} = \Gamma_0(M)\backslash \left\{Ax^2 + Bxy + Cy^2 \quad \text{with } (A, B) \equiv (0, \delta c) \pmod{M}\right\}.$$

For each class $\mathfrak{a} := Ax^2 + Bxy + Cy^2 \in G_{\mathcal{O}}$ as above, let

$$\tau_{\mathfrak{a}} := \frac{-B + c\sqrt{D}}{2A} \in K_p - \mathbb{Q}_p \subset \mathcal{H}_p, \qquad \gamma_{\mathfrak{a}} := \begin{pmatrix} r - Bs & -2Cs \\ 2As & r + Bs \end{pmatrix},$$

where (r, s) is a primitive solution to the Pell equation $x^2 - Dc^2y^2 = 1$. The matrix $\gamma_{\mathfrak{a}} \in \Gamma$ has $\tau_{\mathfrak{a}}$ as a fixed point for its action on $\mathcal{H}_p$. This fact, combined with properties (46) and (47), implies that the period

$$J_{\mathfrak{a}} := \times\!\!\!\!\!\!\int^{\tau_{\mathfrak{a}}}\!\!\int_{x}^{\gamma_{\mathfrak{a}} x} \omega_f \ \in K_p^\times/q^{\mathbb{Z}}$$

does not depend on the choice of $x \in \mathbb{P}_1(\mathbb{Q})$ that was made to define it. Property (46) also shows that $J_{\mathfrak{a}}$ depends only on $\mathfrak{a}$ and not on the choice of Heegner representative that was made in order to define $\tau_{\mathfrak{a}}$ and $\gamma_{\mathfrak{a}}$. The local point

$$y(\mathfrak{a}) := \Phi_{\mathrm{Tate}}(J_{\mathfrak{a}}) \in E(K_p)$$

is called the *Stark-Heegner point* attached to the class $\mathfrak{a} \in G_{\mathcal{O}}$.

Let H denote the narrow ring class field of K attached to $\mathcal{O}$, whose Galois group is canonically identified with $G_{\mathcal{O}}$ via global class field theory. Because p is inert in $K/\mathbb{Q}$ and $\mathrm{Gal}\,(H/K)$ is a generalised dihedral group, this prime splits completely in H/K. The set $\mathcal{P}$ of primes of

H that lie above p has cardinality $[H:K]$ and is endowed with a simply transitive action of $\mathrm{Gal}\,(H/K) = G_{\mathcal{O}}$, denoted $(\mathfrak{a},\mathfrak{p}) \mapsto \mathfrak{a} * \mathfrak{p}$.

Set $K_p^{\mathcal{P}} := \mathrm{Hom}(\mathcal{P}, E(K_p)) \simeq K_p^{[H:K]}$. There is a canonical identification

$$H \otimes \mathbb{Q}_p = K_p^{\mathcal{P}}, \tag{48}$$

sending $x \in H \otimes \mathbb{Q}_p$ to the function $\mathfrak{p} \mapsto x(\mathfrak{p}) := x_{\mathfrak{p}}$, where $x_{\mathfrak{p}}$ denotes the natural image of x in $H_{\mathfrak{p}} = K_p$. The group $\mathrm{Gal}\,(H/K)$ acts compatibly on both sides of (48), acting on the latter via the rule

$$\sigma x(\mathfrak{p}) = x(\sigma^{-1} * \mathfrak{p}). \tag{49}$$

Our fixed embedding of H into $\bar{\mathbb{Q}}_p$ determines a prime $\mathfrak{p}_0 \in \mathcal{P}$. Conjecture 5.6 of [Dar] asserts that the points $y(\mathfrak{a})$ are the images in $E(K_p)$ of global points $P_{\mathfrak{a}}^? \in E(H)$ under this embedding, and Conjecture 5.9 of loc. cit. asserts that these points satisfy the Shimura reciprocity law

$$P_{\mathfrak{b}\mathfrak{a}}^? = \mathrm{rec}(\mathfrak{b})^{-1} P_{\mathfrak{a}}^?, \qquad \text{for all } \mathfrak{b} \in \mathrm{Pic}(\mathcal{O}),$$

where $\mathrm{rec} : \mathrm{Pic}(\mathcal{O}) \longrightarrow \mathrm{Gal}\,(H/K)$ denotes the reciprocity map of global class field theory.

It is convenient to reformulate the conjectures of [Dar] as suggested in the introduction, by parlaying the collection $\{y(\mathfrak{a})\}$ of local points in $E(K_p)$ into a collection of semi-local points

$$P_{\mathfrak{a}} \in E(H \otimes \mathbb{Q}_p) = E(K_p)^{\mathcal{P}}$$

indexed by $\mathfrak{a} \in G_{\mathcal{O}}$. This is done by letting $P_{\mathfrak{a}}$ (viewed as an $E(K_p)$-valued function on the set $\mathcal{P}$) be the element of $E(H \otimes \mathbb{Q}_p)$ given by

$$(P_{\mathfrak{a}})(\mathfrak{b} * \mathfrak{p}_0) := y(\mathfrak{a}\mathfrak{b}),$$

so that, by definition

$$P_{\mathfrak{b}\mathfrak{a}}(\mathfrak{p}) = P_{\mathfrak{a}}(\mathfrak{b} * \mathfrak{p}). \tag{50}$$

This point of view has the pleasant consequence that the Shimura reciprocity law becomes a formal consequence of the definitions:

Lemma 2.1. *The semi-local Stark-Heegner points $P_{\mathfrak{a}} \in E(H \otimes \mathbb{Q}_p)$ satisfy the Shimura reciprocity law*

$$\mathrm{rec}(\mathfrak{b})^{-1}(P_{\mathfrak{a}}) = P_{\mathfrak{b}\mathfrak{a}}.$$

Proof. By (49),

$$\mathrm{rec}(\mathfrak{b})^{-1}(P_{\mathfrak{a}})(\mathfrak{p}) = P_{\mathfrak{a}}(\mathrm{rec}(\mathfrak{b}) * \mathfrak{p}) = P_{\mathfrak{a}}(\mathfrak{b} * \mathfrak{p}), \qquad \text{for all } \mathfrak{p} \in \mathcal{P}.$$

But on the other hand, by (50)

$$P_{\mathfrak{a}}(\mathfrak{b} * \mathfrak{p}) = P_{\mathfrak{b}\mathfrak{a}}(\mathfrak{p}).$$

The result follows from the two displayed identities. □

The modular form f is an eigenvector for the Atkin-Lehner involution W_N acting on $X_0(N)$. Let w_N denote its associated eigenvalue. Note that this is the negative of the sign in the functional equation for $L(E,s)$ and hence that $E(\mathbb{Q})$ is expected to have odd (resp. even) rank if $w_N = 1$ (resp. if $w_N = -1$). Recall the prime $\mathfrak{p}_0$ of H attached to the chosen embedding of H into $\bar{\mathbb{Q}}_p$. The frobenius element at $\mathfrak{p}_0$ in $\mathrm{Gal}\,(H/\mathbb{Q})$ is a reflection in this dihedral group, and is denoted by $\sigma_{\mathfrak{p}_0}$.

Proposition 2.2. *For all $\mathfrak{a} \in G_{\mathcal{O}}$,*

$$\sigma_{\mathfrak{p}_0} P_{\mathfrak{a}} = w_N P_{\mathfrak{a}^{-1}}.$$

Proof. Proposition 5.10 of [Dar] asserts that

$$\sigma_{\mathfrak{p}_0} y(\mathfrak{a}) = w_N y(\mathfrak{c}\mathfrak{a})$$

for some $\mathfrak{c} \in G_{\mathcal{O}}$. The definition of $\mathfrak{c}$ which occurs in equation (177) of loc.cit. directly implies that

$$\sigma_{\mathfrak{p}_0} y(1) = w_N y(1), \qquad \sigma_{\mathfrak{p}_0} y(\mathfrak{a}) = w_N y(\mathfrak{a}^{-1}),$$

and the result follows from this. □

Lemma 2.1 shows that the collection of Stark-Heegner points $P_{\mathfrak{a}}$ is preserved under the action of $\mathrm{Gal}\,(H/K)$, essentially by fiat. A corollary of the less formal Proposition 2.2 is the following invariance of the Stark-Heegner points under the full action of $\mathrm{Gal}\,(H/\mathbb{Q})$:

Corollary 2.3. *For all $\sigma \in \mathrm{Gal}\,(H/\mathbb{Q})$ and all $\mathfrak{a} \in G_{\mathcal{O}}$,*

$$\sigma P_{\mathfrak{a}} = w_N^{\delta_\sigma} P_{\mathfrak{b}}, \qquad \textit{for some } \mathfrak{b} \in G_{\mathcal{O}},$$

where

$$\delta_\sigma = \begin{cases} 0 & \textit{if } \sigma \in \mathrm{Gal}\,(H/K); \\ 1 & \textit{if } \sigma \notin \mathrm{Gal}\,(H/K). \end{cases}$$

Proof. This follows from the fact that $\mathrm{Gal}\,(H/\mathbb{Q})$ is generated by $\mathrm{Gal}\,(H/K)$ together with the reflection $\sigma_{\mathfrak{p}_0}$. □

To each $\mathfrak{p} \in \mathcal{P}$ we have associated an embedding $j_{\mathfrak{p}} : H \longrightarrow K_p$ and a frobenius element $\sigma_{\mathfrak{p}} \in \mathrm{Gal}\,(H/\mathbb{Q})$. If $\mathfrak{p}' = \sigma * \mathfrak{p}$ is another prime in $\mathcal{P}$, then we observe that

$$j_{\mathfrak{p}'} = j_{\mathfrak{p}} \circ \sigma^{-1}, \qquad \sigma_{\mathfrak{p}'} = \sigma\sigma_{\mathfrak{p}}\sigma^{-1}, \qquad j_{\mathfrak{p}'} \circ \sigma_{\mathfrak{p}'} = j_{\mathfrak{p}} \circ \sigma_{\mathfrak{p}} \circ \sigma^{-1}. \tag{51}$$

Let $\psi : \mathrm{Gal}\,(H/K) \longrightarrow L^\times$ be a ring class character, let

$$e_\psi := \frac{1}{\# G_{\mathcal{O}}} \sum_{\sigma \in G_{\mathcal{O}}} \psi(\sigma)\sigma^{-1} \in L[G_{\mathcal{O}}]$$

be the associated idempotent in the group ring, and denote by

$$P_\psi := e_\psi P_1 \in E(H \otimes \mathbb{Q}_p) \otimes L$$

the ψ-component of the Stark-Heegner point. Recall from the introduction the sign $\alpha \in \{-1, 1\}$ which is equal to 1 (resp. -1) if E has split (resp. non-split) multiplicative reduction at the prime p. Following the notations of the introduction, write

$$P_\psi^\alpha = (1 + \alpha\sigma_{\mathfrak{p}})P_\psi.$$

Lemma 2.4. *The local point $j_{\mathfrak{p}}(P_\psi^\alpha)$ is independent of the choice of prime $\mathfrak{p} \in \mathcal{P}$ that was made to define it, up to multiplication by a scalar in $\psi(G_{\mathcal{O}}) \subset L^\times$.*

Proof. Let $\mathfrak{p}' = \sigma * \mathfrak{p}$ be any other element of $\mathcal{P}$. Then by (51),

$$\begin{aligned} j_{\mathfrak{p}'}(1 + \alpha\sigma_{\mathfrak{p}'})P_\psi &= j_{\mathfrak{p}} \circ \sigma^{-1}(1 + \alpha\sigma\sigma_{\mathfrak{p}}\sigma^{-1})e_\psi P_1 = j_{\mathfrak{p}} \circ (1 + \alpha\sigma_{\mathfrak{p}})\sigma^{-1}e_\psi P_1 \\ &= \psi(\sigma)^{-1} j_{\mathfrak{p}} \circ (1 + \alpha\sigma_{\mathfrak{p}})P_\psi. \end{aligned}$$

The result follows. □

2.2. **Examples.** This section describes a few numerical examples illustrating the scope and applicability of the main results of this paper. By way of illustration, suppose that E is an elliptic curve of prime conductor $N = p$, so that $M = 1$. In that special case the Atkin-Lehner sign w_N is related to the local sign α by

$$w_N = -\alpha.$$

The following proposition reveals that the analytic non-vanishing hypothesis fails in the setting of the Stark-Heegner theorem for quadratic characters of [BD2] when $\epsilon = -1$:

Proposition 2.5. *Let ψ be a totally even quadratic ring class character of K of conductor prime to N. Then P_ψ^α is trivial.*

Proof. Let $(\chi_1, \chi_2) = (\chi, \chi\chi_K)$ be the pair of even quadratic Dirichlet characters associated to ψ, ordered in such a way that $L(E, \chi_1, s)$ and $L(E, \chi_2, s)$ have signs 1 and -1 respectively in their functional equations. Writing $\mathrm{sign}(E, \chi) \in \{-1, 1\}$ for the sign in the functional equation of the twisted L-function $L(E, \chi, s)$, it is well-known that, if the conductor of χ is relatively prime to N,

$$\mathrm{sign}(E, \chi) = \mathrm{sign}(E)\chi(-N) = -w_N\chi(-1)\chi(p) = \alpha\chi(p)\chi(-1).$$

It follows that

$$\alpha\chi_1(p) = 1, \qquad \alpha\chi_2(p) = -1,$$

but equation (4) in the Stark-Heegner theorem for quadratic characters implies $P_\psi^\alpha = 0$. $\square$

The systematic vanishing of P_ψ^α for even quadratic ring class characters of K can be traced to the failure of the analytic non-vanishing hypothesis of the introduction, which arises for simple parity reasons. The failure is expected to occur essentially only when E has prime conductor p, i.e., when $M = 1$, and never when M satisfies $\mathrm{ord}_q(M) = 1$ for some prime q. Because of Proposition 2.5, the main theorem of [BD2] gives no information about the Stark-Heegner point P_ψ^α attached to even quadratic ring class characters of conductor prime to p, on an elliptic curve of conductor p.

On the other hand, in the setting of Theorem A of the introduction, where ψ has order > 2, this phenomenon does not occur as the non-vanishing of P_ψ^α and $P_\psi^{-\alpha}$ are *equivalent* to each other, in light of the irreducibility of the induced representation V_ψ. The numerical examples below show many instances of non-vanishing P_ψ^α for ring class characters of both even and odd parity.

Example. Let $E : y^2 + y = x^3 - x$ be the elliptic curve of conductor $p = 37$, whose Mordell-Weil group is generated by the point $(0, 0) \in E(\mathbb{Q})$. Let $K = \mathbb{Q}(\sqrt{5})$ be the real quadratic field of smallest discriminant in which p is inert. It is readily checked that $L(E/K, s)$ has a simple zero at $s = 1$ and that $E(K)$ also has Mordell-Weil rank one. The curve E has non-split multiplicative reduction at p and hence $\alpha = -1$ in this case. It is readily verified that the pair of odd characters (χ_1, χ_2) attached to the quadratic imaginary fields of discriminant -4 and -20 satisfy the three conditions in (4), and hence the analytic non-vanishing hypothesis is satisfied for the triple $(E, K, \epsilon = 1)$. In particular, Theorem A holds for E, K, and all *even* ring class characters of K of conductor prime to 37.

Let $\mathcal{O}$ be an order of $\mathcal{O}_K$ with class number 3, and let H be the corresponding cubic extension of K. The prime $\mathfrak{p}$ of H over p and a generator σ of $\mathrm{Gal}\,(H/K)$ can be chosen so that the components

$$P_1 := P_{\mathfrak{p}}, \qquad P_2 := P_{\sigma\mathfrak{p}}, \qquad P_3 := P_{\sigma^2\mathfrak{p}}$$

in $E(H_{\mathfrak{p}}) = E(K_p)$ of the Stark-Heegner point in $E(H \otimes \mathbb{Q}_p)$ satisfy

$$\overline{P}_1 = P_1, \qquad \overline{P}_2 = P_3, \qquad \overline{P}_3 = P_2.$$

Letting ψ be the cubic character which sends σ to $\zeta := (1+\sqrt{-3})/2$, we find that

$$\begin{aligned} j_{\mathfrak{p}}(P_\psi) &= P_1 + \zeta P_2 + \zeta^2 P_3, \\ \sigma_{\mathfrak{p}}(j_{\mathfrak{p}}(P_\psi)) &= \overline{P}_1 + \zeta \overline{P}_2 + \zeta^2 \overline{P}_3 = P_1 + \zeta P_3 + \zeta^2 P_2, \\ j_{\mathfrak{p}}(P_\psi^\alpha) &= \sqrt{-3} \times (P_2 - P_3) = \sqrt{-3} \times (P_2 - \overline{P}_2). \end{aligned}$$

The following table lists the Stark-Heegner points P_1, P_2, and $P_2 - \overline{P}_2$ attached to the first few orders $\mathcal{O} \subset \mathcal{O}_K$ of conductor $c = c(\mathcal{O})$ and of class number three, calculated to a 37-adic accuracy of 2 significant digits. (The numerical entries in the table below are thus to be understood as elements of $(\mathbb{Z}/37^2\mathbb{Z})[\sqrt{5}]$.)

$c(\mathcal{O})$	P_1	P_2	$P_2 - \overline{P}_2$
18	$(-635, -256)$	$(319 + 678\sqrt{5}, -481230\sqrt{5})$	$(-360, 684 + 27\sqrt{5})$
38	$(-154, 447)$	$(-588 + 1237\sqrt{5}, 367 + 386\sqrt{5})$	$(-437, 684 + 87\sqrt{5})$
46	$(223, 12 \cdot 37)$	$(-112 + 629\sqrt{5}, (-6 + 34\sqrt{5}) \cdot 37)$	∞
47	$(610, -229)$	$(539 + 71\sqrt{5}, 10 + 439\sqrt{5})$	$(-293, 684 + 1132\sqrt{5})$
54	$(533, -561)$	$(679 + 984\sqrt{5}, 391 + 862\sqrt{5})$	$(93, 684 + 673\sqrt{5})$

Since the Mordell-Weil group of $E(K)$ has rank one, the data in this table is enough to conclude that the pro-37-Selmer groups of E over the ring class fields of K attached to the orders of conductors 18, 38, 47 and 54 have rank at least 3. As for the order of conductor 46, a calculation modulo 37^3 reveals that $P_2 - \overline{P}_2$ is non-trivial, and hence the pro-37 Selmer group has rank ≥ 3 over the ring class field of that conductor as well. Under the Stark-Heegner conjecture, more is true: the Stark-Heegner points above are 37-adic approximations of global points rather than mere Selmer classes. But recognising them as such (and thereby proving that the Mordell-Weil ranks are ≥ 3) typically requires a calculations to higher accuracy, depending on the eventual height of the Stark-Heegner point as an algebraic point, about which nothing is known of course a priori, and which can behave somewhat erratically. For example, the x-coordinates of the Stark-Heegner points attached to the order of conductor 47 appear to satisfy the cubic polynomial

$$x^3 - 319x^2 + 190x + 420,$$

while those of the Stark-Heegner points for the order of conductor 46 appear to satisfy the cubic polynomial

$$2352347001x^3 - 34772698791x^2 + 138835821427x - 136501565573$$

with much larger coefficients, whose recognition requires a calculation to at least 7 digits of 37-adic accuracy.

The table above produced many examples of non-vanishing P_ψ^α for ψ even, and in particular it verifies the non-vanishing hypothesis for Stark-Heegner points stated in the introduction, for the sign $\epsilon = -1$. This means that Theorem A is also true for *odd* ring class characters of K, even if the premise of (6) is *never verified* for odd *quadratic* characters of K.

2.3. **p-adic L-functions associated to Hida families over real quadratic fields.** Let

$$\mathbf{f} = \sum_{n \geq 1} a_n(\mathbf{f}) q^n \in \Lambda_{\mathbf{f}}[[q]]$$

be the Hida family of tame level M and trivial tame character passing through f. Let $x_0 \in \mathcal{W}_{\mathbf{f}}^\circ$ denote the point of weight 2 such that $\mathbf{f}_{x_0} = f$. Note that $\mathbf{f}_{x_0} \in S_2(N)$ is new at p, while for any $x \in \mathcal{W}_{\mathbf{f}}^\circ$ with $\mathrm{wt}(x) = k > 2$, $\mathbf{f}_x(q) = \mathbf{f}_x^\circ(q) - \beta \mathbf{f}_x^\circ(q^p)$ is the ordinary p-stabilisation of an eigenform $\mathbf{f}_x^\circ$ of level $M = N/p$. We set $\mathbf{f}_{x_0}^\circ = \mathbf{f}_{x_0} = f$.

Let K be a real quadratic field in which p remains inert and all prime factors of M split, and fix throughout a finite order anticyclotomic character ψ of K of conductor c coprime to

DN, with values in a finite extension $L_p/\mathbb{Q}_p$. Note that $\psi(p) = 1$ as the prime ideal $p\mathcal{O}_K$ is principal.

Under our running assumptions, the sign of the functional equation satisfied by the Hasse-Weil-Artin L-series $L(E/K, \psi, s) = L(f, \psi, s)$ is

$$\varepsilon(E/K, \psi) = -1,$$

and in particular the order of vanishing of $L(E/K, \psi, s)$ at $s = 1$ is odd. In contrast, at every classical point x of even weight $k > 2$ the sign of the functional equation satisfied by $L(\mathbf{f}_x/K, \psi, s)$ is

$$\varepsilon(\mathbf{f}_x/K, \psi) = +1$$

and one expects generic non-vanishing of the central critical value $L(\mathbf{f}_x/K, \psi, k/2)$.

In [BD2, Definition 3.4], a p-adic L-function

$$\mathcal{L}_p(\mathbf{f}/K, \psi) \in \Lambda_{\mathbf{f}}$$

associated to the Hida family $\mathbf{f}$, the ring class character ψ and a choice of collection of periods was defined, by interpolating the algebraic part of (the square-root of) the critical values $L(\mathbf{f}_x/K, \psi, k/2)$ for $x \in \mathcal{W}_{\mathbf{f}}^\circ$ with $\mathrm{wt}(x) = k = k_\circ + 2 \geq 2$. See also [LMY, §4.1] for a more general treatment, encompassing the setting considered here.

In order to describe this p-adic L-function in more detail, let $\Phi_{\mathbf{f}_x,\mathbb{C}}$ denote the classical modular symbol associated to $\mathbf{f}_x$ with values in the space $P_{k_\circ}(\mathbb{C})$ of homogeneous polynomials of degree $k_\circ$ in two variables with coefficients in $\mathbb{C}$. The space of modular symbols is naturally endowed with an action of $\mathrm{GL}_2(\mathbb{Q})$ and we let $\Phi^+_{\mathbf{f}_x,\mathbb{C}}$ and $\Phi^-_{\mathbf{f}_x,\mathbb{C}}$ denote the plus and minus eigencomponents of $\Phi_{\mathbf{f}_x,\mathbb{C}}$ under the involution at infinity induced by $w_\infty = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

As proved in [KZ, §1.1] (with slightly different normalizations as for the powers of the period $2\pi i$ that appear in the formulas, which we have taken into account accordingly), there exists a pair of collections of complex periods

$$\{\Omega^+_{\mathbf{f}_x,\mathbb{C}}\}_{x\in\mathcal{W}_{\mathbf{f}}^\circ}, \quad \{\Omega^-_{\mathbf{f}_x,\mathbb{C}}\}_{x\in\mathcal{W}_{\mathbf{f}}^\circ} \subset \mathbb{C}^\times$$

satisfying the following two conditions:

(i) the modular symbols

$$\Phi^+_{\mathbf{f}_x} := \frac{\Phi^+_{\mathbf{f}_x,\mathbb{C}}}{\Omega^+_{\mathbf{f}_x,\mathbb{C}}}, \quad \Phi^-_{\mathbf{f}_x} := \frac{\Phi^-_{\mathbf{f}_x,\mathbb{C}}}{\Omega^-_{\mathbf{f}_x,\mathbb{C}}} \qquad \text{take values in } \mathbb{Q}(\mathbf{f}_x) = \mathbb{Q}(\{a_n(\mathbf{f}_x)\}_{n\geq 1}),$$

(ii) and $\quad \Omega^+_{\mathbf{f}_x,\mathbb{C}} \cdot \Omega^-_{\mathbf{f}_x,\mathbb{C}} = 4\pi^2\langle \mathbf{f}_x^\circ, \mathbf{f}_x^\circ\rangle.$

Note that conditions (i) and (ii) above only characterize $\Omega^\pm_{\mathbf{f}_x,\mathbb{C}}$ up to multiplication by non-zero scalars in the number field $\mathbb{Q}(\mathbf{f}_x)$.

Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p \subset \mathbb{C}_p$, through which we regard $\Phi^\pm_{\mathbf{f}_x}$ as $\mathbb{C}_p$-valued modular symbols. In [GS], Greenberg and Stevens introduced measure-valued modular symbols $\mu^+_{\mathbf{f}}$ and $\mu^-_{\mathbf{f}}$ interpolating the classical modular symbols $\Phi^+_{\mathbf{f}_x}$ and $\Phi^-_{\mathbf{f}_x}$ as x ranges over the classical specializations of $\mathbf{f}$.

More precisely, they show (cf. [GS, Theorem 5.13] and [BD1, Theorem 1.5]) that for every $x \in \mathcal{W}_{\mathbf{f}}^\circ$, there exist p-adic periods

$$\Omega^+_{\mathbf{f}_x,p}, \Omega^-_{\mathbf{f}_x,p} \in \mathbb{C}_p \tag{52}$$

such that the specialisation of $\mu^+_{\mathbf{f}}$ and $\mu^-_{\mathbf{f}}$ at x satisfy

$$x(\mu^+_{\mathbf{f}}) = \Omega^+_{\mathbf{f}_x,p} \cdot \Phi^+_{\mathbf{f}_x}, \qquad x(\mu^-_{\mathbf{f}}) = \Omega^-_{\mathbf{f}_x,p} \cdot \Phi^-_{\mathbf{f}_x}. \tag{53}$$

Since no natural choice of periods $\Omega^\pm_{\mathbf{f}_x,\mathbb{C}}$ presents itself, the scalars $\Omega^+_{\mathbf{f}_x,p}$ and $\Omega^-_{\mathbf{f}_x,p}$ are not expected to vary p-adically continuously. However, conditions (i) and (ii) above implies that the *product* $\Omega^+_{\mathbf{f}_x,p} \cdot \Omega^-_{\mathbf{f}_x,p} \in \mathbb{C}_p$ is a more canonical quantity, as it may also be characterized by the formula

$$x(\mu^+_{\mathbf{f}}) \cdot x(\mu^-_{\mathbf{f}}) = \Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p} \cdot \frac{\Phi^+_{\mathbf{f}_x,\mathbb{C}} \cdot \Phi^-_{\mathbf{f}_x,\mathbb{C}}}{4\pi^2\langle \mathbf{f}^\circ_x, \mathbf{f}^\circ_x\rangle}, \tag{54}$$

which is independent of any choices of periods.

This suggests that the map $x \mapsto \Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p}$ may extend to a p-adic analytic function, possibly after multiplying it by suitable Euler-like factors at p. And indeed, the following statement was proved in [BD3, Theorem 3.4]:

Proposition 2.6. *There exists a rigid-analytic function $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ on a neighborhood $U_{\mathbf{f}}$ of $\mathcal{W}_{\mathbf{f}}$ around x_0 such that for all classical points $x \in U_{\mathbf{f}} \cap \mathcal{W}^\circ_{\mathbf{f}}$ of weight $k \geq 2$:*

$$\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))(x) = \mathcal{E}_0(\mathbf{f}_x)\mathcal{E}_1(\mathbf{f}_x) \cdot \Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p}, \tag{55}$$

where $\mathcal{E}_0(\mathbf{f}_x)$ and $\mathcal{E}_1(\mathbf{f}_x)$ are the Euler factors introduced in (44).

Remark 2.7. The motivation for denoting $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ the p-adic function appearing above relies on the fact that $\Omega^\pm_{\mathbf{f}_x,p}$ are p-adic analogues of the complex periods $\Omega^\pm_{\mathbf{f}_x,\mathbb{C}}$. As is well-known, the product $\Omega^+_{\mathbf{f}_x,\mathbb{C}} \cdot \Omega^-_{\mathbf{f}_x,\mathbb{C}} = 4\pi^2\langle \mathbf{f}^\circ_x, \mathbf{f}^\circ_x\rangle$ is essentially the near-central critical value of the classical L-function associated to the symmetric square of $\mathbf{f}^\circ_x$. In addition to this, as M. L. Hsieh remarked to us, it might not be difficult to show that $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ is a generator of Hida's congruence ideal in the sense of [Hs, §1.4, p.4].

The result characterizing the p-adic L-function $\mathcal{L}_p(\mathbf{f}/K,\psi)$ alluded to above is [BD2, Theorem 3.5], which we recall below. Although [BD2, Theorem 3.5] is stated in loc. cit. only for genus characters, the proof has been recently generalized to arbitrary (not necessarily quadratic) ring class characters ψ of conductor c with $(c, DN) = 1$ by Longo, Martin and Yan in [LMY, Theorem 4.2], by employing Gross-Prasad test vectors to extend Popa's formula [Po, Theorem 6.3.1] to this setting.

Let $\mathfrak{f}_c \in K^\times$ denote the explicit simple constant introduced at the first display of [LMY, §3.2]. It only depends on the conductor c and its square lies in $\mathbb{Q}^\times$.

Theorem 2.8. *The p-adic L-function $L_p(\mathbf{f}/K,\psi)$ satisfies the following interpolation property: for all $x \in \mathcal{W}^\circ_{\mathbf{f}}$ of weight* $\mathrm{wt}(x) = k = k_\circ + 2 \geq 2$, *we have*

$$\mathcal{L}_p(\mathbf{f}/K,\psi)(x) = \mathfrak{f}_{\mathbf{f},\psi}(x) \times L(\mathbf{f}^\circ_x/K,\psi,k/2)^{1/2}$$

where

$$\mathfrak{f}_{\mathbf{f},\psi}(x) = (1 - \alpha^{-2}_{\mathbf{f}_x} p^{k_\circ}) \cdot \frac{\mathfrak{f}_c \cdot (Dc^2)^{\frac{k_\circ+1}{4}} (\frac{k_\circ}{2})!}{(2\pi i)^{k_\circ/2}} \cdot \frac{\Omega^{\epsilon_\psi}_{\mathbf{f}_x,p}}{\Omega^{\epsilon_\psi}_{\mathbf{f}_x,\mathbb{C}}}.$$

2.4. **A p-adic Gross-Zagier formula for Stark-Heegner points.** One of the main theorems of [BD2] is a formula for the derivative of $\mathcal{L}_p(\mathbf{f}/K,\psi)$ at the point x_0, relating it to the formal group logarithm of a Stark-Heegner point. This formula shall be crucial for relating these points to generalized Kato classes and eventually proving our main results.

Theorem 2.9. *The p-adic L-function $\mathcal{L}_p(\mathbf{f}/K,\psi)$ vanishes at the point x_0 of weight* 2 *and*

$$\frac{d}{dx}\mathscr{L}_p(\mathbf{f}/K,\psi)_{|x=x_0} = \frac{1}{2}\log_p(P^\alpha_\psi). \tag{56}$$

Proof. The vanishing of $\mathcal{L}_p(\mathbf{f}/K,\psi)$ at $x = x_0$ is a direct consequence of the assumptions and definitions, because $x = x_0$ lies in the region of interpolation of the p-adic L-function and therefore $\mathcal{L}_p(\mathbf{f}/K,\psi)(x_0)$ is a non-zero multiple of the central critical value $L(\mathbf{f}/K,\psi,1)$. This L-value vanishes as remarked at the beginning of this section.

The formula for the derivative follows verbatim as in the proof of [BD2, Theorem 4.1]. See also [LMY, Theorem 5.1] for the statement in the generality required here. Finally, we refer to [LV] for a formulation and proof of this formula in the setting of quaternionic Stark-Heegner points, under the general assumption of (2). □

3. Generalised Kato classes

3.1. **A compatible collection of cycles.** This section defines a collection of codimension two cycles in $X_1(Mp^r)^3$ indexed by elements of $(\mathbb{Z}/p^r\mathbb{Z})^{\times 3}$ and records some of their properties.

We retain the notations that were in force in Section 1.3 regarding the meanings of the curves $X = X_1(M)$, $X_r = X_1(Mp^r)$ and $X_{r,s}$. In addition, let

$$\mathbb{Y}(p^r) := Y \times_{X(1)} Y(p^r), \quad \mathbb{X}(p^r) := X \times_{X(1)} X(p^r)$$

denote the (affine and projective, respectively) modular curve over $\mathbb{Q}(\zeta_r)$ with full level p^r structure. The curve $\mathbb{Y}(p^r)$ classifies triples (A, P, Q) in which A is an elliptic curve with $\Gamma_1(M)$ level structure and (P, Q) is a basis for $A[p^r]$ satisfying $\langle P, Q\rangle = \zeta_r$, where $\langle\ ,\ \rangle$ denotes the Weil pairing and ζ_r is a fixed primitive p^r-th root of unity. The curve $\mathbb{X}(p^r)$ is geometrically connected but does not descend to a curve over $\mathbb{Q}$, as can be seen by noting that the description of its moduli problem depends on the choice of ζ_r. The covering $\mathbb{X}(p^r)/X$ is Galois with Galois group $\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$, acting on the left by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (A, P, Q) = (A, aP + bQ, cP + dQ). \tag{57}$$

Consider the natural projection map

$$\varpi_1^r \times \varpi_1^r \times \varpi_1^r : X_r^3 \longrightarrow X^3 \tag{58}$$

induced on triple products by the map ϖ_1^r of (24). Write $\Delta \subset X^3$ for the usual diagonal cycle, namely the image of X under the diagonal embedding $x \mapsto (x, x, x)$. Let Δ_r be the fiber product $\Delta \times_{X^3} X_r^3$ via the natural inclusion and the map of (58), which fits into the cartesian diagram

$$\begin{tikzcd} \Delta_r \arrow[r, hook] \arrow[d, two heads] & X_r^3 \arrow[d, two heads] \\ \Delta \arrow[r, hook] & X^3. \end{tikzcd}$$

An element of a $\mathbb{Z}_p$-module Ω is said to be *primitive* if it does not belong to $p\Omega$, and the set of such primitive elements is denoted Ω'. Let

$$\Sigma_r := ((\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z})')^3 \subset ((\mathbb{Z}/p^r\mathbb{Z})^2)^3$$

be the set of triples of primitive row vectors of length 2 with entries in $\mathbb{Z}/p^r\mathbb{Z}$, equipped with the action of $\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$ acting diagonally by right multiplication.

Lemma 3.1. *The geometrically irreducible components of Δ_r are defined over $\mathbb{Q}(\zeta_r)$ and are in canonical bijection with the set of left orbits*

$$\Sigma_r/\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z}).$$

Proof. Each triple

$$(v_1, v_2, v_3) = ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in \Sigma_r$$

determines a morphism

$$\varphi_{(v_1,v_2,v_3)} : \mathbb{X}(p^r) \longrightarrow \Delta_r \subset X_r^3$$

of curves over $\mathbb{Q}(\zeta_r)$, defined in terms of the moduli descriptions on $\mathbb{Y}(p^r)$ by

$$(A, P, Q) \quad \mapsto \quad (\ (A, x_1P + y_1Q),\ (A, x_2P + y_2Q),\ (A, x_3P + y_3Q)\ ).$$

It is easy to see that if two elements (v_1, v_2, v_3) and $(v_1', v_2', v_3') \in \Sigma_r$ satisfy

$$(v_1', v_2', v_3') = (v_1, v_2, v_3)\gamma, \quad \text{with } \gamma \in \mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z}),$$

then

$$\varphi_{(v_1', v_2', v_3')} = \varphi_{(v_1, v_2, v_3)} \circ \gamma,$$

where γ is being viewed as an automorphism of $\mathbb{X}(p^r)$ as in (57). It follows that the geometrically irreducible cycle

$$\Delta_r(v_1, v_2, v_3) := \varphi_{(v_1,v_2,v_3)*}(\mathbb{X}(p^r))$$

depends only on the $\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$-orbit of (v_1, v_2, v_3).

Since $\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$ acts transitively on $(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z})'$, one further checks that the collection of cycles $\Delta_r(v_1, v_2, v_3)$ for $(v_1, v_2, v_3) \in \Sigma_r/\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$ do not overlap on Y_r^3 and cover Δ_r. Hence the irreducible components of Δ_r are precisely $\Delta_r(v_1, v_2, v_3)$ for $(v_1, v_2, v_3) \in \Sigma_r/\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$. □

The quotient $\Sigma_r/\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$ is equipped with a natural determinant map

$$D : \Sigma_r/\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z}) \longrightarrow (\mathbb{Z}/p^r\mathbb{Z})^3$$

defined by

$$D\left((x_1 y_1), (x_2, y_2), (x_3, y_3)\right) := \left(\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}, \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right).$$

For each $[d_1, d_2, d_3] \in (\mathbb{Z}/p^r\mathbb{Z})^3$, we can then write

$$\Sigma_r[d_1, d_2, d_3] := \{(v_1, v_2, v_3) \in \Sigma_r \text{ with } D(v_1, v_2, v_3) = (d_1, d_2, d_3)\}.$$

The group $\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$ operates *simply transitively* on $\Sigma_r[d_1, d_2, d_3]$ if (and only if)

$$[d_1, d_2, d_3] \in I_r := (\mathbb{Z}/p^r\mathbb{Z})^{\times 3}. \tag{59}$$

In particular, if (v_1, v_2, v_3) belongs to $\Sigma_r[d_1, d_2, d_3]$, then the cycle $\Delta_r(v_1, v_2, v_3)$ depends only on $[d_1, d_2, d_3] \in I_r$ and will henceforth be denoted

$$\Delta_r[d_1, d_2, d_3] \in \mathrm{CH}^2(X_r^3).$$

A somewhat more intrinsic definition of $\Delta_r[d_1, d_2, d_3]$ as a curve embedded in X_r^3 is that it corresponds to the schematic closure of the locus of points $((A, P_1), (A, P_2), (A, P_3))$ satisfying

$$\langle P_2, P_3 \rangle = \zeta_r^{d_1}, \qquad \langle P_3, P_1 \rangle = \zeta_r^{d_2}, \qquad \langle P_1, P_2 \rangle = \zeta_r^{d_3}. \tag{60}$$

This description also makes it apparent that the cycle $\Delta_r[d_1, d_2, d_3]$ is defined over $\mathbb{Q}(\zeta_r)$ but not over $\mathbb{Q}$. Let $\sigma_m \in \mathrm{Gal}\,(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ be the automorphism associated to $m \in (\mathbb{Z}/p^r\mathbb{Z})^\times$, sending ζ_r to ζ_r^m. The threefold X_r^3 is also equipped with an action of the group

$$\tilde{G}_r := ((\mathbb{Z}/p^r\mathbb{Z})^\times)^3 = \{\langle a_1, a_2, a_3 \rangle, \quad a_1, a_2, a_3 \in (\mathbb{Z}/p^r\mathbb{Z})^\times\} \tag{61}$$

of diamond operators, where the automorphism associated to a triple $(\langle a_1 \rangle, \langle a_2 \rangle, \langle a_3 \rangle)$ has simply been denoted $\langle a_1, a_2, a_3 \rangle$.

Lemma 3.2. *For all diamond operators $\langle a_1, a_2, a_3 \rangle \in \tilde{G}_r$ and all $[d_1, d_2, d_3] \in I_r$,*

$$\langle a_1, a_2, a_3 \rangle \Delta_r[d_1, d_2, d_3] = \Delta_r[a_2 a_3 \cdot d_1, a_1 a_3 \cdot d_2, a_1 a_2 \cdot d_3]. \tag{62}$$

For all $\sigma_m \in \mathrm{Gal}\,(\mathbb{Q}(\zeta_r)/\mathbb{Q})$,

$$\sigma_m \Delta_r[d_1, d_2, d_3] = \Delta_r[m \cdot d_1, m \cdot d_2, m \cdot d_3]. \tag{63}$$

Proof. Equation (62) follows directly from the identity

$$D(a_1v_1, a_2v_2, a_3v_3) = [a_2a_3, a_1a_3, a_1a_2]D(v_1, v_2, v_3).$$

The first equality in (63) is most readily seen from the equation (60) defining the cycle $\Delta_r[d_1, d_2, d_3]$, since applying the automorphism $\sigma_m \in \mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ has the effect of replacing ζ_r by ζ_r^m. □

Remark 3.3. Assume m is a quadratic residue in $(\mathbb{Z}/p^r\mathbb{Z})^\times$, which is the case, for instance, when σ_m belongs to $\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}(\zeta_1))$. Then it follows from (62) and (63) that

$$\sigma_m\Delta_r[d_1, d_2, d_3] = \langle m, m, m\rangle^{1/2}\Delta_r[d_1, d_2, d_3]. \tag{64}$$

Let us now turn to the compatibility properties of the cycles $\Delta_r[d_1, d_2, d_3]$ as the level r varies. Recall the modular curve $X_{r,r+1}$ classifying generalised elliptic curves together with a distinguished cyclic subgroup of order p^{r+1} and a point of order p^r in it. The maps μ, ϖ_1, π_1, ϖ_2 and π_2 of (25) induce similar maps on the triple products:

(65)
$$\begin{tikzcd} X^3_{r+1} \arrow[d, "\mu^3"] \arrow[dr, "\varpi_1^3"] & \\ X^3_{r,r+1} \arrow[r, "\pi_1^3"'] & X^3_r, \end{tikzcd} \qquad \begin{tikzcd} X^3_{r+1} \arrow[d, "\mu^3"] \arrow[dr, "\varpi_2^3"] & \\ X^3_{r,r+1} \arrow[r, "\pi_2^3"'] & X^3_r. \end{tikzcd}$$

A finite morphism $j : V_1 \longrightarrow V_2$ of varieties induces maps

$$j_* : \mathrm{CH}^j(V_1) \longrightarrow \mathrm{CH}^j(V_2), \qquad j^* : \mathrm{CH}^j(V_2) \longrightarrow \mathrm{CH}^j(V_1)$$

between Chow groups, and j_*j^* agrees with the multiplication by $\deg(j)$ on $\mathrm{CH}^j(V_2)$. If j is a Galois cover with Galois group G,

$$j^*j_*(\Delta) = \sum_{\sigma\in G} \sigma\Delta. \tag{66}$$

By abuse of notation we will denote the associated maps on cycles (rather than just on cycle classes) by the same symbols.

Lemma 3.4. *For all $r \geq 1$ and all $[d_1', d_2', d_3'] \in I_{r+1}$ whose image in I_r is $[d_1, d_2, d_3]$,*

$$(\varpi_1^3)_*\Delta_{r+1}[d_1', d_2', d_3'] = p^3\Delta_r[d_1, d_2, d_3], \qquad (\varpi_2^3)_*\Delta_{r+1}[d_1', d_2', d_3'] = (U_p)^{\otimes 3}\Delta_r[d_1, d_2, d_3].$$

The cycles $\Delta_r[d_1, d_2, d_3]$ also satisfy the distribution relations

$$\sum_{[d_1',d_2',d_3']} \Delta_{r+1}[d_1', d_2', d_3'] = (\varpi_1^3)^*\Delta_r[d_1, d_2, d_3],$$

where the sum is taken over all triples $[d_1', d_2', d_3'] \in I_{r+1}$ which map to $[d_1, d_2, d_3]$ in I_r.

Proof. A direct verification based on the definitions shows that the morphisms μ^3 and π_1^3 of (65) induce morphisms

$$\Delta_{r+1}[d_1'd_2', d_3'] \xrightarrow{\mu^3} \mu^3_*\Delta_{r+1}[d_1', d_2', d_3'] \xrightarrow{\pi_1^3} \Delta_r[d_1, d_2, d_3],$$

of degrees 1 and p^3 respectively. Hence the restriction of ϖ_1^3 to $\Delta_{r+1}[d_1', d_2', d_3']$ induces a map of degree p^3 from $\Delta_{r+1}[d_1', d_2', d_3']$ to $\Delta_r[d_1, d_2, d_3]$, which implies the first assertion. It also follows from this that

$$\mu^3_*\Delta_{r+1}[d_1', d_2', d_3'] = (\pi_1^3)^*\Delta_r[d_1, d_2, d_3]. \tag{67}$$

Applying $(\pi_2^3)_*$ to this identity implies that

$$(\varpi_2^3)_*\Delta_{r+1}[d_1', d_2', d_3'] = (U_p)^{\otimes 3}\Delta_r[d_1, d_2, d_3].$$

The second compatibility relation follows. To prove the distribution relation, observe that the sum that occurs in it is taken over the p^3 translates of a fixed $\Delta_{r+1}[d_1', d_2', d_3']$ for the action of the Galois group of X_{r+1}^3 over $X_{r,r+1}^3$, and hence, by (66), that

$$\sum_{[d_1',d_2',d_3']} \Delta_{r+1}[d_1', d_2', d_3'] = (\mu^*)^3 \mu_*^3 \Delta_{r+1}[d_1', d_2', d_3'].$$

The result then follows from (67). □

3.2. **Galois cohomology classes.** The goal of this section is to parlay the cycles $\Delta_r[d_1, d_2, d_3]$ into Galois cohomology classes with values in $H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(2)$, essentially by considering their images under the p-adic étale Abel-Jacobi map:

$$\mathrm{AJ}_{\mathrm{et}} : \mathrm{CH}^2(X_r^3)_0 \longrightarrow H^1(\mathbb{Q}, H^3_{\mathrm{et}}(\bar{X}_r^3, \mathbb{Z}_p(2))), \tag{68}$$

where

$$\mathrm{CH}^2(X_r^3)_0 := \ker\left(\mathrm{CH}^2(X_r^3) \longrightarrow H^4_{\mathrm{et}}(\bar{X}_r^3, \mathbb{Z}_p(2))\right)$$

denotes the kernel of the étale cycle class map, i.e., the group of null-homologous algebraic cycles defined over $\mathbb{Q}$. There are two issues that need to be dealt with. Firstly, the cycles $\Delta_r[d_1, d_2, d_2]$ need not be null-homologous and have to be suitably modified so that they lie in the domain of the Abel Jacobi map. Secondly, these cycles are defined over $\mathbb{Q}(\zeta_r)$ and not over $\mathbb{Q}$, and it is desirable to descend the field of definition of the associated extension classes.

To deal with the first issue, let q be any prime not dividing Mp, and let T_q denote the Hecke operator attached to this prime. It can be used to construct an algebraic correspondence on X_r^3 by setting

$$\theta_q := (T_q - (q+1))^{\otimes 3}.$$

Lemma 3.5. *The element θ_q annihilates the target $H^4_{\mathrm{et}}(\bar{X}_r^3, \mathbb{Z}_p)$ of the étale cycle class map on $\mathrm{CH}^2(X_r^3)$.*

Proof. The correspondence T_q acts as multiplication by $(q+1)$ on $H^2_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)$ and θ_q therefore annihilates all the terms in the Künneth decomposition of $H^4_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)$. □

The *modified diagonal cycles* in $\mathrm{CH}^2(X_r^3)$ are defined by the rule

$$\Delta_r^\circ[d_1, d_2, d_3] := \theta_q \Delta_r[d_1, d_2, d_3]. \tag{69}$$

Lemma 3.5 shows that they are null-homologous and defined over $\mathbb{Q}(\zeta_r)$. Define

$$\kappa_r[d_1, d_2, d_3] \quad := \quad \mathrm{AJ}_{\mathrm{et}}(\Delta_r^\circ[d_1, d_2, d_3]) \in H^1(\mathbb{Q}(\zeta_r), H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(2)).$$

To deal with the circumstance that the cycles $\Delta_r^\circ[d_1, d_2, d_3]$ are only defined over $\mathbb{Q}(\zeta_r)$, and hence that the associated cohomology classes $\kappa_r[d_1, d_2, d_3]$ need not (and in fact, do not) extend to $G_{\mathbb{Q}}$, it is necessary to replace the $\mathbb{Z}_p[\tilde{G}_r][G_{\mathbb{Q}}]$-module $H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(2)$ by an appropriate twist over $\mathbb{Q}(\zeta_r)$. Let G_r denote the Sylow p-subgroup of the group $\tilde{G}_r$ of (61), and let $G_\infty := \varprojlim G_r$. Let

$$\Lambda(G_r) := \mathbb{Z}_p[G_r], \qquad \Lambda(G_\infty) = \mathbb{Z}_p[[G_\infty]]$$

be the finite group ring attached to G_r and the associated Iwasawa algebra, respectively.

Let $\Lambda(G_r)(\frac{1}{2})$ denote the Galois module which is isomorphic to $\Lambda(G_r)$ as a $\Lambda(G_r)$-module, and on which the Galois group $G_{\mathbb{Q}(\zeta_1)}$ is made to act via its quotient $\mathrm{Gal}\left(\mathbb{Q}(\zeta_r)/\mathbb{Q}(\zeta_1)\right) = 1 + p\mathbb{Z}/p^r\mathbb{Z}$, the element σ_m acting as multiplication by the group-like element $\langle m, m, m\rangle^{-1/2}$. Let $\Lambda(G_\infty)(\frac{1}{2})$ denote the projective limit of the $\Lambda(G_r)(\frac{1}{2})$. It follows from the definitions that if

$$\nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_r) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}, \qquad \text{or} \qquad \nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_\infty) \longrightarrow \mathbb{Z}_p$$

is the homomorphism sending $\langle a_1, a_2, a_3\rangle$ to $a_1^{k_\circ} a_2^{\ell_\circ} a_3^{m_\circ}$, then

$$\Lambda(G_r)(\tfrac{1}{2}) \otimes_{\nu_{k_\circ,\ell_\circ,m_\circ}} \mathbb{Z}/p^r\mathbb{Z} = (\mathbb{Z}/p^r\mathbb{Z})(\varepsilon_{\mathrm{cyc}}^{-(k_\circ+\ell_\circ+m_\circ)/2}), \tag{70}$$

where the tensor product is taken over $\Lambda(G_r)$, and similarly for G_∞. In particular if $k_\circ + \ell_\circ + m_\circ = 2t$ is an even integer,

$$\Lambda(G_\infty)(\tfrac{1}{2}) \otimes_{\nu_{k_\circ,\ell_\circ,m_\circ}} \mathbb{Z}_p = \mathbb{Z}_p(-t)(\omega^t) \tag{71}$$

as $G_{\mathbb{Q}}$-modules. More generally, if Ω is any $\Lambda(G_\infty)$ module, write

$$\Omega(\tfrac{1}{2}) := \Omega \otimes_{\Lambda(G_\infty)} \Lambda(G_\infty)(\tfrac{1}{2}), \qquad \Omega(\tfrac{-1}{2}) := \Omega \otimes_{\Lambda(G_\infty)} \Lambda(G_\infty)(\tfrac{-1}{2}),$$

for the relevant twists of Ω, which are isomorphic to Ω as a $\Lambda(G_\infty)[G_{\mathbb{Q}(\mu_{p^\infty})}]$-module but are endowed with different actions of $G_{\mathbb{Q}}$.

There is a canonical Galois-equivariant $\Lambda(G_r)$-hermitian bilinear, $\Lambda(G_r)$-valued pairing

$$\langle\!\langle\ ,\ \rangle\!\rangle_r : H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}) \ \times\ H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}) \ \longrightarrow\ \Lambda(G_r), \tag{72}$$

given by the formula

$$\langle\!\langle a, b\rangle\!\rangle_r := \sum_{\sigma = \langle d_1, d_2, d_3\rangle \in G_r} \langle a^\sigma, b\rangle_{X_r} \cdot \langle d_1, d_2, d_3\rangle,$$

where

$$\langle\ ,\ \rangle_{X_r} : H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(2) \times H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(1) \longrightarrow H^2_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p(1))^{\otimes 3} = \mathbb{Z}_p$$

arises from the Poincaré duality between $H^3_{\mathrm{et}}(\bar{X}_r^3, \mathbb{Z}_p)(2)$ and $H^3_{\mathrm{et}}(\bar{X}_r^3, \mathbb{Z}_p)(1)$. This pairing enjoys the following properties:

- For all $\lambda \in \Lambda(G_r)$,

  $$\langle\!\langle \lambda a, b\rangle\!\rangle_r = \lambda^* \langle\!\langle a, b\rangle\!\rangle_r, \qquad \langle\!\langle a, \lambda b\rangle\!\rangle_r = \lambda \langle\!\langle a, b\rangle\!\rangle_r,$$

  where $\lambda^* \in \Lambda(G_r)$ is obtained from λ by applying the involution on the group ring which sends every group-like element to its inverse. In particular, the pairing of (72) can and will also be viewed as a $\Lambda(G_r)$-valued $*$-hermitian pairing

  $$\langle\!\langle\ ,\ \rangle\!\rangle_r : H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(2) \times H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(1) \longrightarrow \Lambda(G_r).$$

- For all $\sigma \in G_{\mathbb{Q}(\zeta_1)}$, we have $\langle\!\langle \sigma a, \sigma b\rangle\!\rangle_r = \langle\!\langle a, b\rangle\!\rangle_r$.
- The U_p and U_p^* operators are adjoint to each other under this pairing, giving rise to a duality (denoted by the same symbol, by an abuse of notation)

  $$\langle\!\langle\ ,\ \rangle\!\rangle_r : e^* H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}) \ \times\ e H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}) \ \longrightarrow\ \Lambda(G_r).$$

Define

$$\begin{aligned}
\mathbb{H}^{111}(X_r) &:= \mathrm{Hom}_{\Lambda(G_r)}(H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_r)) \simeq H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}),\\
\mathbb{H}^{111}_{\mathrm{ord}}(X_r) &:= \mathrm{Hom}_{\Lambda(G_r)}(e H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_r)) \simeq e^* H^1_{\mathrm{et}}(\bar{X}_r, \mathbb{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}).
\end{aligned}$$

The above identifications are isomorphisms of $\Lambda(G_r)$-modules via the pairing (72), which is $\Lambda(G_r)$-linear in the second entry.

To descend the field of definition of the classes $\kappa_r[d_1, d_2, d_3]$, we package them together into elements

$$\boldsymbol{\kappa}_r[a, b, c] \ \in\ H^1(\mathbb{Q}(\zeta_r), \mathbb{H}^{111}(X_r))$$

indexed by triples

$$[a, b, c] \in I_1 = (\mathbb{Z}/p\mathbb{Z})^{\times 3} = \mu_{p-1}(\mathbb{Z}_p)^3 \subset (\mathbb{Z}_p^\times)^3. \tag{73}$$

The class $\boldsymbol{\kappa}_r[a,b,c]$ is defined by setting, for all $\sigma \in G_{\mathbb{Q}(\zeta_r)}$ and all $\gamma_r \in H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1)$,

$$\boldsymbol{\kappa}_r[a,b,c](\sigma)(\gamma_r) = \langle\!\langle \kappa_r[a,b,c](\sigma),\gamma_r\rangle\!\rangle_r, \tag{74}$$

where the elements $a,b,c \in (\mathbb{Z}/p\mathbb{Z})^\times$ are viewed as elements of $(\mathbb{Z}/p^r\mathbb{Z})^\times$ via the Teichmuller lift alluded to in (73). Note that there is a natural identification

$$H^1(\mathbb{Q}(\zeta_r),\mathbb{H}^{111}(X_r)) = \mathrm{Ext}^1_{\Lambda(G_r)[G_{\mathbb{Q}(\zeta_r)}]}(H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1),\Lambda(G_r),$$

because $H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1) = H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1)(\frac{1}{2})$ *as* $G_{\mathbb{Q}(\zeta_r)}$*-modules* and the $\Lambda(G_r)$-dual of the latter is $\mathbb{H}^{111}(X_r)$. With these definitions we have

Lemma 3.6. *The class* $\boldsymbol{\kappa}_r[a,b,c]$ *is the restriction to* $G_{\mathbb{Q}(\zeta_r)}$ *of a class*

$$\boldsymbol{\kappa}_r[a,b,c] \in H^1(\mathbb{Q}(\zeta_1),\mathbb{H}^{111}(X_r)) = \mathrm{Ext}^1_{\Lambda(G_r)[G_{\mathbb{Q}(\zeta_1)}]}(H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}),\Lambda(G_r)).$$

Furthermore, for all $m \in \mu_{p-1}(\mathbb{Z}_p)$,

$$\sigma_m\ \boldsymbol{\kappa}_r[a,b,c] = \boldsymbol{\kappa}_r[ma,mb,mc].$$

Proof. We will prove this by giving a more conceptual description of the cohomology class $\boldsymbol{\kappa}_r[a,b,c]$. Let $|\Delta|$ denote the support of an algebraic cycle Δ, and let

$$\Delta_r^\circ[[a,b,c]] := |\Delta_1^\circ[a,b,c]| \times_{X_1^3} X_r^3 \tag{75}$$

denote the inverse image in X_r^3 of $|\Delta_1^\circ[a,b,c]|$, which fits into the cartesian diagram

$$\begin{tikzcd} \Delta_r^\circ[[a,b,c]] \arrow[r, hook] \arrow[d] & X_r^3 \arrow[d, "(\varpi_1^{r-1})^3"] \\ |\Delta_1^\circ[a,b,c]| \arrow[r, hook] & X_1^3. \end{tikzcd}$$

As in the proof of Lemma 3.1, observe that

$$\Delta_r^\circ[[a,b,c]] = \bigsqcup_{[d_1,d_2,d_3]\in I_r^1} |\Delta_r^\circ[ad_1,bd_2,cd_3]|$$

where I_r^1 denotes the p-Sylow subgroup of I_r. Consider now the commutative diagram of $\Lambda(G_r)[G_{\mathbb{Q}}(\zeta_1)]$-modules with exact rows:

$$\begin{tikzcd} & & \Lambda(G_r)(\frac{-1}{2}) \arrow[d, hook, "j"] \\ H^3_{\mathrm{et}}(\bar{X}_r^3,\mathbb{Z}_p)(2) \arrow[r, hook] \arrow[d, "p"] & H^3_{\mathrm{et}}(\bar{X}_r^3 - \Delta_r^\circ[[a,b,c]],\mathbb{Z}_p)(2) \arrow[r, two heads] & H^0_{\mathrm{et}}(\bar{\Delta}_r^\circ[[a,b,c]],\mathbb{Z}_p)_0 \\ H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(2), & & \end{tikzcd} \tag{76}$$

where

- the map j is the inclusion defined on group-like elements by
$$j\left(\langle d_1,d_2,d_3\rangle\right) = \mathrm{cl}(\Delta_r^\circ[ad_2d_3,bd_1d_3,cd_1d_2]),$$
which is $G_{\mathbb{Q}(\zeta_1)}$-equivariant by Lemma 3.2;
- the middle row arises from the excision exact sequence in étale cohomology (cf. [Ja, (3.6)] and [Mi, p. 108]);

- the subscript of 0 appearing in the rightmost term in the exact sequence denotes the kernel of the cycle class map, i.e.,
$$H^0_{\mathrm{et}}(\bar{\Delta}^\circ_r[[a,b,c]],\mathbb{Z}_p)_0 := \ker\left(H^0_{\mathrm{et}}(\bar{\Delta}^\circ_r[[a,b,c]],\mathbb{Z}_p)_0 \longrightarrow H^4_{\mathrm{et}}(\bar{X}^3_r,\mathbb{Z}_p(2))\right),$$
and the fact that the image of j is contained in $H^0_{\mathrm{et}}(\bar{\Delta}^\circ_r[[a,b,c]],\mathbb{Z}_p)_0$ follows from Lemma 3.5;
- the projection p is the one arising from the Künneth decomposition.

Taking the pushout and pullback of the extension in (76) via the maps p and j yields an exact sequence of $\Lambda(G_r)[G_{\mathbb{Q}}(\zeta_1)]$-modules

$$0 \longrightarrow H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(2) \longrightarrow E_r \longrightarrow \Lambda(G_r)(\tfrac{-1}{2}) \longrightarrow 0. \tag{77}$$

Taking the $\Lambda(G_r)$-dual of this exact sequence, we obtain

$$0 \longrightarrow \Lambda(G_r)(\tfrac{1}{2}) \longrightarrow \check{E}_r \longrightarrow H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1)^* \longrightarrow 0.$$

where M^* means the $\Lambda(G_r)$-module obtained from M by letting act $\Lambda(G_r)$ on it by composing with the involution $\lambda \mapsto \lambda^*$. Twisting this sequence by $(\frac{-1}{2})$ and noting that $M^*(\frac{-1}{2}) \simeq M(\frac{1}{2})^*$ yields an extension

$$0 \longrightarrow \Lambda(G_r) \longrightarrow E'_r \longrightarrow H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2})^* \longrightarrow 0. \tag{78}$$

Since
$$H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2})^* = \mathrm{Hom}_{\Lambda(G_r)}(H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}),\Lambda(G_r)),$$
it follows that the cohomology class realizing the extension E'_r is an element of

$$H^1(\mathbb{Q}(\zeta_1),\mathrm{Hom}_{\Lambda(G_r)}(H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}),\Lambda(G_r))) = H^1(\mathbb{Q}(\zeta_1),\mathbb{H}^{111}(X_r)),$$

because the duality afforded by $\langle\!\langle\ ,\ \rangle\!\rangle_r$ is hermitian (and not Λ-linear). When restricted to $G_{\mathbb{Q}(\zeta_r)}$, this class coincides with $\boldsymbol{\kappa}_r[a,b,c]$, and the first assertion follows.

The second assertion is an immediate consequence of the definitions, using the Galois equivariance properties of the cycles $\Delta_r[d_1,d_2,d_3]$ given in the first assertion of Lemma 3.2. □

Remark 3.7. The extension E'_r of (78) can also be realised as a subquotient of the étale cohomology group $H^3_c(\bar{X}^3_r – \Delta^\circ_r[[a,b,c]],\mathbb{Z}_p)(1)$ with compact supports, in light of the Poincaré duality
$$H^3_{\mathrm{et}}(\bar{X}^3_r – \Delta^\circ_r[[a,b,c]],\mathbb{Z}_p)(2) \ \times\ H^3_c(\bar{X}^3_r – \Delta^\circ_r[[a,b,c]],\mathbb{Z}_p)(1) \longrightarrow \mathbb{Z}_p.$$

3.3. **Λ-adic cohomology classes.** Thanks to Lemma 3.6, we now dispose, for each $[a,b,c] \in \mu_{p-1}(\mathbb{Z}_p)^3$, of a system

$$\boldsymbol{\kappa}_r[a,b,c] \in H^1(\mathbb{Q}(\zeta_1),\mathbb{H}^{111}(X_r)), \qquad e^*\boldsymbol{\kappa}_r[a,b,c] \in H^1(\mathbb{Q}(\zeta_1),\mathbb{H}^{111}_{\mathrm{ord}}(X_r)) \tag{79}$$

of cohomology classes indexed by the integers $r \geq 1$. Let

$$p_{r+1,r} : \Lambda(G_{r+1}) \longrightarrow \Lambda(G_r)$$

be the projection on finite group rings induced from the natural homomorphism $G_{r+1} \longrightarrow G_r$.

Lemma 3.8. *Let $\gamma_{r+1} \in H^1_{\mathrm{et}}(\bar{X}_{r+1},\mathbb{Z}_p)^{\otimes 3}(1)$ and $\gamma_r \in H^1_{\mathrm{et}}(\bar{X}_r,\mathbb{Z}_p)^{\otimes 3}(1)$ be elements that are compatible under the pushforward by ϖ_1^3, i.e., that satisfy $(\varpi_1^3)_*(\gamma_{r+1}) = \gamma_r$. For all $\sigma \in G_{\mathbb{Q}(\zeta_1)}$,*

$$p_{r+1,r}\left(\boldsymbol{\kappa}_{r+1}[a,b,c](\sigma)(\gamma_{r+1})\right) = \boldsymbol{\kappa}_r[a,b,c](\sigma)(\gamma_r).$$

Proof. This amounts to the statement that

$$p_{r+1,r}(\langle\!\langle \kappa_{r+1}[a,b,c],\gamma_{r+1}\rangle\!\rangle_{r+1}) = \langle\!\langle \kappa_r[a,b,c],\gamma_r\rangle\!\rangle_r.$$

But

$$p_{r+1,r}(\langle\!\langle \kappa_{r+1}[a,b,c],\gamma_{r+1}\rangle\!\rangle_{r+1}) = \sum_{G_r} \langle (\mu^3)^*(\mu^3)_*\kappa_{r+1}[ad_2'd_3', bd_1'd_3', cd_1'd_2'], \gamma_{r+1}\rangle_{X_{r+1}} \cdot \langle d_1,d_2,d_3\rangle,$$

where the sum runs over $\langle d_1,d_2,d_3\rangle \in G_r$ and $\langle d_1',d_2',d_3'\rangle$ denotes an (arbitrary) lift of $\langle d_1,d_2,d_3\rangle$ to G_{r+1}. The third assertion in Lemma 3.4 allows us to rewrite the right-hand side as

$$\begin{aligned} p_{r+1,r}(\langle\!\langle \kappa_{r+1}[a,b,c],\gamma_{r+1}\rangle\!\rangle_{r+1}) &= \sum_{G_r} \langle (\varpi_1^3)^*\kappa_r[ad_2d_3, bd_1d_3, cd_1d_2], \gamma_{r+1}\rangle_{X_{r+1}} \cdot \langle d_1,d_2,d_3\rangle \\ &= \sum_{G_r} \langle \kappa_r[ad_2d_3, bd_1d_3, cd_1d_2], (\varpi_1^3)_*\gamma_{r+1}\rangle_{X_r} \cdot \langle d_1,d_2,d_3\rangle \\ &= \sum_{G_r} \langle \kappa_r[ad_2d_3, bd_1d_3, cd_1d_2], \gamma_r\rangle_{X_r} \cdot \langle d_1,d_2,d_3\rangle \\ &= \langle\!\langle \kappa_r[a,b,c],\gamma_r\rangle\!\rangle_r, \end{aligned}$$

and the result follows. □

Define

$$\begin{aligned} \mathbb{H}^{111}(X_\infty^*) := \quad & \mathrm{Hom}_{\Lambda(G_\infty)}(H^1_{\mathrm{et}}(\bar X_\infty^*, \mathbb{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_\infty)) \\ &= \mathrm{Hom}_{\Lambda(G_\infty)}(H^1_{\mathrm{et}}(\bar X_1, \mathcal{L}_\infty^*)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_\infty)), \end{aligned} \tag{80}$$

where the identification follows from (29).

Thanks to Lemma 3.8, the classes $\boldsymbol{\kappa}_r[a,b,c]$ can be packaged into a compatible collection

$$\boldsymbol{\kappa}_\infty[a,b,c] := (\boldsymbol{\kappa}_r[a,b,c])_{r\ge 1} \in \mathbb{H}^{111}(X_\infty^*). \tag{81}$$

It will also be useful to replace the classes $\boldsymbol{\kappa}_\infty[a,b,c]$ by elements that are essentially indexed by triples

$$(\omega_1,\omega_2,\omega_3) : (\mathbb{Z}/p\mathbb{Z}^\times)^3 \longrightarrow \mathbb{Z}_p^\times$$

of tame characters of $\tilde G_r/G_r$. Assume that the product $\omega_1\omega_2\omega_3$ is an *even* character. This assumption is equivalent to requiring that

$$\omega_1\omega_2\omega_3 = \delta^2, \qquad \text{for some } \delta : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{Z}_p^\times.$$

Note that for a given $(\omega_1,\omega_2,\omega_3)$, there are in fact two characters δ as above, which differ by the unique quadratic character of conductor p. With the choices of $\omega_1,\omega_2,\omega_3$ and δ in hand, we set

$$\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta) := \frac{p^3}{(p-1)^3} \cdot \sum_{[a,b,c]} \delta^{-1}(abc)\cdot\omega_1(a)\omega_2(b)\omega_3(c)\cdot \boldsymbol{\kappa}_\infty[bc,ac,ab], \tag{82}$$

where the sum is taken over the triples $[a,b,c]$ of $(p-1)$st roots of unity in $\mathbb{Z}_p^\times$. The classes $\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta)$ satisfy the following properties.

Lemma 3.9. *For all $\sigma_m \in \mathrm{Gal}\,(\mathbb{Q}(\zeta_\infty)/\mathbb{Q})$,*

$$\sigma_m\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta) = \delta(m)\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta).$$

For all diamond operators $\langle a_1,a_2,a_3\rangle \in \mu_{p-1}(\mathbb{Z}_p)^3$

$$\langle a_1,a_2,a_3\rangle\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta) = \omega_{123}(a_1,a_2,a_3)\cdot\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta).$$

Proof. This follows from a direct calculation based on the definitions, using the compatibilities of Lemma 3.2 satisfied by the cycles $\Delta_r[d_1,d_2,d_3]$. □

The classes $\boldsymbol{\kappa}_\infty[a,b,c]$ and $\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta)$ are called the Λ-*adic cohomology classes* attached to the triple $[a,b,c]\in\mu_{p-1}(\mathbb{Z}_p)^3$ or the quadruple $(\omega_1,\omega_2,\omega_3;\delta)$. As will be explained in the next section, they are three variable families of cohomology classes parametrised by points in the triple product $\mathcal{W}\times\mathcal{W}\times\mathcal{W}$ of weight spaces, and taking values in the three-parameter family of self-dual Tate twists of the Galois representations attached to the different specialisations of a triple of Hida families.

Remark 3.10. It is instructive to compare the construction of $\boldsymbol{\kappa}_\infty[a,b,c]$ to the approach taken in [DR2], which associated to a triple $(f,\mathbf{g},\mathbf{h})$ consisting of a *fixed* newform f and a pair $(\mathbf{g},\mathbf{h})$ of Hida families a *one-variable* family of cohomology classes instead of the two-variable family that one might have felt entitled to *a priori*. This shortcoming of the earlier approach can be understood by noting that the space of embeddings of $\mathbb{X}(p^r)$ into $X\times X_r\times X_r$ as above in which the projection to the first factor is fixed is naturally parametrized by the coset space $M_2(\mathbb{Z}/p^r\mathbb{Z})'/\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})$, where $M_2(\mathbb{Z}/p^r\mathbb{Z})'$ denotes the set of 2×2 matrices whose rows are not divisible by p. The analogue of the cycles $\Delta_r[d_1,d_2,d_3]$ above are therefore parametrised by the coset space $\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})=(\mathbb{Z}/p^r\mathbb{Z})^\times$, whose inverse limit with r is the one dimensional p-adic space $\mathbb{Z}_p^\times$ rather than a two-dimensional one.

3.4. **Higher weight balanced specialisations.** For every integer $k_\circ\geq 0$ define

$$W_1^{k_\circ}:=H^1_{\mathrm{et}}(\bar{X}_1,\mathcal{H}^{k_\circ})$$

and recall from the combination of (29), (31) and (32) the specialisation map

$$\mathrm{sp}^*_{k_\circ}:H^1_{\mathrm{et}}(\bar{X}^*_\infty,\mathbb{Z}_p)=H^1_{\mathrm{et}}(\bar{X}_1,\mathcal{L}^*_\infty)\longrightarrow W_1^{k_\circ}. \tag{83}$$

Fix throughout this section a triple

$$k=k_\circ+2,\qquad \ell=\ell_\circ+2,\qquad m=m_\circ+2$$

of integers ≥ 2 for which $k_\circ+\ell_\circ+m_\circ=2t$ is even. Let

$$\mathcal{H}^{k_\circ,\ell_\circ,m_\circ}:=\mathcal{H}^{k_\circ}\boxtimes\mathcal{H}^{\ell_\circ}\boxtimes\mathcal{H}^{m_\circ}$$

viewed as a sheaf on X_1^3, and

$$W_1^{k_\circ,\ell_\circ,m_\circ}:=W_1^{k_\circ}\otimes W_1^{\ell_\circ}\otimes W_1^{m_\circ}(2-t).$$

As one readily checks, the p-adic Galois representation $W_1^{k_\circ,\ell_\circ,m_\circ}$ is Kummer self-dual, i.e., there is an isomorphism of $G_{\mathbb{Q}}$-modules

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(W_1^{k_\circ,\ell_\circ,m_\circ},\mathbb{Z}_p(1))\simeq W_1^{k_\circ,\ell_\circ,m_\circ}.$$

The specialisation maps give rise, in light of (71), to the triple product specialisation map

$$\mathrm{sp}^*_{k_\circ,\ell_\circ,m_\circ}:=\mathrm{sp}^*_{k_\circ}\otimes\mathrm{sp}^*_{\ell_\circ}\otimes\mathrm{sp}^*_{m_\circ}:\mathbb{H}^{111}(X^*_\infty)\longrightarrow W_1^{k_\circ,\ell_\circ,m_\circ} \tag{84}$$

and to the associated collection of specialised classes

$$\kappa_1(k_\circ,\ell_\circ,m_\circ)[a,b,c]:=\mathrm{sp}_{k_\circ,\ell_\circ,m_\circ}(\boldsymbol{\kappa}_\infty[a,b,c])\in H^1(\mathbb{Q}(\zeta_1),W_1^{k_\circ,\ell_\circ,m_\circ}). \tag{85}$$

Note that for $(k_\circ,\ell_\circ,m_\circ)=(0,0,0)$, it follows from the definitions (cf. e.g. the proof of Lemma 3.6) that the class $\kappa_1(k_\circ,\ell_\circ,m_\circ)[a,b,c]$ is simply the image under the étale Abel-Jacobi map of the cycle $\Delta_1^\circ[a,b,c]$.

The main goal of this section is to offer a similar geometric description for the above classes also when (k,ℓ,m) is *balanced* and $k_\circ,\ell_\circ,m_\circ>0$, which we assume henceforth for the remainder of this section.

In order to do this, it shall be useful to dispose of an alternate description of the extension (77) in terms of the étale cohomology of the (open) three-fold $X_1^3-|\Delta_1^\circ[a,b,c]|$ with values in appropriate sheaves.

Lemma 3.11. *Let $\mathcal{L}_r^{*\boxtimes 3}$ denote the exterior tensor product of $\mathcal{L}_r^*$, over the triple product X_1^3. There is a commutative diagram*

$$\begin{array}{ccccccc} 0 \longrightarrow & H^3_{\mathrm{et}}(X_r^3, \mathbb{Z}_p)(2) & \longrightarrow & H^3_{\mathrm{et}}(X_r^3 - \Delta_r^\circ[[a,b,c]], \mathbb{Z}_p)(2) & \longrightarrow & H^0_{\mathrm{et}}(\Delta_r^\circ[[a,b,c]], \mathbb{Z}_p) \\ & \| & & \| & & \| \\ 0 \longrightarrow & H^3_{\mathrm{et}}(\bar{X}_1^3, \mathcal{L}_r^{*\boxtimes 3})(2) & \longrightarrow & H^3_{\mathrm{et}}(\bar{X}_1^3 - |\Delta_1^\circ[a,b,c]|, \mathcal{L}_r^{*\boxtimes 3})(2) & \longrightarrow & H^0_{\mathrm{et}}(|\Delta_1^\circ[a,b,c]|, \mathcal{L}_r^{*\otimes 3})), \end{array}$$

in which the horizontal sequences are exact.

Proof. Recall from (27) that

$$\mathcal{L}_r^{*\boxtimes 3} = (\varpi_1^{r-1} \times \varpi_1^{r-1} \times \varpi_1^{r-1})_* \mathbb{Z}_p,$$

where

$$\varpi_1^{r-1} \times \varpi_1^{r-1} \times \varpi_1^{r-1} : X_r^3 \longrightarrow X_1^3$$

is defined as in (58). The vertical isomorphisms then follow from Shapiro's lemma and the definition of $\Delta_r^\circ[[a,b,c]]$ in (75). The horizontal sequence arises from the excision exact sequence in étale cohomology of [Ja, (3.6)] and [Mi, p. 108]. □

Lemma 3.12. *For all $[a,b,c] \in I_1$,*

$$H^0_{\mathrm{et}}(\bar{\Delta}_1[a,b,c], \mathcal{H}^{k_\circ,\ell_\circ,m_\circ}) = \mathbb{Z}_p(t).$$

Proof. The Clebsch-Gordan formula asserts that the space of tri-homogenous polynomials in $6 = 2+2+2$ variables of tridegree $(k_\circ, \ell_\circ, m_\circ)$ has a unique SL_2-invariant element, namely, the polynomial

$$P_{k_\circ,\ell_\circ,m_\circ}(x_1,y_1,x_2,y_2,x_3,y_3) = \left|\begin{array}{cc} x_2 & y_2 \\ x_3 & y_3 \end{array}\right|^{k_\circ{}'} \left|\begin{array}{cc} x_3 & y_3 \\ x_1 & y_1 \end{array}\right|^{\ell_\circ{}'} \left|\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \end{array}\right|^{m_\circ{}'},$$

where

$$k_\circ{}' = \frac{-k_\circ + \ell_\circ + m_\circ}{2}, \qquad \ell_\circ{}' = \frac{k_\circ - \ell_\circ + m_\circ}{2} \qquad m_\circ{}' = \frac{k_\circ + \ell_\circ - m_\circ}{2}.$$

Since the triplet of weights is balanced, it follows that $k_\circ{}', \ell_\circ{}', m_\circ{}' \geq 0$. From the Clebsch-Gordan formula it follows that $H^0_{\mathrm{et}}(\bar{\Delta}_1[a,b,c], \mathcal{H}^{k_\circ,\ell_\circ,m_\circ})$ is spanned by the global section whose stalk at a point $((A,P_1),(A,P_2),(A,P_3)) \in \Delta_1[a,b,c]$ as in (60) is given by

$$(X_2 \otimes Y_3 - Y_2 \otimes X_3)^{\otimes k_\circ{}'} \otimes (X_1 \otimes Y_3 - Y_1 \otimes X_3)^{\otimes \ell_\circ{}'} \otimes (X_1 \otimes Y_2 - Y_1 \otimes X_2)^{\otimes m_\circ{}'},$$

where (X_i, Y_i), $i = 1,2,3$, is a basis of the stalk of $\mathcal{H}$ at the point (A, P_i) in X_1. The Galois action is given by the t-th power of the cyclotomic character because the Weil pairing takes values in $\mathbb{Z}_p(1)$ and $k_\circ{}' + \ell_\circ{}' + m_\circ{}' = t$. □

Write $\mathrm{cl}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1[a,b,c]) \in H^0_{\mathrm{et}}(|\bar{\Delta}_1^\circ[a,b,c]|, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ})$ for the standard generator given by Lemma 3.12. Define

$$\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1[a,b,c]) \in H^1(\mathbb{Q}(\zeta_1), W_1^{k_\circ,\ell_\circ,m_\circ}) \tag{86}$$

to be the extension class constructed by pulling back by j and pushing forward by p in the exact sequence of the middle row of the following diagram:

$$\begin{tikzcd} & & \mathbb{Z}_p(t) \arrow[d, hook, "j"] \\ H^3_{\mathrm{et}}(\bar{X}_1^3, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ})(2) \arrow[r, hook] \arrow[d, "p"] & H^3_{\mathrm{et}}(\bar{X}_1^3 - \bar{\Delta}, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ})(2) \arrow[r, two heads] & H^0_{\mathrm{et}}(\bar{\Delta}, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ}) \\ W_1^{k_\circ,\ell_\circ,m_\circ}(t), & & \end{tikzcd} \tag{87}$$

where

- $\Delta = \Delta_1[a,b,c]$;
- the map j is the $G_{\mathbb{Q}(\zeta_1)}$-equivariant inclusion defined by $j(1) = \mathrm{cl}_{k_\circ,\ell_\circ,m_\circ}(\Delta)$;
- the surjectivity of the right-most horizontal row follows from the vanishing of the group $H^4_{\mathrm{et}}(\bar{X}_1^3, \mathcal{H}^{k_\circ,\ell_\circ,m_\circ})$, which in turn is a consequence of the Künneth formula and the vanishing of the terms $H^2_{\mathrm{et}}(\bar{X}_1, \mathcal{H}^{k_\circ})$ when $k_\circ > 0$ (cf. [BDP1, Lemmas 2.1, 2.2]).

In particular the image of j lies in the image of the right-most horizontal row and this holds regardless whether the cycle is null-homologous or not. The reader may compare this construction with (76), where the cycle $\Delta_r^\circ[[a,b,c]]$ is null-homologous and this property was crucially exploited.

Theorem 3.13. *Set* $\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1^\circ[a,b,c]) = \theta_q \mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1[a,b,c])$. *Then the identity*

$$\kappa_1(k_\circ,\ell_\circ,m_\circ)[a,b,c] = \mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1^\circ[a,b,c])$$

holds in $H^1(\mathbb{Q}(\zeta_1), W_1^{k_\circ,\ell_\circ,m_\circ})$.

Proof. Set $\Delta := \Delta_1^\circ[a,b,c]$ in order to alleviate notations. Thanks to Lemma 3.11, the diagram in (76) used to construct the extension E_r realising the class $\boldsymbol{\kappa}_r[a,b,c]$ is *the same* as the diagram

(88)
$$\begin{tikzcd}
 & & & \Lambda(G_r)(\frac{-1}{2}) \arrow[d, hook, "j"] \\
0 \arrow[r] & H^3_{\mathrm{et}}(\bar{X}_1^3, \mathcal{L}_r^{*\boxtimes 3})(2) \arrow[r] \arrow[d, "p"] & H^3_{\mathrm{et}}(\bar{X}_1^3 - |\bar{\Delta}|, \mathcal{L}_r^{*\boxtimes 3})(2) \arrow[r] & H^0_{\mathrm{et}}(|\bar{\Delta}|, \mathcal{L}_r^{*\otimes 3}) \\
 & H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{L}_r^*)^{\otimes 3}(2). & &
\end{tikzcd}$$

Let

$$\nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_r) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$$

be the algebra homomorphism sending the group like element $\langle d_1, d_2, d_3\rangle$ to $d_1^{k_\circ} d_2^{\ell_\circ} d_3^{m_\circ}$, and observe that the moment maps of (30) allow us to identify

$$\mathcal{L}_r^{*\boxtimes 3} \otimes_{\nu_{k_\circ,\ell_\circ,m_\circ}} (\mathbb{Z}/p^r\mathbb{Z}) = \mathcal{H}_r^{k_\circ,\ell_\circ,m_\circ}.$$

Tensoring (88) over $\Lambda(G_r)$ with $\mathbb{Z}/p^r\mathbb{Z}$ via the map $\nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_r) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$, yields the specialised diagram which coincides exactly with the mod p^r reduction of (87), with $\Delta = \Delta_1^\circ[a,b,c]$. The result follows by passing to the limit with r. □

Corollary 3.14. *Let*

(89)
$$\Delta_1^\circ(\omega_1,\omega_2,\omega_3;\delta) := \frac{p^3}{(p-1)^3} \cdot \sum_{[a,b,c]\in I_1} \delta^{-1}(abc)\omega_1(a)\omega_2(b)\omega_3(c)\Delta_1^\circ[a,b,c].$$

Then

$$\mathrm{sp}^*_{k_\circ,\ell_\circ,m_\circ}(\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta) = \mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1^\circ(\omega_1,\omega_2,\omega_3;\delta)).$$

Proof. This follows directly from the definitions. □

3.5. **Cristalline specialisations.** Let $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ be three arbitrary primitive, residually irreducible p-adic Hida families of tame levels M_f, M_g, M_h and tame characters χ_f, χ_g, χ_h, respectively, with associated weight space $\mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}}$. Assume $\chi_f\chi_g\chi_h = 1$ and set $M = \mathrm{lcm}(M_f, M_g, M_h)$. Let $(x, y, z) \in \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}}$ be a point lying above a classical triple $(\nu_{k_\circ,\epsilon_1}, \nu_{\ell_\circ,\epsilon_2}, \nu_{m_\circ,\epsilon_3}) \in \mathcal{W}^3$ of weight space. As in Definition 1.2, the point (x, y, z) is said to be *tamely ramified* if the three characters ϵ_1, ϵ_2 and ϵ_3 are tamely ramified, i.e., factor through the quotient $(\mathbb{Z}/p\mathbb{Z})^\times$ of $\mathbb{Z}_p^\times$, and is said to be *cristalline* if $\epsilon_1\omega^{-k_\circ} = \epsilon_2\omega^{-\ell_\circ} = \epsilon_3\omega^{-m_\circ} = 1$.

Fix such a cristalline point (x, y, z) of balanced weight $(k, \ell, m) = (k_\circ + 2, \ell_\circ + 2, m_\circ + 2)$, and let $(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ be the specialisations of $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at (x, y, z). The ordinariness hypothesis implies that, for all but finitely many exceptions, these eigenforms are the p-stabilisations of newforms of level dividing M, denoted f, g and h respectively:

$$\mathbf{f}_x(q) = f(q) - \beta_f f(q^p), \qquad \mathbf{g}_y = g(q) - \beta_g g(q^p), \qquad \mathbf{h}_z(q) = h(q) - \beta_h h(q^p).$$

Since the point (x, y, z) is fixed throughout this section, the dependency of (f, g, h) on (x, y, z) has been suppressed from the notations, and we also write $(f_\alpha, g_\alpha, h_\alpha) := (\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ for the ordinary p-stabilisations of f, g and h.

Recall the quotient X_{01} of X_1, having $\Gamma_0(p)$-level structure at p, and the projection map $\mu : X_1 \longrightarrow X_{01}$ introduced in (25). By an abuse of notation, the symbol $\mathcal{H}^{k_\circ}$ is also used to denote the étale sheaves appearing in (14) over any quotient of X_1, such as X_{01}. Let

$$\begin{aligned} W_1 &:= H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{H}^{k_\circ}) \otimes H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{H}^{\ell_\circ}) \otimes H^1_{\mathrm{et}}(\bar{X}_1, \mathcal{H}^{m_\circ})(2-t), \\ W_{01} &:= H^1_{\mathrm{et}}(\bar{X}_{01}, \mathcal{H}^{k_\circ}) \otimes H^1_{\mathrm{et}}(\bar{X}_{01}, \mathcal{H}^{\ell_\circ}) \otimes H^1_{\mathrm{et}}(\bar{X}_{01}, \mathcal{H}^{m_\circ})(2-t), \end{aligned}$$

be the Galois representations arising from the cohomology of X_1 and X_{01} with values in these sheaves. They are endowed with a natural action of the triple tensor product of the Hecke algebras of weight $k_\circ$, $\ell_\circ$, $m_\circ$ and level Mp.

Let $W_1[f_\alpha, g_\alpha, h_\alpha]$ denote the $(f_\alpha, g_\alpha, h_\alpha)$-isotypic component of W_1 on which the Hecke operators act with the same eigenvalues as on $f_\alpha \otimes g_\alpha \otimes h_\alpha$. Let $\pi_{f_\alpha,g_\alpha,h_\alpha} : W_1 \to W_1[f_\alpha, g_\alpha, h_\alpha]$ denote the associated projection. Use similar notations for W_{01}.

Recall the family

$$\boldsymbol{\kappa}_\infty(\epsilon_1\omega^{-k_\circ}, \epsilon_2\omega^{-\ell_\circ}, \epsilon_3\omega^{-m_\circ}; 1) = \boldsymbol{\kappa}_\infty(1, 1, 1; 1) \tag{90}$$

that was introduced in (82). By Lemma 3.9, this class lies in $H^1(\mathbb{Q}, \mathbb{H}^{111}(X_\infty^*))$.

Recall the choice of auxiliary prime q made in the definition of the modified diagonal cycle (69). We assume now that q is chosen so that $C_q := (a_q(f) - q - 1)(a_q(g) - q - 1)(a_q(h) - q - 1)$ is a p-adic unit. Note that this is possible because the Galois representations $\varrho_{\mathbf{f}}$, $\varrho_{\mathbf{g}}$ and $\varrho_{\mathbf{h}}$ were assumed to be residually irreducible and hence f, g and h are non-Eisenstein mod p. Let

$$\kappa_1(f_\alpha, g_\alpha, h_\alpha) := \frac{1}{C_q} \cdot \pi_{f_\alpha,g_\alpha,h_\alpha} \mathrm{sp}^*_{x,y,z} \boldsymbol{\kappa}_\infty(1, 1, 1; 1)) \in H^1(\mathbb{Q}, W_1[f_\alpha, g_\alpha, h_\alpha]) \tag{91}$$

be the specialisation at the cristalline point (x, y, z) of (90), after projecting it to the $(f_\alpha, g_\alpha, h_\alpha)$-isotypic component of W_1 via $\pi_{f_\alpha,g_\alpha,h_\alpha}$. We normalize the class by multiplying it by the above constant in order to remove the dependency on the choice of q.

The main goal of this section is to relate this class to the generalised Gross-Schoen diagonal cycles that were studied in [DR1], arising from cycles in Kuga-Sato varieties which are fibered over X^3 and have *good reduction* at p.

The fact that (x, y, z) is a cristalline point implies that the diamond operators in $\mathrm{Gal}\,(X_1/X_{01})$ act trivially on the $(f_\alpha, g_\alpha, h_\alpha)$-eigencomponents, and hence the Hecke-equivariant projection $\mu^3_* : W_1 \longrightarrow W_{01}$ induces an isomorphism

$$\mu^3_* : W_1[f_\alpha, g_\alpha, h_\alpha] \longrightarrow W_{01}[f_\alpha, g_\alpha, h_\alpha].$$

Our first aim is to give a geometric description of the class

$$\kappa_{01}(f_\alpha, g_\alpha, h_\alpha) := \mu_*^3 \kappa_1(f_\alpha, g_\alpha, h_\alpha)$$

in terms of appropriate algebraic cycles. To this end, recall the cycles $\Delta_1[a,b,c] \in \mathrm{CH}^2(X_1^3)$ introduced before, and let $p^* := \pm p$ be such that $\mathbb{Q}(\sqrt{p^*})$ is the quadratic subfield of $\mathbb{Q}(\zeta_1)$.

Lemma 3.15. *The cycle $\mu_*^3\Delta_1[a,b,c]$ depends only on the quadratic residue symbol $(\frac{abc}{p})$ attached to $abc \in (\mathbb{Z}/p\mathbb{Z})^\times$. The cycles*

$$\begin{aligned} \Delta_{01}^+ &:= \mu_*^3\Delta_1[a,b,c] \quad \textit{for any } a,b,c \textit{ with } \left(\frac{abc}{p}\right) = 1, \\ \Delta_{01}^- &:= \mu_*^3\Delta_1[a,b,c] \quad \textit{for any } a,b,c \textit{ with } \left(\frac{abc}{p}\right) = -1, \end{aligned}$$

belong to $\mathrm{CH}^2(X_{01}^3/\mathbb{Q}(\sqrt{p^*}))$ and are interchanged by the non-trivial automorphism of $\mathbb{Q}(\sqrt{p^*})$.

Proof. Arguing as in Lemma 3.2,

$$\langle d_1, d_2, d_3\rangle\Delta_1[a,b,c] = \Delta_1[d_2d_3a, d_1d_3b, d_1d_2c], \qquad \text{for all } (d_1,d_2,d_3) \in I_1 = (\mathbb{Z}/p\mathbb{Z})^{\times 3}.$$

The orbit of the triple $[a,b,c]$ under the action of I_1 is precisely the set of triples $[a',b',c']$ for which $(\frac{a'b'c'}{p}) = (\frac{abc}{p})$. Since X_{01} is the quotient of X_1 by the group I_1, it follows that $\mu_*^3\Delta_1[a,b,c]$ depends only on this quadratic residue symbol, and hence that the classes Δ_{01}^+ and Δ_{01}^- in the statement of Lemma 3.15 are well-defined. Furthermore, Lemma 3.6 implies that, for all $m \in (\mathbb{Z}/p\mathbb{Z})^\times$, the Galois automorphism σ_m fixes Δ_{01}^+ and Δ_{01}^- if m is a square modulo p, and interchanges these two cycle classes otherwise. It follows that they are invariant under the Galois group $\mathrm{Gal}\,(\mathbb{Q}(\zeta_1)/\mathbb{Q}(\sqrt{p^*}))$ and hence descend to a pair of conjugate cycles $\Delta_{01}^\pm$ defined over $\mathbb{Q}(\sqrt{p^*})$, as claimed. □

It follows from this lemma that the algebraic cycle

$$\Delta_{01} := \Delta_{01}^+ + \Delta_{01}^- \in \mathrm{CH}^2(X_{01}^3/\mathbb{Q}). \tag{92}$$

is defined over $\mathbb{Q}$. To describe it concretely, note that a triple (C_1, C_2, C_3) of distinct cyclic subgroups of order p in an elliptic curve A admits a somewhat subtle discrete invariant in $(\mu_p^{\otimes 3} - \{1\})$ modulo the action of $(\mathbb{Z}/p\mathbb{Z})^{\times 2}$, denoted $o(C_1,C_2,C_3)$ and called the *orientation* of (C_1,C_2,C_3). This orientation is defined by choosing generators P_1, P_2, P_3 of C_1, C_2 and C_3 respectively and setting

$$o(C_1,C_2,C_3) := \langle P_2, P_3\rangle \otimes \langle P_3, P_1\rangle \otimes \langle P_1, P_2\rangle \in \mu_p^{\otimes 3} - \{1\}.$$

It is easy to check that the value of $o(C_1,C_2,C_3)$ in $\mu_p^{\otimes 3} - \{1\}$ only depends on the choices of generators P_1, P_2 and P_3, up to multiplication by a *non-zero square* in $(\mathbb{Z}/p\mathbb{Z})^\times$. In view of (60), we then have

$$\begin{aligned} \Delta_{01}^+ &= \left\{((A,C_1),(A,C_2),(A,C_3)) \quad \text{with } o(C_1,C_2,C_3) = a\zeta_1^{\otimes 3}, \quad a \in (\mathbb{Z}/p\mathbb{Z})^{\times 2}\right\}, \\ \Delta_{01}^- &= \left\{((A,C_1),(A,C_2),(A,C_3)) \quad \text{with } o(C_1,C_2,C_3) = a\zeta_1^{\otimes 3}, \quad a \notin (\mathbb{Z}/p\mathbb{Z})^{\times 2}\right\}, \\ \Delta_{01} &= \{((A,C_1),(A,C_2),(A,C_3)) \quad \text{with} \quad C_1 \neq C_2 \neq C_3\}. \qquad (93) \end{aligned}$$

Recall the natural projections

$$\pi_1, \pi_2 : X_{01} \longrightarrow X, \qquad \varpi_1, \varpi_2 : X_1 \longrightarrow X$$

to the curve $X = X_0(M)$ of prime to p level, and set

$$W_0 \ := \ H^1_{\mathrm{et}}(\bar{X}_0, \mathcal{H}^{k_\circ}) \otimes H^1_{\mathrm{et}}(\bar{X}_0, \mathcal{H}^{\ell_\circ}) \otimes H^1_{\mathrm{et}}(\bar{X}_0, \mathcal{H}^{m_\circ})(2-t),$$

The Galois representation W_0 is endowed with a natural action of the triple tensor product of the Hecke algebras of weight $k_\circ$, $\ell_\circ$, $m_\circ$ and level M. Let $W_0[f,g,h]$ denote the (f,g,h)-

isotypic component of W_0, on which the Hecke operators act with the same eigenvalues as on $f \otimes g \otimes h$. Note that the U_p^* operator does not act naturally on W_0 and hence one cannot speak of the $(f_\alpha, g_\alpha, h_\alpha)$-eigenspace of this Hecke module. One can, however, denote by $W_1[f,g,h]$ and $W_{01}[f,g,h]$ the (f,g,h)-isotypic component of these Galois representations, in which the action of the U_p^* operators on the three factors are not taken into account. Thus, $W_{01}[f_\alpha, g_\alpha, h_\alpha]$ is the image of $W_{01}[f,g,h]$ under the ordinary projection, and likewise for W_1. In other words, denoting by $\pi_{f,g,h}$ the projection to the (f,g,h)-isotypic component on any of these modules, one has

$$\pi_{f_\alpha, g_\alpha, h_\alpha} = e^* \pi_{f,g,h}$$

whenever the left-hand projection is defined.

The projection maps

$$(\pi_1, \pi_1, \pi_1) : X_{01}^3 \longrightarrow X^3, \quad (\varpi_1, \varpi_1, \varpi_1) : X_1^3 \longrightarrow X^3$$

induces push-forward maps

$$(\pi_1, \pi_1, \pi_1)_* : W_{01}[f_\alpha, g_\alpha, h_\alpha] \longrightarrow W_0[f,g,h], \qquad (\varpi_1, \varpi_1, \varpi_1)_* : W_1[f_\alpha, g_\alpha, h_\alpha] \longrightarrow W_0[f,g,h]$$

on cohomology, as well as maps on the associated Galois cohomology groups.

Our goal is now to relate the class

$$(\varpi_1, \varpi_1, \varpi_1)_*(\kappa_1(f_\alpha, g_\alpha, h_\alpha)) = (\pi_1, \pi_1, \pi_1)_*(\kappa_{01}(f_\alpha, g_\alpha, h_\alpha)) \tag{94}$$

to those arising from the diagonal cycles on the curve $X_0 = X$, whose level is prime to p.

To do this, it is key to understand how the maps π_{1*} and $(\pi_1, \pi_1, \pi_1)_*$ interact with the Hecke operators, especially with the ordinary and anti-ordinary projectors e and e^*, which do not act naturally on the target of π_{1*}. Consider the map

$$(\pi_1, \pi_2) : W_{01}^{k_\circ} := H^1_{\mathrm{et}}(\bar{X}_{01}, \mathcal{H}^{k_\circ}) \longrightarrow W_0^{k_\circ} := H^1_{\mathrm{et}}(\bar{X}_0, \mathcal{H}^{k_\circ}).$$

It is compatible in the obvious way with the good Hecke operators arising from primes $\ell \nmid Mp$, and therefore induces a map

$$(\pi_1, \pi_2) : W_{01}^{k_\circ}[f] \longrightarrow W_0^{k_\circ}[f] \oplus W_0^{k_\circ}[f] \tag{95}$$

on the f-isotypic components for this Hecke action. As before, note that $W_{01}^{k_\circ}[f]$ is a priori larger than $W_{01}^{k_\circ}[f_\alpha]$, which is its ordinary quotient.

Let $\xi_f := \chi_f(p)p^{k-1}$ be the determinant of the frobenius at p acting on the two-dimensional Galois representation attached to f, and likewise for g and h.

Lemma 3.16. *For the map* (π_1, π_2) *as in* (95),

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \circ U_p = \begin{pmatrix} a_p(f) & -1 \\ \xi_f & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \qquad \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \circ U_p^* = \begin{pmatrix} 0 & p \\ -\xi_f p^{-1} & a_p(f) \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}.$$

Proof. The definitions π_1 and π_2 imply that, viewing U_p and U_p^* (resp. T_p) as correspondences on a Kuga-Sato variety fibered over X_{01} (resp. over X_0), we have

$$\begin{aligned} \pi_1 U_p &= T_p \pi_1 - \pi_2, & \pi_1 U_p^* &= p\pi_2 \\ \pi_2 U_p &= p[p]\pi_1 & \pi_2 U_p^* &= -[p]\pi_1 + T_p \pi_2, \end{aligned}$$

where $[p]$ is the correspondence induced by the multiplication by p on the fibers and on the prime-to-p part of the level structure. The result follows by passing to the f-isotypic parts, using the fact that $[p]$ induces multiplication by $\xi_f p^{-1}$ on this isotypic part. $\square$

For the next calculations, it shall be notationally convenient to introduce the notations

$$\delta_f = \alpha_f - \beta_f, \qquad \delta_g = \alpha_g - \beta_g, \qquad \delta_h = \alpha_h - \beta_h, \qquad \delta_{fgh} = \delta_f \delta_g \delta_h.$$

Lemma 3.17. *For (π_1, π_2) as in Lemma 3.16,*

$$\pi_1 \circ e = \frac{\alpha_f \pi_1 - \pi_2}{\delta_f}, \qquad \pi_2 \circ e = \frac{\xi_f \pi_1 - \beta_f \pi_2}{\delta_f} = \beta_f \cdot (\pi_1 \circ e),$$
$$\pi_1 \circ e^* = \frac{-\beta_f \pi_1 + p\pi_2}{\delta_f}, \qquad \pi_2 \circ e^* = \frac{-\xi_f p^{-1} \pi_1 + \alpha_f \pi_2}{\delta_f} = p\alpha_f^{-1} \cdot (\pi_1 \circ e^*).$$

Proof. The matrix identities

$$\begin{pmatrix} a_p(f) & -1 \\ \xi_f & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix} \times \begin{pmatrix} \alpha_f & 0 \\ 0 & \beta_f \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix}^{-1},$$
$$\begin{pmatrix} 0 & p \\ -\xi_f p^{-1} & a_p(f) \end{pmatrix} = \begin{pmatrix} \beta_f & \alpha_f \\ \xi_f p^{-1} & \xi_f p^{-1} \end{pmatrix} \times \begin{pmatrix} \alpha_f & 0 \\ 0 & \beta_f \end{pmatrix} \times \begin{pmatrix} \beta_f & \alpha_f \\ \xi_f p^{-1} & \xi_f(p) p^{-1} \end{pmatrix}^{-1},$$

show that

$$\lim \begin{pmatrix} a_p(f) & -1 \\ \xi_f & 0 \end{pmatrix}^{n!} = \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix}^{-1} = \delta_f^{-1} \begin{pmatrix} \alpha_f & -1 \\ \xi_f & -\beta_f \end{pmatrix},$$
$$\lim \begin{pmatrix} 0 & p \\ -\xi_f p^{-1} & a_p(f) \end{pmatrix}^{n!} = \delta_f^{-1} \begin{pmatrix} -\beta_f & p \\ -\xi_f p^{-1} & \alpha_f \end{pmatrix},$$

and the result now follows from Lemma 3.16. □

Lemma 3.18. *Let $\kappa \in H^1(\mathbb{Q}, W_{01}[f,g,h])$ be any cohomology class with values in the (f,g,h)-isotypic subspace of W_{01}, and let $e, e^* : H^1(\mathbb{Q}, W_{01}[fgh]) \longrightarrow H^1(\mathbb{Q}, W_{01}[f_\alpha, g_\alpha, h_\alpha])$ denote the ordinary and anti-ordinary projections. Then*

$$\begin{aligned} (\pi_1,\pi_1,\pi_1)_*(e\kappa) = \delta_{fgh}^{-1} \times \Big\{ & \alpha_f\alpha_g\alpha_h(\pi_1,\pi_1,\pi_1)_* \\ & -\alpha_g\alpha_h(\pi_2,\pi_1,\pi_1)_* - \alpha_f\alpha_h(\pi_1,\pi_2,\pi_1)_* - \alpha_f\alpha_g(\pi_1,\pi_1,\pi_2)_* \\ & +\alpha_f(\pi_1,\pi_2,\pi_2)_* + \alpha_g(\pi_2,\pi_1,\pi_2)_* + \alpha_h \cdot (\pi_2,\pi_2,\pi_1)_* \\ & -(\pi_2,\pi_2,\pi_2)_* \Big\}(\kappa). \end{aligned}$$

$$\begin{aligned} (\pi_1,\pi_1,\pi_1)_*(e^*\kappa) = \delta_{fgh}^{-1} \times \Big\{ & -\beta_f\beta_g\beta_h(\pi_1,\pi_1,\pi_1)_* \\ & +p\beta_g\beta_h(\pi_2,\pi_1,\pi_1)_* + p\beta_f\beta_h(\pi_1,\pi_2,\pi_1)_* + p\beta_f\beta_g(\pi_1,\pi_1,\pi_2)_* \\ & -p^2\beta_f(\pi_1,\pi_2,\pi_2)_* - p^2\beta_g(\pi_2,\pi_1,\pi_2)_* - p^2\beta_h(\pi_2,\pi_2,\pi_1)_* \\ & +p^3(\pi_2,\pi_2,\pi_2)_* \Big\}(\kappa), \end{aligned}$$

where we recall that $\delta_{fgh} := ((\alpha_f - \beta_f)(\alpha_g - \beta_g)(\alpha_h - \beta_h))$.

Proof. This follows directly from Lemma 3.17. □

Recall the notations

$$k_\circ := k-2, \qquad \ell_\circ = \ell - 2, \qquad m_\circ := m-2, \qquad r := (k_\circ + \ell_\circ + m_\circ)/2.$$

Let $\mathcal{A}$ denote the Kuga-Sato variety over X as introduced in 1.2. In [DR1, Definitions 3.1,3.2 and 3.3], a generalized diagonal cycle

$$\Delta^{k_\circ,\ell_\circ,m_\circ} = \Delta_0^{k_\circ,\ell_\circ,m_\circ} \in \mathrm{CH}^{r+2}(\mathcal{A}^{k_\circ} \times \mathcal{A}^{\ell_\circ} \times \mathcal{A}^{m_\circ}, \mathbb{Q})$$

is associated to the triple $(k_\circ, \ell_\circ, m_\circ)$.

When $k_\circ, \ell_\circ, m_\circ > 0$, $\Delta^{k_\circ,\ell_\circ,m_\circ}$ is obtained by choosing subsets A, B and C of the set $\{1,\ldots,r\}$ which satisfy:

$$\#A = k_\circ, \qquad \#B = \ell_\circ, \qquad \#C = m_\circ, \qquad A \cap B \cap C = \emptyset,$$

$$\#(B\cap C) = r - k_\circ, \qquad \#(A\cap C) = r-\ell_\circ, \qquad \#(A\cap B) = r - m_\circ.$$

The cycle $\Delta^{k_\circ,\ell_\circ,m_\circ}$ is defined as the image of the embedding $\mathcal{A}^r$ into $\mathcal{A}^{k_\circ}\times\mathcal{A}^{\ell_\circ}\times\mathcal{A}^{m_\circ}$ given by sending $(E,(P_1,\dots,P_r))$ to $((E,P_A),(E,P_B),(E,P_C))$, where for instance P_A is a shorthand for the $k_\circ$-tuple of points P_j with $j\in A$.

Let also $\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}\in \mathrm{CH}^{r+2}(\mathcal{A}^{k_\circ}\times\mathcal{A}^{\ell_\circ}\times\mathcal{A}^{m_\circ})$ denote the generalised diagonal cycle in the product of the three Kuga-Sato varieties of weights (k,ℓ,m) fibered over X_{01}, defined in a similar way as in (93) and along the same lines as recalled above.

More precisely, $\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ is defined as the schematic closure in $\mathcal{A}^{k_\circ}\times\mathcal{A}^{\ell_\circ}\times\mathcal{A}^{m_\circ}$ of the set of tuples $((E,C_1,P_A),(E,C_2,P_B),(E,C_3,P_C))$ where P_A,P_B,P_C are as above, and C_1,C_2,C_3 is a triple of pairwise distinct subgroups of order p in the elliptic curve E.

Since the triple $(k_\circ,\ell_\circ,m_\circ)$ is fixed throughout this section, in order to alleviate notations in the statements below we shall simply denote $\Delta^\sharp$ and $\Delta^\sharp_{01}$ for $\Delta^{k_\circ,\ell_\circ,m_\circ}$ and $\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ respectively.

Lemma 3.19. *The following identities hold in* $\mathrm{CH}^{r+2}(\mathcal{A}^{k_\circ}\times\mathcal{A}^{\ell_\circ}\times\mathcal{A}^{m_\circ})$*:*

$$\begin{aligned}
(\pi_1,\pi_1,\pi_1)_*(\Delta^\sharp_{01}) &= (p+1)p(p-1)(\Delta^\sharp),\\
(\pi_2,\pi_1,\pi_1)_*(\Delta^\sharp_{01}) &= p(p-1)\times(T_p,1,1)(\Delta^\sharp),\\
(\pi_1,\pi_2,\pi_1)_*(\Delta^\sharp_{01}) &= p(p-1)\times(1,T_p,1)(\Delta^\sharp),\\
(\pi_1,\pi_1,\pi_2)_*(\Delta^\sharp_{01}) &= p(p-1)\times(1,1,T_p)(\Delta^\sharp),\\
(\pi_1,\pi_2,\pi_2)_*(\Delta^\sharp_{01}) &= (p-1)\times((1,T_p,T_p)(\Delta^\sharp)-p^{r-k_\circ}D_1)\\
(\pi_2,\pi_1,\pi_2)_*(\Delta^\sharp_{01}) &= (p-1)\times((T_p,1,T_p)(\Delta^\sharp)-p^{r-\ell_\circ}D_2)\\
(\pi_2,\pi_2,\pi_1)_*(\Delta^\sharp_{01}) &= (p-1)\times((T_p,T_p,1)(\Delta^\sharp)-p^{r-m_\circ}D_3)\\
(\pi_2,\pi_2,\pi_2)_*(\Delta^\sharp_{01}) &= (T_p,T_p,T_p)(\Delta^\sharp)-p^{r-k_\circ}E_1-p^{r-\ell_\circ}E_2-p^{r-m_\circ}E_3-p^r(p+1)\Delta^\sharp,
\end{aligned}$$

where the cycles D_1, D_2 and D_3 satisfy

$$([p],1,1)_*(D_1) = p^{k_\circ}(T_p,1,1)_*(\Delta^\sharp),\qquad (1,[p],1)_*(D_2)=p^{\ell_\circ}(1,T_p,1)_*(\Delta^\sharp),$$
$$(1,1,[p])_*(D_3)=p^{m_\circ}(1,1,T_p)(\Delta^\sharp),$$

and the cycles E_1, E_2 and E_3 satisfy

$$([p],1,1)_*(E_1) = p^{k_\circ}(T_{p^2},1,1)(\Delta^\sharp),\qquad (1,[p],1)_*(E_2)=p^{\ell_\circ}(1,T_{p^2},1)(\Delta^\sharp),$$
$$(1,1,[p])_*(E_3)=p^{m_\circ}(1,1,T_{p^2})(\Delta^\sharp),$$

and $T_{p^2}:=T_p^2-(p+1)[p]$*.*

Proof. The first four identities are straightforward: the map $\pi_1\times\pi_1\times\pi_1$ induces a finite map from $\Delta^\sharp_{01}$ to $\Delta^\sharp$ of degree $(p+1)p(p-1)$, which is the number of possible choices of an ordered triple of distinct subgroups of order p on an elliptic curve, and likewise $\pi_2\times\pi_1\times\pi_1$ induces a map of degree $p(p-1)$ from $\Delta^\sharp_{01}$ to $(T_p,1,1)\Delta^\sharp$. The remaining identities follow from combinatorial reasonings based on the explicit description of the cycles $\Delta^\sharp_{01}$ and $\Delta^\sharp$. For the 5th identity, observe that $(\pi_1,\pi_2,\pi_2)_*$ induces a degree $(p-1)$ map from $\Delta^\sharp_{01}$ to the variety X parametrising triples $((E,P_A),(E',P'_B),(E'',P''_C))$ for which there are distinct cyclic p-isogenies $\varphi':E\longrightarrow E'$ and $\varphi':E\longrightarrow E''$, the system of points $P'_B\subset E'$ and $P''_C\subset E''$ indexed by the sets B and C satisfy

$$\varphi'(P_{A\cap B}) = P'_{A\cap B},\quad \varphi''(P_{A\cap C}) = P''_{A\cap C},$$

and for which there exists points $Q_{B\cap C}\subset E$ indexed by $B\cap C$ satisfying

$$\varphi'(Q_{B\cap C}) = P'_{B\cap C},\qquad \varphi''(Q_{B\cap C}) = P''_{B\cap C}.$$

On the other hand, $(1,T_p,T_p)$ parametrises triples of the same type, in which E' and E'' need not be distinct. It follows that

$$(1,T_p,T_p)(\Delta^\sharp) = X + p^{r-k_\circ} D_1, \tag{96}$$

where the closed points of D_1 correspond to triples of the form $((E,P_A),(E',P'_B),(E',P'_C))$ for which there is a cyclic p-isogeny $\varphi' : E \longrightarrow E'$ satisfying

$$\varphi'(P_{A\cap B}) = P'_{A\cap B}, \qquad \varphi'(P_{A\cap C}) = P'_{A\cap C}.$$

The coefficient of $p^{r-k_\circ}$ in (96) arises because each closed point of D_1 comes from $p^{\#(B\cap C)}$ distinct closed points on $(1,T_p,T_p)(\Delta^\sharp)$, obtained by translating the points $P_j \in P_{B\cap C}$ with $j \in B\cap C$ by arbitrary elements of $\ker(\varphi)$. The fifth identity now follows after noting that the map $([p],1,1])$ induces a map of degree $p^{k_\circ}$ from D_1 to $(T_p,1,1)_*\Delta^\sharp$. The 6th and 7th identity can be treated with an identical reasoning by interchanging the three factors in $W^{k_\circ} \times W^{\ell_\circ} \times W^{m_\circ}$. As for the last identity, the map (π_2,π_2,π_2) induces a map of degree 1 to the variety Y consisting of triples (E',E'',E''') of elliptic curves which are p-isogenous to a common elliptic curve E and distinct. But it is not hard to see that

$$(T_p,T_p,T_p)(\Delta^\sharp) = Y + p^{r-k_\circ} E_1 + p^{r-\ell_\circ} E_2 + p^{r-m_\circ} E_3 + p^r(p+1)\Delta^\sharp$$

where the additional terms on the right hand side account for triples (E',E'',E''') where $E' \neq E'' = E'''$, where $E'' \neq E' = E'''$, where $E''' \neq E' = E''$, and where $E' = E'' = E'''$ respectively. $\square$

Assume for the remainder of the section that $k_\circ, \ell_\circ, m_\circ > 0$. Recall the projectors $\epsilon_{k_\circ}$ of (15). It was shown in [DR1, §3.1] that $(\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta^{k_\circ,\ell_\circ,m_\circ}$ is a null-homologous cycle and we may define

$$\kappa(f,g,h) := \pi_{f,g,h}\,\mathrm{AJ}_{\mathrm{et}}((\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta^{k_\circ,\ell_\circ,m_\circ}) \in H^1(\mathbb{Q}, W_0[f,g,h]) \tag{97}$$

as the image of this cycle under the p-adic étale Abel-Jacobi map, followed by the natural projection from $H^{2c-1}_{\mathrm{et}}(\bar{\mathcal{A}}^{k_\circ} \times \bar{\mathcal{A}}^{\ell_\circ} \times \bar{\mathcal{A}}^{m_\circ}, \mathbb{Q}_p(c))$ to $W_0^{k_\circ,\ell_\circ,m_\circ}$ induced by the Künneth decomposition and the projection $\pi_{f,g,h}$.

It follows from [DR1, (66)], (15) and the vanishing of the terms $H^i_{\mathrm{et}}(\bar{X}_1, \mathcal{H}^{k_\circ})$ for $i \neq 1$ when $k_\circ > 0$, that the class $\kappa(f,g,h)$ is realized by the (f,g,h)-isotypic component of the same extension class as in (87), after replacing X_1 by the curve $X = X_0$ and $\Delta = \Delta^{0,0,0}$ is taken to be the usual diagonal cycle in X^3. In the notations of (86), this amounts to

$$\kappa(f,g,h) = \pi_{f,g,h}\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta). \tag{98}$$

Similar statements holds over the curve X_{01}. Namely, we also have the following:

Proposition 3.20. *The cycle $(\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ is null-homologous and the following equality of classes holds in $H^1(\mathbb{Q}, W_{01}[f_\alpha,g_\alpha,h_\alpha])$:*

$$\kappa_{01}(f_\alpha,g_\alpha,h_\alpha) = p^3 \cdot \pi_{f_\alpha,g_\alpha,h_\alpha}\,\mathrm{AJ}_{\mathrm{et}}((\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}). \tag{99}$$

Proof. Corollary 3.14 together with (91) imply that

$$\kappa_1(f_\alpha,g_\alpha,h_\alpha) = \frac{1}{C_q} \cdot \pi_{f_\alpha,g_\alpha,h_\alpha}\,\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1^\circ(1,1,1;\delta)),$$

in which $\delta = 1$ is the trivial character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Since μ^3 induces a finite map of degree $(p-1)^3$ from the support of $\Delta_1(1,1,1;\delta)$ to Δ_{01}, it follows from the convention adopted in (89) that

$$\kappa_{01}(f_\alpha,g_\alpha,h_\alpha) := \mu^3_*\,\kappa_1(f_\alpha,g_\alpha,h_\alpha) = \frac{p^3}{C_q} \cdot \pi_{f_\alpha,g_\alpha,h_\alpha}\,\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_{01}^\circ),$$

where $\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_{01}^\circ)$ is defined to be the class realized by the same extension class as in (87), after replacing X_1 by the curve X_{01} and replacing Δ by the cycle Δ_{01}° arising from (93). Since $\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ is fibered over Δ_{01}, the same argument as in (98) then shows that

$$\mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_{01}) = \mathrm{AJ}_{\mathrm{et}}((\epsilon_{k_\circ},\epsilon_{\ell_\circ},\epsilon_{m_\circ})\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}).$$

Since $\pi_{f_\alpha,g_\alpha,h_\alpha}(\Delta_{01}) = \frac{1}{C_q}\pi_{f_\alpha,g_\alpha,h_\alpha}(\Delta_{01}^\circ)$, the proposition follows. □

Theorem 3.21. *With notations as before, letting $c = r+2$, we have*

$$(\varpi_1,\varpi_1,\varpi_1)_*\ \kappa_1(f_\alpha,g_\alpha,h_\alpha) = \frac{\mathcal{E}^{\mathrm{bal}}(f_\alpha,g_\alpha,h_\alpha)}{\mathcal{E}(f_\alpha)\mathcal{E}(g_\alpha)\mathcal{E}(h_\alpha)} \times \kappa(f,g,h),$$

where

$$\mathcal{E}^{\mathrm{bal}}(f_\alpha,g_\alpha,h_\alpha) = (1-\alpha_f\beta_g\beta_h p^{-c})(1-\beta_f\alpha_g\beta_h p^{-c})(1-\beta_f\beta_g\alpha_h p^{-c})(1-\beta_f\beta_g\beta_h p^{-c}),$$

and

$$\mathcal{E}(f_\alpha) = 1-\chi_f^{-1}(p)\beta_f^2 p^{1-k}, \qquad \mathcal{E}(g_\alpha) = 1-\chi_g^{-1}(p)\beta_g^2 p^{1-\ell}, \qquad \mathcal{E}(h_\alpha) = 1-\chi_h^{-1}(p)\beta_h^2 p^{1-m}.$$

Proof. In view of (94), (97) and (99), it suffices to prove the claim for the cycles $\Delta^{k_\circ,\ell_\circ,m_\circ}$ and $(\pi_1,\pi_1,\pi_1)_*\ e^*\Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$. Since $k_\circ,\ell_\circ,m_\circ$ are fixed throughout the discussion, we again denote $\Delta^\sharp = \Delta^{k_\circ,\ell_\circ,m_\circ}$ and $\Delta_{01}^\sharp = \Delta_{01}^{k_\circ,\ell_\circ,m_\circ}$ to lighten notations.

When combined with Lemma 3.18, Lemma 3.19 equips us with a completely explicit formula for comparing $(\pi_1,\pi_1,\pi_1)_*e^*(\Delta_{01}^\sharp)$ with the generalised diagonal cycle $\Delta^\sharp$. Namely, since the correspondences $([p],1,1)$, $(1,[p],1)$ and $(1,1,[p])$ induce multiplication by $p^{k_\circ}$, $p^{\ell_\circ}$ and $p^{m_\circ}$ respectively on the (f,g,h)-isotypic parts, while $(T_p,1,1)$, $(1,T_p,1)$, and $(1,1,T_p)$ induce multiplication by $a_p(f)$, $a_p(g)$, and $a_p(h)$ respectively, it follows that, with notations as in the proof of Lemma 3.19,

$$\pi_{f,g,h}(D_1) = a_p(f)\pi_{f,g,h}(\Delta^\sharp), \quad \pi_{f,g,h}(D_2) = a_p(g)\pi_{f,g,h}(\Delta^\sharp), \quad \pi_{f,g,h}(D_3) = a_p(h)\pi_{f,g,h}(\Delta^\sharp),$$

and that

$$\begin{aligned}
\pi_{f,g,h}(E_1) &= (a_p^2(f)-(p+1)p^{k_\circ})\pi_{f,g,h}(\Delta^\sharp),\\
\pi_{f,g,h}(E_2) &= (a_p^2(g)-(p+1)p^{\ell_\circ})\pi_{f,g,h}(\Delta^\sharp),\\
\pi_{f,g,h}(E_3) &= (a_p^2(h)-(p+1)p^{m_\circ})\pi_{f,g,h}(\Delta^\sharp).
\end{aligned}$$

By projecting to the (f,g,h)-isotypic component the various formulae for $(\pi_1,\pi_2,\pi_1)_*(\Delta_{01}^\sharp)$ that are given in Lemma 3.19 and substituting them into Lemma 3.18, one obtains a expression for $e_{f,g,h}(\pi_1,\pi_1,\pi_1)_*e^*(\Delta_{01}^\sharp)$ as a multiple of $\pi_{f,g,h}(\Delta^\sharp)$ by an explicit factor, which is a rational function in α_f, α_g and α_h. This explicit factor is somewhat tedious to calculate by hand, but the identity asserted in Theorem 3.21 is readily checked with the help of a symbolic algebra package. □

3.6. **Triple product p-adic L-functions.** Let $(\mathbf{f},\mathbf{g},\mathbf{h})$ be a triple of p-adic Hida families of tame levels M_f, M_g, M_h and tame characters χ_f, χ_g, χ_h as in the previous section. Let also $(\mathbf{f}^*,\mathbf{g}^*,\mathbf{h}^*) = (\mathbf{f}\otimes\bar\chi_f,\mathbf{g}\otimes\bar\chi_g,\mathbf{h}\otimes\bar\chi_h)$ denote the conjugate triple. As before, we assume $\chi_f\chi_g\chi_h = 1$ and set $M = \mathrm{lcm}(M_f,M_g,M_h)$.

Let $\Lambda_{\mathbf{f}}$, $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ be the finite flat extensions of Λ generated by the coefficients of the Hida families $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, and set $\Lambda_{\mathbf{fgh}} = \Lambda_{\mathbf{f}}\hat\otimes_{\mathbb{Z}_p}\Lambda_{\mathbf{g}}\hat\otimes_{\mathbb{Z}_p}\Lambda_{\mathbf{h}}$. Let also $\mathcal{Q}_{\mathbf{f}}$ denote the fraction field of $\Lambda_{\mathbf{f}}$ and define

$$\mathcal{Q}_{\mathbf{f},\mathbf{gh}} := \mathcal{Q}_{\mathbf{f}}\hat\otimes\Lambda_{\mathbf{g}}\hat\otimes\Lambda_{\mathbf{h}}.$$

Let $\mathcal{W}_{\mathbf{fgh}}^\circ := \mathcal{W}_{\mathbf{f}}^\circ\times\mathcal{W}_{\mathbf{g}}^\circ\times\mathcal{W}_{\mathbf{h}}^\circ \subset \mathcal{W}_{\mathbf{fgh}} = \mathrm{Spf}(\Lambda_{\mathbf{fgh}})$ denote the set of triples of *cristalline* classical points, at which the three Hida families specialize to modular forms with trivial nebentype at p (and may be either old or new at p). This set admits a natural partition, namely

$$\mathcal{W}^{\circ}_{\mathbf{fgh}} = \mathcal{W}^{f}_{\mathbf{fgh}} \sqcup \mathcal{W}^{g}_{\mathbf{fgh}} \sqcup \mathcal{W}^{h}_{\mathbf{fgh}} \sqcup \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$$

where

- $\mathcal{W}^{f}_{\mathbf{fgh}}$ denotes the set of points $(x,y,z) \in \mathcal{W}^{\circ}_{\mathbf{fgh}}$ of weights (k,ℓ,m) such that $k \geq \ell+m$.
- $\mathcal{W}^{g}_{\mathbf{fgh}}$ and $\mathcal{W}^{h}_{\mathbf{fgh}}$ are defined similarly, replacing the role of $\mathbf{f}$ with $\mathbf{g}$ (resp. $\mathbf{h}$).
- $\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$ is the set of *balanced* triples, consisting of points (x,y,z) of weights (k,ℓ,m) such that each of the weights is strictly smaller than the sum of the other two.

Each of the four subsets appearing in the above partition is dense in $\mathcal{W}_{\mathbf{fgh}}$ for the rigid-analytic topology.

Recall from (42) the spaces of Λ-adic test vectors $S^{\mathrm{ord}}_{\Lambda}(M,\chi_f)[\mathbf{f}]$. For any choice of a triple

$$(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) \in S^{\mathrm{ord}}_{\Lambda}(M,\chi_f)[\mathbf{f}] \times S^{\mathrm{ord}}_{\Lambda}(M,\chi_g)[\mathbf{g}] \times S^{\mathrm{ord}}_{\Lambda}(M,\chi_h)[\mathbf{h}]$$

of Λ-adic test vectors of tame level M, in [DR1, Lemma 2.19 and Definition 4.4] we constructed a p-adic L-function $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ in $\mathcal{Q}_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$, giving rise to a meromorphic rigid-analytic function

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) : \mathcal{W}_{\mathbf{fgh}} \longrightarrow \mathbb{C}_p. \tag{100}$$

As shown in [DR1, §4], this p-adic L-function is characterized by an interpolation property relating its values at classical points $(x,y,z) \in \mathcal{W}^{f}_{\mathbf{fgh}}$ to the central critical value of Garrett's triple-product complex L-function $L(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z, s)$ associated to the triple of classical eigenforms $(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$. The fudge factors appearing in the interpolation property depend heavily on the choice of test vectors: cf. [DR1, §4] and [DLR, §2] for more details. In a recent preprint, Hsieh [Hs] has found an explicit choice of test vectors, which yields a very optimal interpolation formula which shall be very useful for our purposes. We describe it below:

Proposition 3.22. *(Hsieh) Fix test vectors* $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ *as in* [Hs, §3]. *Then* $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ *lies in* $\Lambda_{\mathbf{fgh}}$ *and for every* $(x,y,z) \in \mathcal{W}^{f}_{\mathbf{fgh}}$ *of weights* (k,ℓ,m) *we have*

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})^2(x,y,z) = \frac{\mathfrak{a}(k,\ell,m)}{\langle \mathbf{f}^{\circ}_x, \mathbf{f}^{\circ}_x \rangle^2} \cdot \mathfrak{e}^2(x,y,z) \times L(\mathbf{f}^{\circ}_x, \mathbf{g}^{\circ}_y, \mathbf{h}^{\circ}_z, c) \tag{101}$$

where

i) $c = \frac{k+\ell+m-2}{2}$,

ii) $\mathfrak{a}(k,\ell,m) = (2\pi i)^{-2k} \cdot (\frac{k+\ell+m-4}{2})! \cdot (\frac{k+\ell-m-2}{2})! \cdot (\frac{k-\ell+m-2}{2})! \cdot (\frac{k-\ell-m}{2})!$,

iii) $\mathfrak{e}(x,y,z) = \mathcal{E}(x,y,z)/\mathcal{E}_0(x)\mathcal{E}_1(x)$ *with*

$$\begin{aligned}
\mathcal{E}_0(x) &:= 1 - \chi_f^{-1}(p)\beta^2_{\mathbf{f}_x} p^{1-k}, \\
\mathcal{E}_1(x) &:= 1 - \chi_f(p)\alpha^{-2}_{\mathbf{f}_x} p^{k_\circ}, \\
\mathcal{E}(x,y,z) &:= \left(1 - \chi_f(p)\alpha^{-1}_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z} p^{\frac{k-\ell-m}{2}}\right) \times \left(1 - \chi_f(p)\alpha^{-1}_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\beta_{\mathbf{h}_z} p^{\frac{k-\ell-m}{2}}\right) \\
&\quad \times \left(1 - \chi_f(p)\alpha^{-1}_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\alpha_{\mathbf{h}_z} p^{\frac{k-\ell-m}{2}}\right) \times \left(1 - \chi_f(p)\alpha^{-1}_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z} p^{\frac{k-\ell-m}{2}}\right).
\end{aligned}$$

Proof. This follows from [Hs, Theorem A], after spelling out explicitly the definitions involved in Hsieh's formulation.

Let us remark that throughout the whole article [DR1], it was implicitly assumed that $\mathbf{f}_x$, $\mathbf{g}_\ell$ and $\mathbf{h}_m$ are all old at p, and note that the definition we have given here of the terms $\mathcal{E}_0(x)$, $\mathcal{E}_1(x)$ and $\mathcal{E}(x,y,z)$ is exactly the same as in [DR1] in such cases, because $\beta_{\mathbf{f}_x} = \chi_f(p)\alpha^{-1}_{\mathbf{f}_x} p^{k-1}$ when $\mathbf{f}_x$ is old at p.

In contrast with loc. cit., in the above proposition we also allow any of the eigenforms $\mathbf{f}_x$, $\mathbf{g}_\ell$ and $\mathbf{h}_m$ to be new at p (which can only occur when the weight is 2); in such case, recall the usual convention adopted in §1.2 to set $\beta_\phi = 0$ when p divides the primitive level of an eigenform ϕ. With these notations, the current formulation of $\mathcal{E}(x,y,z)$, $\mathcal{E}_0(x)$ and $\mathcal{E}_1(x)$ is the correct one, as one can readily verify by rewriting the proof of [DR1, Lemma 4.10] in this setting.

□

3.7. **Perrin-Riou's regulator and the triple product p-adic L-function.** Recall the Λ-adic cyclotomic character $\underline{\varepsilon}_{\rm cyc}$ and the unramified characters $\Psi_{\mathbf{f}}$, $\Psi_{\mathbf{g}}$, $\Psi_{\mathbf{h}}$ of $G_{\mathbb{Q}_p}$ introduced in Theorem 1.3. As a piece of notation, let $\underline{\varepsilon}_{\mathbf{f}} : G_{\mathbb{Q}_p} \longrightarrow \Lambda_{\mathbf{f}}^\times$ denote the composition of $\underline{\varepsilon}_{\rm cyc}$ and the natural inclusion $\Lambda^\times \subset \Lambda_{\mathbf{f}}^\times$, and likewise for $\underline{\varepsilon}_{\mathbf{g}}$ and $\underline{\varepsilon}_{\mathbf{h}}$. Expressions like $\Psi_{\mathbf{f}}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}}$ or $\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}}$ are a short-hand notation for the $\Lambda_{\mathbf{fgh}}^\times$-valued character of $G_{\mathbb{Q}_p}$ given by the tensor product of the three characters.

Let $\mathbb{V}_{\mathbf{f}}$, $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{h}}$ be the Galois representations associated to $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ in Theorem 1.3.

The purpose of this section is describing in precise terms the close connection between the Euler system of diagonal cycles constructed above and the three-variable triple-product p-adic L-function. In order to do that, let us introduce the $\Lambda_{\mathbf{fgh}}$-modules

$$\mathbb{V}^\dagger_{\mathbf{fgh}} := \mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}(-1)(\tfrac{-}{2}) = \mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}(\varepsilon_{\rm cyc}^{-1}\underline{\varepsilon}_{\mathbf{f}}^{-1/2}\underline{\varepsilon}_{\mathbf{g}}^{-1/2}\underline{\varepsilon}_{\mathbf{h}}^{-1/2}). \tag{102}$$

and

$$\mathbb{V}^\dagger_{\mathbf{fgh}}(M) := \mathbb{V}_{\mathbf{f}}(M) \otimes \mathbb{V}_{\mathbf{g}}(M) \otimes \mathbb{V}_{\mathbf{h}}(M)(-1)(\tfrac{-}{2}). \tag{103}$$

As explained in the paragraphs following (34), $\mathbb{V}^\dagger_{\mathbf{fgh}}(M)$ is isomorphic to the direct sum of several copies of $\mathbb{V}^\dagger_{\mathbf{fgh}}$ and there are canonical projections $\varpi_{\mathbf{f}}$, $\varpi_{\mathbf{g}}$, $\varpi_{\mathbf{h}}$ which assemble into a $G_{\mathbb{Q}}$-equivariant map

$$\varpi_{\mathbf{f},\mathbf{g},\mathbf{h}} : \mathbb{H}^{111}(X_\infty^*) \longrightarrow \mathbb{V}^\dagger_{\mathbf{fgh}}(M).$$

Recall the three-variable Λ-adic global cohomology class

$$\boldsymbol{\kappa}_\infty(\epsilon_1\omega^{-k_\circ}, \epsilon_2\omega^{-\ell_\circ}, \epsilon_3\omega^{-m_\circ}; 1) = \boldsymbol{\kappa}_\infty(1,1,1;1) \in H^1(\mathbb{Q}, \mathbb{H}^{111}(X_\infty^*))$$

introduced in (90).

Set $C_q(\mathbf{f},\mathbf{g},\mathbf{h}) := (a_q(\mathbf{f}) - q - 1)(a_q(\mathbf{g}) - q - 1)(a_q(\mathbf{h}) - q - 1)$. Note that $C_q(\mathbf{f},\mathbf{g},\mathbf{h})$ is a unit in $\Lambda_{\mathbf{fgh}}$, because its classical specializations are p-adic units (cf. (91)). Define

$$\boldsymbol{\kappa}(\mathbf{f},\mathbf{g},\mathbf{h}) := \frac{1}{C_q(\mathbf{f},\mathbf{g},\mathbf{h})} \cdot \varpi_{\mathbf{f},\mathbf{g},\mathbf{h}*}(\boldsymbol{\kappa}_\infty(\epsilon_1\omega^{-k_\circ}, \epsilon_2\omega^{-\ell_\circ}, \epsilon_3\omega^{-m_\circ}; 1)) \in H^1(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)) \tag{104}$$

to be the projection of the above class to the $(\mathbf{f},\mathbf{g},\mathbf{h})$-isotypical component. We normalize it by the above constant so that the classical specializations of $\boldsymbol{\kappa}(\mathbf{f},\mathbf{g},\mathbf{h})$ at classical points coincide with the classes $\kappa_1(f_\alpha, g_\alpha, h_\alpha)$ introduced in (91).

Let

$$\mathrm{res}_p : H^1(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)) \to H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M))$$

denote the restriction map to the local cohomology at p and set

$$\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h}) := \mathrm{res}_p(\boldsymbol{\kappa}(\mathbf{f},\mathbf{g},\mathbf{h})) \in H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)).$$

The main result of this section asserts that the p-adic L-function $\mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ introduced in §3.6 can be recast as the image of the Λ-adic class $\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ under a suitable three-variable Perrin-Riou regulator map whose formulation relies on a choice of families of periods which depends on the test vectors $\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}$.

The recipe we are about to describe depends solely only on the projection of $\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ to a suitable sub-quotient of $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ which is free of rank one over $\Lambda_{\mathbf{fgh}}$, and whose definition requires the following lemma.

Lemma 3.23. *The Galois representation $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ is endowed with a four-step filtration*

$$0 \subset \mathbb{V}^{++}_{\mathbf{fgh}} \subset \mathbb{V}^{+}_{\mathbf{fgh}} \subset \mathbb{V}^{-}_{\mathbf{fgh}} \subset \mathbb{V}^{\dagger}_{\mathbf{fgh}}$$

by $G_{\mathbb{Q}_p}$-stable $\Lambda_{\mathbf{fgh}}$-submodules of ranks 0, 1, 4, 7 and 8 respectively.

The group $G_{\mathbb{Q}_p}$ acts on the successive quotients for this filtration (which are free over $\Lambda_{\mathbf{fgh}}$ of ranks 1, 3, 3 and 1 respectively) as a direct sum of one dimensional characters,

$$\mathbb{V}^{++}_{\mathbf{fgh}} = \eta^{\mathbf{fgh}}, \qquad \frac{\mathbb{V}^{+}_{\mathbf{fgh}}}{\mathbb{V}^{++}_{\mathbf{fgh}}} = \eta^{\mathbf{gh}}_{\mathbf{f}} \oplus \eta^{\mathbf{fh}}_{\mathbf{g}} \oplus \eta^{\mathbf{fg}}_{\mathbf{h}}, \qquad \frac{\mathbb{V}^{-}_{\mathbf{fgh}}}{\mathbb{V}^{+}_{\mathbf{fgh}}} = \eta^{\mathbf{f}}_{\mathbf{gh}} \oplus \eta^{\mathbf{g}}_{\mathbf{fh}} \oplus \eta^{\mathbf{h}}_{\mathbf{fg}}, \qquad \frac{\mathbb{V}^{\dagger}_{\mathbf{fgh}}}{\mathbb{V}^{-}_{\mathbf{fgh}}} = \eta_{\mathbf{fgh}},$$

where

$$\eta^{\mathbf{fgh}} = (\Psi_{\mathbf{f}}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}} \times \varepsilon^2_{\mathrm{cyc}}(\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}})^{1/2}, \qquad \eta_{\mathbf{fgh}} = \Psi_{\mathbf{f}}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}} \times \varepsilon^{-1}_{\mathrm{cyc}}(\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}})^{-1/2},$$

$$\eta^{\mathbf{gh}}_{\mathbf{f}} = \chi_f^{-1}\Psi_{\mathbf{f}}\Psi_{\mathbf{g}}^{-1}\Psi_{\mathbf{h}}^{-1} \times \varepsilon_{\mathrm{cyc}}(\underline{\varepsilon}_{\mathbf{f}}^{-1}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}})^{1/2}, \qquad \eta^{\mathbf{f}}_{\mathbf{gh}} = \chi_f\Psi_{\mathbf{f}}^{-1}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}} \times (\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}^{-1}\underline{\varepsilon}_{\mathbf{h}}^{-1})^{1/2},$$

$$\eta^{\mathbf{fh}}_{\mathbf{g}} = \chi_g^{-1}\Psi_{\mathbf{g}}\Psi_{\mathbf{f}}^{-1}\Psi_{\mathbf{h}}^{-1} \times \varepsilon_{\mathrm{cyc}}(\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}^{-1}\underline{\varepsilon}_{\mathbf{h}})^{1/2}, \qquad \eta^{\mathbf{g}}_{\mathbf{fh}} = \chi_g\Psi_{\mathbf{f}}\Psi_{\mathbf{h}}\Psi_{\mathbf{g}}^{-1} \times (\underline{\varepsilon}_{\mathbf{f}}^{-1}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}}^{-1})^{1/2},$$

$$\eta^{\mathbf{fg}}_{\mathbf{h}} = \chi_h^{-1}\Psi_{\mathbf{h}}\Psi_{\mathbf{f}}^{-1}\Psi_{\mathbf{g}}^{-1} \times \varepsilon_{\mathrm{cyc}}(\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}}^{-1})^{1/2}, \qquad \eta^{\mathbf{h}}_{\mathbf{fg}} = \chi_h\Psi_{\mathbf{f}}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}}^{-1} \times (\underline{\varepsilon}_{\mathbf{f}}^{-1}\underline{\varepsilon}_{\mathbf{g}}^{-1}\underline{\varepsilon}_{\mathbf{h}})^{1/2}.$$

Proof. Let $\boldsymbol{\phi}$ be a Hida family of tame character χ as in §1.3. Let $\psi_{\boldsymbol{\phi}}$ denote the unramified character of $G_{\mathbb{Q}_p}$ sending a Frobenius element Fr_p to $\mathbf{a}_p(\boldsymbol{\phi})$ and recall from (22) that the restriction of $\mathbb{V}_{\boldsymbol{\phi}}$ to $G_{\mathbb{Q}_p}$ admits a filtration

$$0 \to \mathbb{V}^{+}_{\boldsymbol{\phi}} \to \mathbb{V}_{\boldsymbol{\phi}} \to \mathbb{V}^{-}_{\boldsymbol{\phi}} \to 0 \quad \text{with } \mathbb{V}^{+}_{\boldsymbol{\phi}} \simeq \Lambda_{\boldsymbol{\phi}}(\psi^{-1}_{\boldsymbol{\phi}}\chi\varepsilon^{-1}_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}) \text{ and } \mathbb{V}^{-}_{\boldsymbol{\phi}} \simeq \Lambda_{\boldsymbol{\phi}}(\psi_{\boldsymbol{\phi}}).$$

Set

$$\begin{aligned}
\mathbb{V}^{++}_{\mathbf{fgh}} &= \mathbb{V}^{+}_{\mathbf{f}} \otimes \mathbb{V}^{+}_{\mathbf{g}} \otimes \mathbb{V}^{+}_{\mathbf{h}}(\varepsilon^{-1}_{\mathrm{cyc}}\underline{\varepsilon}_{\mathbf{f}}^{-1/2}\underline{\varepsilon}_{\mathbf{g}}^{-1/2}\underline{\varepsilon}_{\mathbf{h}}^{-1/2})\\
\mathbb{V}^{+}_{\mathbf{fgh}} &= \big(\mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}^{+}_{\mathbf{g}} \otimes \mathbb{V}^{+}_{\mathbf{h}} + \mathbb{V}^{+}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}^{+}_{\mathbf{h}} + \mathbb{V}^{+}_{\mathbf{f}} \otimes \mathbb{V}^{+}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}\big)(\varepsilon^{-1}_{\mathrm{cyc}}\underline{\varepsilon}_{\mathbf{f}}^{-1/2}\underline{\varepsilon}_{\mathbf{g}}^{-1/2}\underline{\varepsilon}_{\mathbf{h}}^{-1/2})\\
\mathbb{V}^{-}_{\mathbf{fgh}} &= \big(\mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}^{+}_{\mathbf{h}} + \mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}^{+}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}} + \mathbb{V}^{+}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}\big)(\varepsilon^{-1}_{\mathrm{cyc}}\underline{\varepsilon}_{\mathbf{f}}^{-1/2}\underline{\varepsilon}_{\mathbf{g}}^{-1/2}\underline{\varepsilon}_{\mathbf{h}}^{-1/2})
\end{aligned}$$

It follows from the definitions that these three representations are $\Lambda_{\mathbf{fgh}}[G_{\mathbb{Q}_p}]$-submodules of $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ of ranks 1, 4, 7 as claimed. Moreover, since $\chi_f\chi_g\chi_h = 1$, the rest of the lemma follows from (22). □

A one-dimensional character $\eta : G_{\mathbb{Q}_p} \longrightarrow \mathbb{C}_p^{\times}$ is said to be *of Hodge-Tate weight $-j$* if it is equal to a finite order character times the j-th power of the cyclotomic character. The following is an immediate corollary of Lemma 3.23.

Corollary 3.24. *Let $(x,y,z) \in \mathcal{W}^{\circ}_{\mathbf{fgh}}$ be a triple of classical points of weights (k,ℓ,m). The Galois representation $V^{\dagger}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$ is endowed with a four-step $G_{\mathbb{Q}_p}$-stable filtration*

$$0 \subset V^{++}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z} \subset V^{+}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z} \subset V^{-}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z} \subset V^{\dagger}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z},$$

and the Hodge-Tate weights of its successive quotients are:

Subquotient	Hodge-Tate weights
$V^{++}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$	$\frac{-k-\ell-m}{2}+1$
$V^{+}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}/V^{++}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$	$\frac{k-\ell-m}{2}, \frac{-k+\ell-m}{2}, \frac{-k-\ell+m}{2}$
$V^{-}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}/V^{+}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$	$\frac{-k+\ell+m}{2}-1, \frac{k-\ell+m}{2}-1, \frac{k+\ell-m}{2}-1$
$V_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}/V^{-}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$	$\frac{k+\ell+m}{2}-2$

Corollary 3.25. *The Hodge-Tate weights of $V^{+}_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}$ are all strictly negative if and only if (k,ℓ,m) is balanced.*

Let $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}$ and $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)$ be the subquotient of $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ (resp. of $\mathbb{V}^{\dagger}_{\mathbf{fgh}}(M)$) on which $G_{\mathbb{Q}_p}$ acts via (several copies of) the character

$$\eta^{\mathbf{gh}}_{\mathbf{f}} := \Psi^{\mathbf{gh}}_{\mathbf{f}} \times \Theta^{\mathbf{gh}}_{\mathbf{f}} \tag{105}$$

where

- $\Psi^{\mathbf{gh}}_{\mathbf{f}}$ is the unramified character of $G_{\mathbb{Q}_p}$ sending Fr_p to $\chi_f^{-1}(p)\mathbf{a}_p(\mathbf{f})\mathbf{a}_p(\mathbf{g})^{-1}\mathbf{a}_p(\mathbf{h})^{-1}$, and
- $\Theta^{\mathbf{gh}}_{\mathbf{f}}$ is the $\Lambda_{\mathbf{fgh}}$-adic cyclotomic character whose specialization at a point of weight (k,ℓ,m) is $\varepsilon^t_{\mathrm{cyc}}$ with $t := (-k+\ell+m)/2$.

The classical specializations of $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}$ are

$$V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x} := V^{-}_{\mathbf{f}_x} \otimes V^{+}_{\mathbf{g}_y} \otimes V^{+}_{\mathbf{h}_z}(\frac{-k-\ell-m+4}{2}) \simeq L_p\big(\chi_f^{-1}\psi_{\mathbf{f}_x}\psi^{-1}_{\mathbf{g}_y}\psi^{-1}_{\mathbf{h}_z}\big)(t), \tag{106}$$

where the coefficient field is $L_p = \mathbb{Q}_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$. Note that $t>0$ when $(x,y,z)\in\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$, while $t\le 0$ when $(x,y,z)\in\mathcal{W}^{f}_{\mathbf{fgh}}$.

Recall now from §1.4 the Dieudonné module $D(V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}(Mp))$ associated to (106). As it follows from loc. cit., every triple $(\eta_1,\omega_2,\omega_3) \in D(V^{+}_{\mathbf{f}^*_x}(Mp)) \times D(V^{-}_{\mathbf{g}^*_y}(Mp)) \times D(V^{-}_{\mathbf{h}^*_z}(Mp))$ gives rise to a linear functional $\eta_1\otimes\omega_2\otimes\omega_3 : D(V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}(Mp)) \longrightarrow L_p$.

In order to deal with the p-adic variation of these Dieudonné modules, write $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)$ as

$$\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M) = \mathbb{U}(\Theta^{\mathbf{gh}}_{\mathbf{f}})$$

where $\mathbb{U}$ is the unramified $\Lambda_{\mathbf{fgh}}$-adic representation of $G_{\mathbb{Q}_p}$ given by (several copies of) the character $\Psi^{\mathbf{gh}}_{\mathbf{f}}$.

As in §1.4, define the Λ-adic Dieudonné module

$$\mathbb{D}(\mathbb{U}) := (\mathbb{U}\hat{\otimes}\hat{\mathbb{Z}}^{\mathrm{nr}}_p)^{G_{\mathbb{Q}_p}}.$$

In view of (38), for every $(x,y,z)\in\mathcal{W}^{\circ}_{\mathbf{fgh}}$ there is a natural specialisation map

$$\nu_{x,y,z} : \mathbb{D}(\mathbb{U}) \longrightarrow D(U^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x})$$

where $U^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x} := \mathbb{U}\otimes_{\Lambda_{\mathbf{fgh}}}\mathbb{Q}_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z) \simeq V^{\mathbf{g}_y\mathbf{h}_z}_{\mathbf{f}_x}(Mp)(-t)$.

Proposition 3.26. *For any triple of test vectors*

$$(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}) \in S^{\mathrm{ord}}_{\Lambda}(M,\chi_f)[\mathbf{f}] \times S^{\mathrm{ord}}_{\Lambda}(M,\chi_g)[\mathbf{g}] \times S^{\mathrm{ord}}_{\Lambda}(M,\chi_h)[\mathbf{h}],$$

there exists a homomorphism of $\Lambda_{\mathbf{fgh}}$-modules

$$\langle\ \ ,\eta_{\breve{\mathbf{f}}^*}\otimes\omega_{\breve{\mathbf{g}}^*}\otimes\omega_{\breve{\mathbf{h}}^*}\rangle : \mathbb{D}(\mathbb{U}) \longrightarrow \mathcal{Q}_{\mathbf{f},\mathbf{gh}}$$

such that for all $\boldsymbol{\lambda} \in \mathbb{D}(\mathbb{U})$ *and all* $(x,y,z) \in \mathcal{W}_{\mathbf{fgh}}^{\mathrm{bal}}$ *such that* $\mathbf{f}_x$ *is the ordinary stabilization of an eigenform* $\mathbf{f}_x^\circ$ *of level* M*:*

$$\nu_{x,y,z}\big(\langle \boldsymbol{\lambda}, \eta_{\breve{\mathbf{f}}^*} \otimes \omega_{\breve{\mathbf{g}}^*} \otimes \omega_{\breve{\mathbf{h}}^*}\rangle\big) = \frac{1}{\mathcal{E}_0(\mathbf{f}_x^\circ)\mathcal{E}_1(\mathbf{f}_x^\circ)} \times \langle \nu_{x,y,z}(\boldsymbol{\lambda}), \eta_{\breve{\mathbf{f}}_x^*} \otimes \omega_{\breve{\mathbf{g}}_y^*} \otimes \omega_{\breve{\mathbf{h}}_z^*}\rangle.$$

Recall from (44) *that*

$$\mathcal{E}_0(\mathbf{f}_x^\circ) = 1 - \chi^{-1}(p)\beta_{\mathbf{f}_x^\circ}^2 p^{1-k}, \quad \mathcal{E}_1(\mathbf{f}_x^\circ) = 1 - \chi(p)\alpha_{\mathbf{f}_x^\circ}^{-2} p^{k-2}.$$

Proof. Since $\mathbb{U}$ is isomorphic to the unramified twist of $\mathbb{V}_{\mathbf{f}}^- \otimes \mathbb{V}_{\mathbf{g}}^+ \otimes \mathbb{V}_{\mathbf{h}}^+$, this follows from Proposition 1.7 because $\mathcal{E}_0(\mathbf{f}_x^\circ) = \mathcal{E}_0(\mathbf{f}_x^{\circ *})$ and $\mathcal{E}_1(\mathbf{f}_x^\circ) = \mathcal{E}_1(\mathbf{f}_x^{\circ *})$. □

It follows from Example 1.6 (a) and (b) that the Bloch-Kato logarithm and dual exponential maps yield isomorphisms

$$\log_{\mathrm{BK}} : H^1(\mathbb{Q}_p, V_{\mathbf{f}_x}^{\mathbf{g}_y\mathbf{h}_z}) \xrightarrow{\sim} D(V_{\mathbf{f}_x}^{\mathbf{g}_y\mathbf{h}_z}), \qquad \text{if } t > 0,$$
$$\exp_{\mathrm{BK}}^* : H^1(\mathbb{Q}_p, V_{\mathbf{f}_x}^{\mathbf{g}_y\mathbf{h}_z}) \xrightarrow{\sim} D(V_{\mathbf{f}_x}^{\mathbf{g}_y\mathbf{h}_z}), \qquad \text{if } t \leq 0.$$

Define

$$\mathcal{E}^{\mathrm{PR}}(x,y,z) = \frac{1 - p^{\frac{k-\ell-m}{2}}\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{1 - p^{\frac{\ell+m-k-2}{2}}\alpha_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}^{-1}\alpha_{\mathbf{h}_z}^{-1}} = \frac{1 - p^{-c}\beta_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{1 - p^{-c}\alpha_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}}. \tag{107}$$

The following is a three-variable version of Perrin-Riou's regulator map constructed in [PR] and [LZ14].

Proposition 3.27. *There is a homomorphism*

$$\mathcal{L}_{\mathbf{f},\mathbf{gh}} : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}(M)) \longrightarrow \mathbb{D}(\mathbb{U})$$

such that for all $\boldsymbol{\kappa}_p \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}(M))$ *the image* $\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p)$ *satisfies the following interpolation properties:*

(i) *For all balanced points* $(x,y,z) \in \mathcal{W}_{\mathbf{fgh}}^{\mathrm{bal}}$*,*

$$\nu_{x,y,z}\big(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p)\big) = \frac{(-1)^t}{t!} \cdot \mathcal{E}^{\mathrm{PR}}(x,y,z) \cdot \log_{\mathrm{BK}}(\nu_{x,y,z}(\boldsymbol{\kappa}_p)),$$

(ii) *For all points* $(x,y,z) \in \mathcal{W}_{\mathbf{fgh}}^{f}$*,*

$$\nu_{x,y,z}\big(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p)\big) = (-1)^t \cdot (1-t)! \cdot \mathcal{E}^{\mathrm{PR}}(x,y,z) \cdot \exp_{\mathrm{BK}}^*(\nu_{x,y,z}(\boldsymbol{\kappa}_p)).$$

Proof. This follows by standard methods as in [KLZ, Theorem 8.2.8], [LZ14, Appendix B], [DR2, §5.1]. □

Proposition 3.28. *The class* $\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ *belongs to the image of* $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^+(M))$ *in* $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger(M))$ *under the map induced from the inclusion* $\mathbb{V}_{\mathbf{fgh}}^+(M) \hookrightarrow \mathbb{V}_{\mathbf{fgh}}^\dagger(M)$*.*

Proof. Let $(x,y,z) \in \mathcal{W}_{\mathbf{fgh}}^\circ$ be a triple of classical points of weights (k,ℓ,m). By the results proved in §3.5, the cohomology class $\kappa_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$ is proportional to the image under the p-adic étale Abel-Jacobi map of the cycles appearing in (97), that were introduced in [DR1, §3]. The purity conjecture for the monodromy filtration is known to hold for the variety $\mathcal{A}^{k_\circ} \times \mathcal{A}^{\ell_\circ} \times \mathcal{A}^{m_\circ}$ by the work of Saito (cf. [Sa97], [Ne98, (3.2)]). By Theorem 3.1 of loc.cit., it follows that the extension $\kappa_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$ is cristalline. Hence $\kappa_p(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$ belongs to $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}^\dagger(Mp)) \subset H^1(\mathbb{Q}_p, V_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}^\dagger(Mp))$.

Since (k,ℓ,m) is balanced, Corollary 3.25 implies that $V_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}^+$ is the subrepresentation of $V_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}^\dagger$ on which the Hodge-Tate weights are all *strictly negative*. As is well-known (cf. [Fl90, Lemma 2, p. 125], [LZ16, §3.3] for similar results), the finite Bloch-Kato local Selmer group of our ordinary representation can be recast à la Greenberg [Gr89] as

$$H^1_{\mathrm{f}}(\mathbb{Q}_p, V^\dagger_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}) = \ker\left(H^1(\mathbb{Q}_p, V^\dagger_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}) \longrightarrow H^1(I_p, V^\dagger_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z}/V^+_{\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z})\right),$$

where I_p denotes the inertia group at p.

Since the set of balanced classical points is dense in $\mathcal{W}_{\mathbf{fgh}}$ for the rigid-analytic topology, it follows that the Λ-adic class $\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})$ belongs to the kernel of the natural map

$$H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)) \longrightarrow H^1(I_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)/\mathbb{V}^+_{\mathbf{fgh}}(M)).$$

Since the kernel of the restriction map

$$H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)/\mathbb{V}^+_{\mathbf{fgh}}(M)) \longrightarrow H^1(I_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)/\mathbb{V}^+_{\mathbf{fgh}}(M))$$

is trivial by Lemma 3.23, the result follows. □

Thanks to Lemma 3.23 and Proposition 3.28, we are entitled to define

$$\boldsymbol{\kappa}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})^- \quad\in\quad H^1(\mathbb{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)) \tag{108}$$

as the projection of the local class $\boldsymbol{\kappa}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ to $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)$.

Theorem 3.29. *For any triple of Λ-adic test vectors $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$, the following equality holds in the ring $\mathcal{Q}_{\mathbf{f},\mathbf{gh}}$:*

$$\langle \mathcal{L}_{\mathbf{f},\mathbf{gh}}(\,\boldsymbol{\kappa}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})^-\,),\, \eta_{\breve{\mathbf{f}}^*}\otimes\omega_{\breve{\mathbf{g}}^*}\otimes\omega_{\breve{\mathbf{h}}^*}\,\rangle \;=\; \mathscr{L}_p{}^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}).$$

Proof. It is enough to prove this equality for a subset of classical points that is dense for the rigid-analytic topology, and we shall do so for all balanced triple of cristalline classical points $(x,y,z)\in\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$ such that $\mathbf{f}_x$, $\mathbf{g}_\ell$ and $\mathbf{h}_m$ are respectively the ordinary stabilization of an eigenform $f := \mathbf{f}^\circ_x$, $g := \mathbf{g}^\circ_y$ and $h := \mathbf{h}^\circ_z$ of level M.

Set $\boldsymbol{\kappa}_p^- := \boldsymbol{\kappa}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})^-$ and $\mathcal{L} = \langle \mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p^-), \eta_{\breve{\mathbf{f}}^*}\otimes\omega_{\breve{\mathbf{g}}^*}\otimes\omega_{\breve{\mathbf{h}}^*}\rangle$ for notational simplicity. Proposition 3.26 asserts that the following identity holds in L_p:

$$\nu_{x,y,z}(\mathcal{L}) \quad=\quad \frac{1}{\mathcal{E}_0(f)\mathcal{E}_1(f)}\langle \nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p^-)), \eta_{\breve{\mathbf{f}}^*_x}\otimes\omega_{\breve{\mathbf{g}}^*_y}\otimes\omega_{\breve{\mathbf{h}}^*_z}\rangle.$$

Recall also from Remark 1.8 that

$$\eta_{\breve{\mathbf{f}}^*_x} = (1-\beta_f/\alpha_f)\varpi_1^*(\eta_{\breve{f}^*}),\quad \omega_{\breve{\mathbf{g}}^*_y} = (1-\beta_g/\alpha_g)\varpi_1^*(\omega_{\breve{g}^*}),\quad \omega_{\breve{\mathbf{h}}^*_z} = (1-\beta_h/\alpha_h)\varpi_1^*(\omega_{\breve{h}^*})$$

and

$$\nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p^-)) = \frac{(-1)^t}{t!}\cdot\mathcal{E}^{\mathrm{PR}}(x,y,z)\log_{\mathrm{BK}}(\nu_{x,y,z}(\boldsymbol{\kappa}_p^-))$$

by Proposition 3.27.

Recall the class $\kappa(f,g,h) = \kappa(\mathbf{f}^\circ_x, \mathbf{g}^\circ_y, \mathbf{h}^\circ_z)$ introduced in (97) arising from the generalized diagonal cycles of [DR1]. As in (108), we may define $\kappa_p^f(f,g,h)^- \in H^1(\mathbb{Q}_p, \mathbb{V}^{gh}_f(M))$ as the projection to $V_f^{gh}(M)$ of the restriction at p of the global class $\kappa(f,g,h)$.

It follows from Theorem 3.21 that

$$(\varpi_1,\varpi_1,\varpi_1)_*\nu_{x,y,z}(\boldsymbol{\kappa}_p^-) \quad=\quad \frac{\mathcal{E}^{\mathrm{bal}}(x,y,z)}{(1-\beta_f/\alpha_f)(1-\beta_g/\alpha_g)(1-\beta_h/\alpha_h)}\times\kappa_p^f(f,g,h)^-$$

where

$$\mathcal{E}^{\mathrm{bal}}(x,y,z) = (1-\alpha_f\beta_g\beta_h p^{-c})(1-\beta_f\alpha_g\beta_h p^{-c})(1-\beta_f\beta_g\alpha_h p^{-c})(1-\beta_f\beta_g\beta_h p^{-c}).$$

The combination of the above identities shows that the value of $\mathcal{L}$ at the balanced triple (x,y,z) is

$$\nu_{x,y,z}(\mathcal{L}) = \frac{(-1)^t\cdot \mathcal{E}^{\mathrm{bal}}(x,y,z)\mathcal{E}^{\mathrm{PR}}(x,y,z)}{t!\cdot \mathcal{E}_0(f)\mathcal{E}_1(f)} \times \langle \log_{\mathrm{BK}}(\kappa_p^f(f,g,h)), \eta_{\breve{f}^*}\otimes\omega_{\breve{g}^*}\otimes\omega_{\breve{h}^*}\rangle$$

Besides, since the syntomic Abel-Jacobi map appearing in [DR1] is the composition of the étale Abel-Jacobi map and the Bloch-Kato logarithm, the main theorem of loc. cit. asserts in our notations that

$$\nu_{x,y,z}\big(\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})\big) \quad = \quad \frac{(-1)^t}{t!}\frac{\mathcal{E}^{\mathrm{f}}(x,y,z)}{\mathcal{E}_0(f)\mathcal{E}_1(f)}\langle \log_{\mathrm{BK}}(\kappa_p^f(f,g,h)^-), \eta_{\breve{f}^*}\otimes\omega_{\breve{g}_y^*}\otimes\omega_{\breve{h}^*}\rangle$$

where

$$\mathcal{E}^{\mathrm{f}}(x,y,z) = \left(1-\beta_f\alpha_g\alpha_h p^{-c}\right)\left(1-\beta_f\alpha_g\beta_h p^{-c}\right)\left(1-\beta_f\beta_g\alpha_h p^{-c}\right)\left(1-\beta_f\beta_g\beta_h p^{-c}\right).$$

Since

$$\mathcal{E}^{\mathrm{f}}(x,y,z) = \mathcal{E}^{\mathrm{bal}}(x,y,z)\times \mathcal{E}^{\mathrm{PR}}(x,y,z)$$

and the sign and factorial terms also cancel, we have

$$\nu_{x,y,z}(\mathcal{L}) = \nu_{x,y,z}\big(\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})\big),$$

as we wanted to show. The theorem follows. □

4. The main results

We are finally in position to prove the main theorems of this article. Let $E/\mathbb{Q}$ be an elliptic curve having multiplicative reduction at a prime p and set $\alpha = a_p(E) = \pm 1$. Let

$$\psi : \mathrm{Gal}\,(H/K) \longrightarrow L^\times$$

be an anticyclotomic character of a real quadratic field K satisfying the hypotheses stated in the introduction.

In particular we assume that a prime ideal $\mathfrak{p}$ above p in H has been fixed and either of the *non-vanishing hypothesis* stated in loc. cit. holds; as explained in *Step 1* of the strategy of proof of Theorem A in the introduction, these hypotheses give rise to a character ξ of K that we fix for the remainder of this note, satisfying that the local Stark-Heegner point $P^{\alpha}_{\xi,\mathfrak{p}}$ is non-zero.

As shown in [DR2, Lemma 6.9], there exists a (non-necessarily anti-cyclotomic) character ψ_0 of finite order of K and conductor prime to DN_E such that

$$\psi_0/\psi_0' = \xi/\psi. \tag{109}$$

Since by hypothesis ξ/ψ is totally odd, it follows that ψ_0 has mixed signature $(+,-)$ with respect to the two real embeddings of K.

Let $\mathfrak{c}\subset\mathcal{O}_K$ denote the conductor of ψ_0 and let χ denote the odd central Dirichlet character of ψ_0. Let χ_K also denote the quadratic Dirichlet character associated to $K/\mathbb{Q}$.

Let $f\in S_2(pM_f)$ denote the modular form associated to E by modularity. Likewise, set

$$M_g = Dc^2\cdot \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{c}) \quad \text{and} \quad M_h = D\cdot \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{c})$$

and define the eigenforms

$$g = \theta(\psi_0\psi)\in S_1(M_g,\chi\chi_K) \qquad \text{and} \qquad h=\theta(\psi_0^{-1})\in S_1(M_h,\chi^{-1}\chi_K)$$

to be the theta series associated to the characters $\psi_0\psi$ and ψ_0^{-1}, respectively.

Recall from the introduction that $E[p]$ is assumed to be irreducible as a $G_{\mathbb{Q}}$-module implies that the mod p residual Galois representation attached to f is irreducible, and thus also non-Eisenstein mod p. The same claim holds for g and h because ψ and ξ have opposite signs and p is odd, hence $\xi\not\equiv\psi^{\pm1}\,(\mathrm{mod}\,p)$.

Note that $p \nmid M_f M_g M_h$. As in previous sections, we let M denote the least common multiple of M_f, M_g and M_h. The Artin representations V_g and V_h associated to g and h are both odd and unramified at the prime p. Since p remains inert in K, the arithmetic frobenius Fr_p acts on V_g and V_h with eigenvalues

$$\{\alpha_g, \beta_g\} = \{\zeta, -\zeta\}, \qquad \{\alpha_h, \beta_h\} = \{\zeta^{-1}, -\zeta^{-1}\},$$

where ζ is a root of unity satisfying $\chi(p) = -\zeta^2$.

In light of (109) we have $\psi_0\psi/\psi_0 = \psi$ and $\psi_0\psi/\psi_0' = \xi$, hence the tensor product of V_g and V_h decomposes as

$$V_{gh} = V_g \otimes V_h \simeq \mathrm{Ind}_K^{\mathbb{Q}}(\psi) \oplus \mathrm{Ind}_K^{\mathbb{Q}}(\xi) \quad \text{as } G_{\mathbb{Q}}\text{-modules} \tag{110}$$

and

$$V_g = V_g^{\alpha_g} \oplus V_g^{\beta_g}, \quad V_h = V_h^{\alpha_h} \oplus V_h^{\beta_h}, \quad V_{gh} = \bigoplus_{(a,b)} V_{gh}^{ab} \quad \text{as } G_{\mathbb{Q}_p}\text{-modules}$$

where (a,b) ranges through the four pairs $(\alpha_g, \alpha_h), (\alpha_g, \beta_h), (\beta_g, \alpha_h), (\beta_g, \beta_h)$. Here $V_g^{\alpha_g}$, say, is the $G_{\mathbb{Q}_p}$-submodule of V_g on which Fr_p acts with eigenvalue α_g, and similarly for the remaining terms.

4.1. **Selmer groups.** Let W_p be an arbitrary self-dual Artin representation with coefficients in L_p and factoring through the Galois group of a finite extension H of $\mathbb{Q}$. Assume W_p is unramified at p. There is a canonical isomorphism

$$\begin{aligned} H^1(\mathbb{Q}, V_p(E) \otimes W_p) &\simeq (H^1(H, V_p(E)) \otimes W_p)^{\mathrm{Gal}\,(H/\mathbb{Q})} \\ &= \mathrm{Hom}_{\mathrm{Gal}\,(H/\mathbb{Q})}(W_p, H^1(H, V_p(E))), \end{aligned} \tag{111}$$

where the the second equality follows from the self-duality of W_p. Kummer theory gives rise to a homomorphism

$$\delta : E(H)^{W_p} := \mathrm{Hom}_{\mathrm{Gal}\,(H/\mathbb{Q})}(W_p, E(H) \otimes L_p) \longrightarrow H^1(\mathbb{Q}, V_p(E) \otimes W_p). \tag{112}$$

For each rational prime ℓ, the maps (111) and (112) admit local counterparts

$$\begin{aligned} H^1(\mathbb{Q}_\ell, V_p(E) \otimes W_p) &\simeq \mathrm{Hom}_{\mathrm{Gal}\,(H/\mathbb{Q})}(W_p, \oplus_{\lambda|\ell} H^1(H_\lambda, V_p(E))), \\ \delta_\ell : \left(\oplus_{\lambda|\ell} E(H_\lambda)\right)^{W_p} &\longrightarrow H^1(\mathbb{Q}_\ell, V_p(E) \otimes W_p), \end{aligned}$$

for which the following diagram commutes:

$$\begin{tikzcd} E(H)^{W_p} \arrow[r, "\delta"] \arrow[d, "\mathrm{res}_\ell"] & H^1(\mathbb{Q}, V_p(E) \otimes W_p) \arrow[d, "\mathrm{res}_\ell"] \\ \left(\oplus_{\lambda|\ell} E(H_\lambda)\right)^{W_p} \arrow[r, "\delta_\ell"] & H^1(\mathbb{Q}_\ell, V_p(E) \otimes W_p). \end{tikzcd} \tag{113}$$

For primes $\ell \neq p$, it follows from [Ne98, (2.5) and (3.2)] that $H^1(\mathbb{Q}_\ell, V_p(E) \otimes W_p) = 0$. (We warn however that if we were working with integral coefficients, these cohomology groups may contain non-trivial torsion.) For $\ell = p$, the Bloch-Kato submodule $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E) \otimes W_p)$ is the subgroup of $H^1(\mathbb{Q}_p, V_p(E) \otimes W_p)$ formed by classes of *cristalline* extensions of Galois representations of $V_p(E) \otimes W_p$ by $\mathbb{Q}_p$. It may also be identified with the image of the local connecting homomorphism δ_p.

Lemma 4.1. *There is a natural isomorphism of L_p-vector spaces*

$$H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E) \otimes W_p) = H^1(\mathbb{Q}_p, V_f^+ \otimes W_p^{\mathrm{Fr}_p \neq \alpha}) \oplus H^1_{\mathrm{f}}(\mathbb{Q}_p, V_f^+ \otimes W_p^{\mathrm{Fr}_p = \alpha}),$$

where recall $\alpha = a_p(E) = \pm 1$.

Proof. We firstly observe that $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E)\otimes W_p) = H^1_{\mathrm{g}}(\mathbb{Q}_p, V_p(E)\otimes W_p)$ by e.g. [Bel, Prop. 2.0 and Ex. 2.20], because $V_p(E)\otimes W_p$ contains no unramified *submodule*. As shown in [Fl90, Lemma , p.125], it follows that

$$H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E)\otimes W_p) = \mathrm{Ker}\big(H^1(\mathbb{Q}_p, V_p(E)\otimes W_p) \longrightarrow H^1(I_p, V_p^-(E)\otimes W_p)\big)$$

is the kernel of the composition of the homomorphism in cohomology induced by the natural projection $V_p(E)\longrightarrow V_p^-(E)$ and restriction to the inertia subgroup $I_p\subset G_{\mathbb{Q}_p}$.

The long exact sequence in Galois cohomology arising from (18) shows that the kernel of the map $H^1(\mathbb{Q}_p, V_p(E)\otimes W_p)\longrightarrow H^1(\mathbb{Q}_p, V_p^-(E)\otimes W_p)$ is naturally identified with $H^1(\mathbb{Q}_p, V_p^+(E)\otimes W_p)$. We have $H^1(I_p, \mathbb{Q}_p(\psi\varepsilon_{\mathrm{cyc}})) = 0$ for any nontrivial unramified character ψ. Besides, it follows from Example 1.6 that $H^1_{\mathrm{f}}(\mathbb{Q}_p, \mathbb{Q}_p(\varepsilon_{\mathrm{cyc}})) = \ker\big(H^1(\mathbb{Q}_p, \mathbb{Q}_p(\varepsilon_{\mathrm{cyc}}))\to H^1(I_p, \mathbb{Q}_p(\varepsilon_{\mathrm{cyc}}))\big)$ is a line in the two-dimensional space $H^1(\mathbb{Q}_p, \mathbb{Q}_p(\varepsilon_{\mathrm{cyc}}))$, which Kummer theory identifies with $\mathbb{Z}_p^\times\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Q}_p$ sitting inside $\mathbb{Q}_p^\times\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Q}_p$.

Recall from (18) that $V_p^+(E) = L_p(\psi_f\varepsilon_{\mathrm{cyc}})$ and $V_p^-(E)\simeq L_p(\psi_f)$ where ψ_f is the unramified quadratic character of $G_{\mathbb{Q}_p}$ sending Fr_p to α. The lemma follows. □

The Selmer group $\mathrm{Sel}_p(E, W_p)$ is defined as

$$\mathrm{Sel}_p(E, W_p) := \{\lambda\in H^1(\mathbb{Q}, V_p(E)\otimes W_p) : \mathrm{res}_p(\lambda)\in H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E)\otimes W_p)\}.$$

Here res_p stands for the natural map in cohomology induced by restriction from $G_{\mathbb{Q}}$ to $G_{\mathbb{Q}_p}$.

4.2. **Factorisation of p-adic L-series.** The goal of this section is proving a factorisation formula of p-adic L-functions which shall be crucial in the proof of our main theorem.

Recall the sign $\alpha := a_p(f)\in\{\pm 1\}$ associated to E. Let g_ζ and $h_{\alpha\zeta^{-1}}$ denote the ordinary p-stabilizations of g and h on which the Hecke operator U_p acts with eigenvalue

$$\alpha_g := \zeta \quad\text{and}\quad \alpha_h := \alpha\zeta^{-1}, \tag{114}$$

respectively.

Let $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ be the Hida families of tame levels M_f, M_g, M_h and tame characters 1, $\chi\chi_K$, $\chi^{-1}\chi_K$ passing respectively through f, g_ζ and $h_{\alpha\zeta^{-1}}$. The existence of these families is a theorem of Wiles [W88], and their uniqueness follows from a recent result of Bellaïche and Dimitrov [BeDi] (note that the main theorem of loc. cit. indeed applies because $\alpha_g\neq\beta_g$, $\alpha_h\neq\beta_h$ and p does not split in K). Let x_0, y_0, z_0 denote the classical points in $\mathcal{W}_{\mathbf{f}}$, $\mathcal{W}_{\mathbf{g}}$ and $\mathcal{W}_{\mathbf{h}}$ respectively such that $\mathbf{f}_{x_0} = f$, $\mathbf{g}_{y_0} = g_\zeta$ and $\mathbf{h}_{z_0} = h_{\alpha\zeta^{-1}}$.

Let

$$\breve{\mathbf{f}}\in S^{\mathrm{ord}}_{\Lambda_{\mathbf{f}}}(M)[\mathbf{f}],\quad \breve{\mathbf{g}}\in S^{\mathrm{ord}}_{\Lambda_{\mathbf{g}}}(M,\chi\chi_K)[\mathbf{g}],\quad \breve{\mathbf{h}}\in S^{\mathrm{ord}}_{\Lambda_{\mathbf{h}}}(M,\chi^{-1}\chi_K)[\mathbf{h}]$$

be Hsieh's choice of Λ-adic test vectors of tame level M as in Proposition 3.22. Associated to it there is the three-variable p-adic L-function $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ introduced in (100), and we define

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}})\in\Lambda_{\mathbf{f}} \tag{115}$$

to be the one-variable p-adic L-function arising as the restriction of $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ to the rigid analytic curve $\mathcal{W}_{\mathbf{f}}\times\{y_0, z_0\}$.

In addition to it, recall the p-adic L-functions described in §2.3 associated to the twist of E/K by an anticyclotomic character of K, and set $\mathfrak{f}_{\mathcal{O}}(k_\circ) := (Dc^2)^{\frac{1-k}{2}}/\mathfrak{f}_c^2$, where $\mathfrak{f}_c$ is the simple constant invoked in that section. Note that the rule $k\mapsto\mathfrak{f}_{\mathcal{O}}(k_\circ)$ extends to an Iwasawa function, that we continue to denote $\mathfrak{f}_{\mathcal{O}}$, because p does not divide Dc^2. Recall also the rigid-analytic function $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ in a neighborhood $U_{\mathbf{f}}\subset\mathcal{W}_{\mathbf{f}}$ of x_0 introduced in (55).

Theorem 4.2. *The following factorization of p-adic L-functions holds in $\Lambda_{\mathbf{f}}$:*

$$\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))\times\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}}) = \mathfrak{f}_{\mathcal{O}}\cdot\mathscr{L}_p(\mathbf{f}/K,\psi)\times\mathscr{L}_p(\mathbf{f}/K,\xi).$$

Proof. Write

$$\mathfrak{a}(k)=\mathfrak{a}(k,1,1),\quad \mathfrak{e}(x)=\mathfrak{e}(x,y_0,z_0)$$

for the factors appearing in the interpolation formula satisfied by $\mathscr{L}_p{}^f(\tilde{\mathbf{f}},\tilde{g}_\zeta,\tilde{h}_{\alpha\zeta^{-1}})$ described in Proposition 3.22.

Recall we set $k=k_\circ+2$. It directly follows from our definitions and running assumptions that

$$\mathfrak{a}(k)=(2\pi i)^{-2k}\cdot(\frac{k_\circ}{2}!)^4$$

and

$$\mathfrak{e}(x)=\frac{(1-\alpha_{\mathbf{f}_x}^{-1}p^{\frac{k_\circ}{2}})^2\cdot(1+\alpha_{\mathbf{f}_x}^{-1}p^{\frac{k_\circ}{2}})^2}{(1-\beta_{\mathbf{f}_x}^2p^{1-k})(1-\alpha_{\mathbf{f}_x}^{-2}p^{k_\circ})}=\frac{1-\alpha_{\mathbf{f}_x}^{-2}p^{k_\circ}}{1-\beta_{\mathbf{f}_x}^2p^{1-k}}.$$

By Proposition 3.22, it follows that $\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}})$ satisfies the following interpolation property for all $x\in\mathcal{W}_{\mathbf{f}}^\circ$ of weight $k\geq 2$:

$$\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}})(x)=(2\pi i)^{-k}\cdot(\frac{k_\circ}{2}!)^2\cdot\frac{1-\alpha_{\mathbf{f}_x}^{-2}p^{k_\circ}}{1-\beta_{\mathbf{f}_x}^2p^{1-k}}\cdot\frac{L(\mathbf{f}_x^\circ,g,h,\frac{k}{2})^{1/2}}{\langle\mathbf{f}_x^\circ,\mathbf{f}_x^\circ\rangle}.$$

Besides, it follows from Theorem 2.8 that the product of $\mathscr{L}_p(\mathbf{f}/K,\psi)$ and $\mathscr{L}_p(\mathbf{f}/K,\xi)$ satisfies that for all $x\in\mathcal{W}_{\mathbf{f}}^\circ$ of weight $k\geq 2$:

$$\mathscr{L}_p(\mathbf{f}/K,\psi)\mathscr{L}_p(\mathbf{f}/K,\xi)(x)=\mathfrak{f}_{\mathbf{f},\psi}(x)\cdot\mathfrak{f}_{\mathbf{f},\xi}(x)\times L(\mathbf{f}_x^\circ/K,\psi,k/2)^{1/2}\cdot L(\mathbf{f}_x^\circ/K,\xi,k/2)^{1/2}$$

where

$$\mathfrak{f}_{\mathbf{f},\psi}(x)\cdot\mathfrak{f}_\xi(x)=(1-\alpha_{\mathbf{f}_x}^{-2}p^{k_\circ})^2\cdot\frac{\mathfrak{f}_c^2\cdot(Dc^2)^{\frac{k_\circ+1}{2}}\cdot(\frac{k_\circ}{2})!^2}{(2\pi i)^{k_\circ}}\cdot\frac{\Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p}}{\Omega^+_{\mathbf{f}_x,\mathbb{C}}\Omega^-_{\mathbf{f}_x,\mathbb{C}}}.$$

A direct inspection to the Euler factors shows that for all $x\in\mathcal{W}_{\mathbf{f}}^\circ$ of weight $k\geq 2$:

$$L(\mathbf{f}_x^\circ,g,h,k/2)=L(\mathbf{f}_x^\circ/K,\psi,k/2)\cdot L(\mathbf{f}_x^\circ/K,\xi,k/2). \tag{116}$$

Recall finally that the value of the function $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ at a classical point $x\in U_{\mathbf{f}}\cap\mathcal{W}_{\mathbf{f}}^\circ$ is

$$\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))(x)=(1-\beta_{\mathbf{f}_x}^2p^{1-k})(1-\alpha_{\mathbf{f}_x}^{-2}p^{k_\circ})\Omega^+_{\mathbf{f}_x,p}\Omega^-_{\mathbf{f}_x,p}.$$

Combining the above formulae together with the equality

$$\Omega^+_{\mathbf{f}_x,\mathbb{C}}\cdot\Omega^-_{\mathbf{f}_x,\mathbb{C}}=4\pi^2\langle\mathbf{f}_x^\circ,\mathbf{f}_x^\circ\rangle,$$

described in §2.3, it follows that the following formula holds for all $x\in\mathcal{W}_{\mathbf{f}}^\circ$ of weight $k\geq 2$:

$$\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))(x)\times\mathscr{L}_p{}^f(\breve{\mathbf{f}},\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}})(x)=\lambda_{\mathcal{O}}(k_\circ)\cdot\mathscr{L}_p(\mathbf{f}/K,\psi)(x)\times\mathscr{L}_p(\mathbf{f}/K,\xi)(x). \tag{117}$$

Since $\mathcal{W}_{\mathbf{f}}^\circ$ is dense in $\mathcal{W}_{\mathbf{f}}$ for the rigid-analytic topology, the factorization formula claimed in the theorem follows. □

In Theorem 2.8 we showed that $\mathcal{L}_p(\mathbf{f}/K,\psi)$ and $\mathcal{L}_p(\mathbf{f}/K,\xi)$ both vanish at x_0 and

$$\frac{d}{dx}\mathcal{L}_p(\mathbf{f}/K,\psi)_{|x=x_0}=\frac{1}{2}\cdot\log_p(P_\psi^\alpha),\quad \frac{d}{dx}\mathcal{L}_p(\mathbf{f}/K,\xi)_{|x=x_0}=\frac{1}{2}\cdot\log_p(P_\xi^\alpha). \tag{118}$$

As remarked e.g. in the remarks preceding [BD3, Theorem 3.4], the function $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{f}))$ does not vanish at x_0 and takes an algebraic value in $L^\times$. It thus follows from Theorem 4.2 that the order of vanishing of $\mathcal{L}_p^f(\breve{\mathbf{f}}^\vee,\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}})$ at $x=x_0$ is at least two and

$$\frac{d^2}{dx^2}\mathcal{L}_p^f(\breve{\mathbf{f}}^\vee,\breve{g}_\zeta,\breve{h}_{\alpha\zeta^{-1}})_{|x=x_0}=C_1\cdot\log_p(P_\psi^\alpha)\cdot\log_p(P_\xi^\alpha), \tag{119}$$

where $C_1\in L^\times$ is a non-zero simple algebraic constant.

As recalled at the beginning of this chapter, $P^{\alpha}_{\xi,\mathfrak{p}}$ is non-zero. We can also suppose that $P^{\alpha}_{\psi,\mathfrak{p}}$ is non-zero, as otherwise there is nothing to prove. Hence (119) shows that the order of vanishing of $\mathcal{L}_p^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})$ at $x = x_0$ is exactly two.

4.3. **Proof of Theorems A and B.** Let

$$\kappa(\mathbf{f},\mathbf{g},\mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{fgh}}(M))$$

be the Λ-adic global cohomology class introduced in (104).

Define $\mathbb{V}^\dagger_{\mathbf{f}gh}(M)$ as the $\Lambda_{\mathbf{f}}[G_{\mathbb{Q}}]$-module obtained by specialising the $\Lambda_{\mathbf{fgh}}[G_{\mathbb{Q}}]$-module $\mathbb{V}^\dagger_{\mathbf{fgh}}(M)$ at (y_0, z_0). Let

$$\kappa(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}}) := \nu_{y_0,z_0}\kappa(\mathbf{f},\mathbf{g},\mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{f}gh}(M)) \tag{120}$$

denote the specialisation of $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})$ at (y_0, z_0), and

$$\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1(\mathbb{Q}, V_{fgh}(M))$$

denote the class obtained by specializing (120) further at x_0.

The goal of this section is proving that $\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}})$ belongs to the Selmer group, and computing its logarithm along a suitable direction, showing that it factors as the product of logarithms of two Stark-Heegner points. This will allow us to prove Theorem C, from which Theorems A and B also follow.

To this end, define the $\Lambda_{\mathbf{f}}[G_{\mathbb{Q}_p}]$-modules

$$\mathbb{W} := \mathbb{V}_{\mathbf{f}}(M)(\underline{\varepsilon}_{\mathbf{f}}^{-1/2}) \otimes V_{gh}^{\beta_g\beta_h}(M), \quad \mathbb{W}^- := \mathbb{V}^-_{\mathbf{f}}(M)(\underline{\varepsilon}_{\mathbf{f}}^{-1/2}) \otimes V_{gh}^{\beta_g\beta_h}(M).$$

It follows from (114) that $V_{gh}^{\beta\beta} = L_p(\alpha)$ is the one-dimensional representation afforded by character of $\mathrm{Gal}\,(K_p/\mathbb{Q}_p)$ sending Fr_p to $\alpha = a_p(E)$.

Hence $\mathbb{W}^-$ is the sub-quotient of $\mathbb{V}^\dagger_{\mathbf{f}gh}(M)$ that is isomorphic to several copies of $\Lambda_{\mathbf{f}}(\Psi^{gh}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{f}}^{-1/2})$, where as in (105), $\Psi^{gh}_{\mathbf{f}}$ denotes the unramified character of $G_{\mathbb{Q}_p}$ satisfying

$$\Psi^{gh}_{\mathbf{f}}(\mathrm{Fr}_p) = \mathbf{a}_p(\mathbf{f})a_p^{-1}(\mathbf{g}_1)a_p^{-1}(\mathbf{h}_1) = \alpha\cdot\mathbf{a}_p(\mathbf{f}).$$

Let

$$\boldsymbol{\kappa}_p^f(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1(\mathbb{Q}_p, \mathbb{W}), \quad \boldsymbol{\kappa}_p^f(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}})^- \in H^1(\mathbb{Q}_p, \mathbb{W}^-) \tag{121}$$

denote the image of the restriction at p of $\kappa(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}})$ under the map induced by the projection $\mathbb{V}^+_{\mathbf{f}gh}(M) \to \mathbb{W}$, and further to $\mathbb{W}^-$ respectively.

Equivalently and in consonance with our notations, $\boldsymbol{\kappa}_p^f(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}})^-$ is the specialization at (y_0, z_0) of the local class $\boldsymbol{\kappa}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})^-$ introduced in (108) and invoked in Theorem 3.29. Hence this theorem applies, and asserts that the following identity holds in $\Lambda_{\mathbf{f}}$ for any triple $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ of Λ-adic test vectors:

$$\langle \mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p^f(\mathbf{f}, g_\zeta, h_{\alpha\zeta^{-1}})^-), \eta_{\breve{\mathbf{f}}^*} \otimes \omega_{\breve{g}^*_\zeta} \otimes \omega_{\breve{h}^*_{\alpha\zeta^{-1}}}\rangle = \mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}}). \tag{122}$$

Let now $\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})$ and $\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})^-$ denote the specializations at x_0 of the classes in (121). Since $a_p(f) = \alpha \in \{\pm 1\}$ and $\underline{\varepsilon}_{\mathbf{f}}(x_0) = 1$, it follows from the above description of $\mathbb{W}$ and the character $\Psi^{gh}_{\mathbf{f}}$ that, as $G_{\mathbb{Q}_p}$-modules,

$$\mathbb{W}(x_0) \simeq V_p(E_+)(M), \quad \mathbb{W}^-(x_0) \simeq L_p(M),$$

where E_+ is the (trivial or quadratic) twist of E given by α, and L_p stands for the trivial representation.

Hence

$$\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1(\mathbb{Q}_p, V_p(E_+)(M)) \quad \text{and} \quad \kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})^- \in H^1(\mathbb{Q}_p, L_p(M)).$$

The Bloch-Kato dual exponential and logarithm maps associated to the p-adic representation $V_p(E_+)(M)$ take values in a space $L_p(M)$ consisting of several copies of the base field L_p. The choice of test vectors gives rise to a projection $L_p(M) \longrightarrow L_p$. Since the test vectors are fixed throughout, we shall denote by a slight abuse of notation

$$\exp^*_{\mathrm{BK}} : H^1(\mathbb{Q}_p, V_p(E_+)(M)) \longrightarrow L_p, \quad \log_{\mathrm{BK}} : H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E_+)(M)) \longrightarrow L_p$$

the composition of Bloch-Kato dual exponential and logarithm maps, respectively, with the aforementioned projection to L_p

Venerucci [Ve] has recently proved a variant of a conjecture of Perrin-Riou for elliptic curves A having *split* multiplicative reduction at a prime p, exploiting the fact due to Kato and Ochiai that the two-variable Mazur-Kitagawa p-adic L-function associated to A can be recast as the image under the Perrin-Riou Λ-adic regulator of Kato's Euler system of Siegel modular units on the K_2-group of a tower of modular curves. Although Kato's original setting is different from ours, some of the results that Venerucci proves in the technical core of his article are purely local, and can be applied to arbitrary local Λ-adic classes satisfying suitable conditions that are also met in the present scenario. Using them we can prove the following result:

Theorem 4.3. *The class $\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})$ belongs to $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E_+)(M))$. In addition we have*

$$\frac{d^2}{dx^2}\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})_{|x=x_0} = C_2 \cdot \log_{\mathrm{BK}}(\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})) \tag{123}$$

for some nonzero rational number $C_2 \in \mathbb{Q}^\times$.

Proof. Let $f_+ = f \otimes \alpha$ denote the (trivial or quadratic) twist of f over K such that $a_p(f_+) = 1$, and $\mathbf{f}_+$ be the Hida family passing through f_+. Then $\mathbb{W}$ is isomorphic to several copies of $\mathbb{V}_{\mathbf{f}_+}(\varepsilon_{\mathbf{f}_+}^{-1/2})$ and thus we are placed in the setting covered by Theorem 3.1 of [Ve].

To be more precise, [Ve, Theorem 3.1] can be applied to any local two-variable Λ-adic class in $H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, \mathbb{V}_{\mathbf{f}_+}) := H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}_+} \otimes \Lambda(\varepsilon_{\mathrm{cyc}}))$; the two variables in play are the weight k of the Hida family $\mathbf{f}_+$ (or rather the points x in the finite flat cover $\mathcal{W}_{\mathbf{f}_+}$ of weight space) and the cyclotomic variable s, although the notations for the variables employed in loc. cit. differ from ours here.

For our purposes it will suffice to apply loc. cit. restricted to the *central critical line*

$$s - 1 = -k(x)/2,$$

which is the one characterized by the fact that all classical specializations of $\mathbb{V}_{\mathbf{f}_+} \otimes \Lambda(\varepsilon_{\mathrm{cyc}})$ at points on this line are Kummer self-dual. Moreover, the restriction of $\mathbb{V}^-_{\mathbf{f}_+} \otimes \Lambda(\varepsilon_{\mathrm{cyc}})$ to this line is precisely the $\Lambda_{\mathbf{f}}[G_{\mathbb{Q}_p}]$-module $\mathbb{V}^-_{\mathbf{f}_+}(\varepsilon_{\mathbf{f}_+}^{-1/2})$ invoked above, whose specialization at a point x of weight k is $V^-_{\mathbf{f}_+,x}(k/2)$.

Now that we have clarified the notational passage from [Ve] to our setting, the restriction to the central critical line of [Ve, Theorem 3.1] applies, and combining it with (122) asserts that

$$(1 - \frac{1}{p})\frac{d}{dx}\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})_{|x=x_0} = \mathcal{L}_p(E_+)\exp^*_{\mathrm{BK}}(\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})).$$

Since we concluded at the end of §4.2 that $\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})$ vanishes at $x = x_0$ with order exatly two, and the $\mathcal{L}$-invariant of E_+ is non-zero, it follows that Bloch-Kato's dual exponential vanishes at $\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})$ for all choices of test vectors, and hence this class belongs to $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E_+)(M))$, as claimed.

We are hence in position to apply the second part of Theorem 5.1 of [Ve], which in combination with the displayed equation (6) of loc. cit. states that

$$\log(\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})) \times \frac{d^2}{dx^2}\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})_{|x=x_0} = \log^2(\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}}))$$

up to a nonzero rational number. The argument of [Ve, Lemma 6.1] applies in this setting and hence $\log(\kappa_p^f(f, g_\zeta, h_{\alpha\zeta^{-1}})) \neq 0$. The theorem follows. □

Corollary 4.4. *The global class $\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}})$ lies in the Selmer group $\mathrm{Sel}_p(E, V_{gh}(M))$.*

Proof. Write $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1(\mathbb{Q}_p, V_f \otimes V_{gh}(M))$ for the restriction of $\kappa(f, g_\zeta, h_{\alpha\zeta^{-1}}))$ at p and $\kappa_p^-(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}(M))$ for its projection to $V_f^- \otimes V_{gh}(M)$.

After setting $V_{gh}^{ab} = V_g^a \otimes V_h^b$, we find that there is a natural decomposition

$$H^1(\mathbb{Q}_p, V_p(E) \otimes V_{gh}) = \bigoplus_{(a,b)} H^1(\mathbb{Q}_p, V_p(E) \otimes V_{gh}^{ab}) \tag{124}$$

where (a, b) ranges through the four pairs $(\alpha_g, \alpha_h), (\alpha_g, \beta_h), (\beta_g, \alpha_h), (\beta_g, \beta_h)$. There are similar decompositions of course for $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E) \otimes V_{gh})$ and $H^1(\mathbb{Q}_p, V_p^+(E) \otimes V_{gh})$.

Note that

$$\alpha_g\alpha_h = \beta_g\beta_h = \alpha, \qquad \alpha_g\beta_h = \beta_g\alpha_h = -\alpha. \tag{125}$$

Hence, according to Lemma 4.1 and the discussion preceding it, in order to prove the statement we must show that $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})$ lies in $H^1(\mathbb{Q}_p, V_p^+(E) \otimes V_{gh}(M))$ and its (α_g, α_h) and (β_g, β_h)-components lie in the finite Bloch-Kato submodule.

By Proposition 3.28, the local class $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})$ is the specialization at (x_0, y_0, z_0) of a Λ-adic cohomology class with values in the Λ-adic representation $\mathbb{V}^+_{\mathbf{fgh}}(M)$, which recall is defined as the span in $\mathbb{V}^\dagger_{\mathbf{fgh}}(M)$ of (suitably twisted) triple tensor products of the form $\mathbb{V}^\pm_{\mathbf{f}} \otimes \mathbb{V}^\pm_{\mathbf{g}} \otimes \mathbb{V}^\pm_{\mathbf{h}}$, with at least two $+$'s in the exponents.

Since $V_g^{\beta_g} = V_g^+$ and $V_g^{\alpha_g} = V_g^-$, and similarly for V_h, it follows from the very definition of $\mathbb{V}^+_{\mathbf{fgh}}(M)$ that the (α_g, α_h)-component of $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})$ in $H^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\alpha_g\alpha_h}(M))$ vanishes, and the (α_g, β_h) and (β_g, α_h)-components of $\kappa_p^-(f, g_\zeta, h_{\alpha\zeta^{-1}})$ in $H^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\alpha_g\alpha_h}(M))$ vanish. In addition to that, Theorem 4.3 amounts to saying that the (β_g, β_h)-component of $\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})$ lies in $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_f \otimes V_{gh}^{\beta_g\beta_h}(M))$, and hence its projection to $V_f^- \otimes V_{gh}^{\beta_g\beta_h}(M)$ also vanishes. Putting it together, the corollary follows. □

We are finally in position to prove the main theorems stated in the introduction. We start with Theorem C. In order to recall its statement, recall from (110) that $V_{gh} = V_\psi \oplus V_\xi$ decomposes as the direct sum of the induced representations of ψ and ξ. Write

$$\kappa_\psi(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E) \otimes V_\psi(M)), \quad \kappa_\xi(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E) \otimes V_\xi(M)) \tag{126}$$

for the projections of the class appearing in Corollary 4.4 to the corresponding quotients. We denote as in the introduction

$$\kappa_\psi^\alpha(f, g_\zeta, h_{\alpha\zeta^{-1}}) = (1 + \alpha\sigma_{\mathfrak{p}})\kappa_\psi(f, g_\zeta, h_{\alpha\zeta^{-1}}) \in H^1_{\mathrm{f}}(H, V_p(E)(M))^{\psi\oplus\bar{\psi}}$$

the component of $\kappa_\psi(f, g_\zeta, h_{\alpha\zeta^{-1}})$ on which $\sigma_{\mathfrak{p}}$ acts with eigenvalue α, and likewise with ψ replaced by the auxiliary character ξ.

Lemma 4.5. *We have*

$$\log_{E,\mathfrak{p}} \kappa_\psi^\alpha(f, g_\zeta, h_{\alpha\zeta^{-1}}) = \log_{E,\mathfrak{p}} \kappa_\xi^\alpha(f, g_\zeta, h_{\alpha\zeta^{-1}}).$$

Proof. We may decompose the local class

$$\kappa_p := \kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}}) = (\kappa_p^{\alpha_g\alpha_h}, \kappa_p^{\alpha_g\beta_h}, \kappa_p^{\beta_g\alpha_h}, \kappa_p^{\beta_g\beta_h})$$

in $H^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\alpha_g\alpha_h}(M))$ as the sum of four contributions with respect to the decomposition (124) afforded by the eigenspaces for the action of $\sigma_{\mathfrak{p}}$. In addition to that, κ_p also decomposes as

$$\kappa_p = (\kappa_{\psi,p}, \kappa_{\xi,p}) \in H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E) \otimes V_\psi(M)) \oplus H^1_{\mathrm{f}}(\mathbb{Q}_p, V_p(E) \otimes V_\xi(M)),$$

where $\kappa_{\psi,p}$, $\kappa_{\xi,p}$ are the local components at p of the classes in (126). An easy exercise in linear algebra shows that

$$\kappa_p^{\alpha_g\alpha_h} = \kappa_{\psi,p}^\alpha - \kappa_{\xi,p}^\alpha, \quad \kappa_p^{\beta_g\beta_h} = \kappa_{\psi,p}^\alpha + \kappa_{\xi,p}^\alpha. \tag{127}$$

It was shown in the proof of the previous corollary that $\kappa_p^{\alpha_g\alpha_h} = 0$. Hence the above display implies that $\kappa_{\psi,p}^\alpha = \kappa_{\xi,p}^\alpha$ are the same element in $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{f+}(M))$. The lemma follows. $\square$

Let

$$\log_{\beta_g\beta_h} : H^1_{\mathrm{f}}(\mathbb{Q}_p, V_f \otimes V_{gh}(M)) \stackrel{\mathrm{pr}_{\beta_g\beta_h}}{\longrightarrow} H^1_{\mathrm{f}}(\mathbb{Q}_p, V_f \otimes V_{gh}^{\beta_g\beta_h}(M)) \stackrel{\log_{\mathrm{BK}}}{\longrightarrow} L_p$$

denote the composition of the natural projection to the (β_g, β_h)-component with the Bloch-Kato logarithm map associated to the p-adic representation $V_f \otimes V_{gh}^{\beta_g\beta_h}(M) \simeq V_{f+}(M)$ and the choice of test vectors. Note that $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{f+}) = H^1_{\mathrm{f}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$, which as recalled in Example 1.6 (c) is naturally identified with the completion of $\mathbb{Z}_p^\times$, and the Bloch-Kato logarithm is nothing but the usual p-adic logarithm on $\mathbb{Z}_p^\times$ under this identification. Lemma 4.5 together with the second identity in (127) imply that

$$(i) \qquad \log_{E,\mathfrak{p}} \kappa_\psi^\alpha(f, g_\zeta, h_{\alpha\zeta^{-1}}) = \log_{\beta_g\beta_h}(\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})).$$

Theorem 4.3 shows that

$$(ii) \qquad \log_{\beta_g\beta_h}(\kappa_p(f, g_\zeta, h_{\alpha\zeta^{-1}})) = \frac{d^2}{dx^2}\mathscr{L}_p{}^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})_{|x=x_0} \pmod{L^\times}.$$

Finally, fix $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ to be Hsieh's choice of Λ-adic test vectors satisfying the properties stated in Theorem 4.2. Recall from (119) that, with this choice, we have

$$(iii) \qquad \frac{d^2}{dx^2}\mathcal{L}_p^f(\breve{\mathbf{f}}^\vee, \breve{g}_\zeta, \breve{h}_{\alpha\zeta^{-1}})_{|x=x_0} = \log_p(P_\psi^\alpha) \cdot \log_p(P_\xi^\alpha) \pmod{L^\times}.$$

Putting together (i)-(ii)-(iii) it follows that

$$\log_{E,\mathfrak{p}} \kappa_\psi^\alpha(f, g_\zeta, h_{\alpha\zeta^{-1}}) = \log_{E,\mathfrak{p}}(P_\psi^\alpha) \times \log_{E,\mathfrak{p}}(P_\xi^\alpha) \pmod{L^\times}.$$

This is precisely the content of Theorem C, which we just proved.

Theorem A is now a direct consequence of Theorem C, if we take

$$\kappa_\psi := \log_{E,\mathfrak{p}}(P_\xi^\alpha)^{-1} \times \kappa_\psi^\alpha(f, g_\alpha, h_\alpha).$$

Theorem B also follows, because the non-vanishing of the first derivative $\frac{d}{dx}\mathscr{L}_p(\mathbf{f}/K, \psi)_{|x=x_0}$ implies that $P_{\psi,\mathfrak{p}}^\alpha \neq 0$. Theorem A then implies that the class $\kappa_\psi \in H^1_{\mathrm{f}}(H, V_p(E)(M))^{\psi\oplus\bar{\psi}}$ is non-trivial.

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