ELLIPTIC CURVES OF RANK TWO AND GENERALIZED KATO CLASSES

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In memory of Robert Coleman

ABSTRACT. Heegner points play an outstanding role in the study of the Birch and Swinnerton-Dyer conjecture, providing canonical Mordell-Weil generators whose heights encode first derivatives of the associated Hasse-Weil L-series. Yet the fruitful connection between Heegner points and L-series also accounts for their main limitation, namely, that they are torsion in (analytic) rank > 1. This partly expository article discusses the *generalised Kato classes* introduced in [BDR2] and [DR2], stressing their analogy with Heegner points but explaining why they are expected to give non-trivial, canonical elements of the idoneous Selmer group in settings where the classical L-function (of Hasse-Weil-Artin type) that governs their behaviour has a *double zero* at the center.

The generalized Kato class denoted $\kappa(f,g,h)$ is associated to a triple (f,g,h) consisting of an eigenform f of weight two and classical p-stabilised eigenforms g and h of weight one, corresponding to odd two-dimensional Artin representations V_g and V_h of $\mathrm{Gal}\,(H/\mathbb{Q})$ with p-adic coefficients for a suitable number field H. This class is germane to the Birch and Swinnerton-Dyer conjecture over H for the modular abelian variety E over $\mathbb{Q}$ attached to f. One of the main results of [BDR2] and [DR2] is that $\kappa(f,g,h)$ lies in the pro-p Selmer group of E over H precisely when $L(E,V_{gh},1)=0$, where $L(E,V_{gh},s)$ is the L-function of E twisted by $V_{gh}:=V_g\otimes V_h$. In the setting of interest, parity considerations imply that $L(E,V_{gh},s)$ vanishes to even order at $s=1$, and the Selmer class $\kappa(f,g,h)$ is expected to be trivial when $\mathrm{ord}_{s=1}L(E,V_{gh},s)>2$. The main new contribution of this article is a conjecture expressing $\kappa(f,g,h)$ as a canonical point in $(E(H)\otimes V_{gh})^{G_{\mathbb{Q}}}$ when $\mathrm{ord}_{s=1}L(E,V_{gh},s)=2$. This conjecture strengthens and refines the main conjecture of [DLR1], and supplies a framework for understanding the results of [DLR1], [BDR2] and [DR2].

CONTENTS

1. BACKGROUND AND MOTIVATION

The theme of modularity of p-adic Galois representations has occupied center stage in number theory for the last several decades, and Robert Coleman has been a major figure in many of its key developments, notably through the theory of *Coleman families* of p-adic modular forms and of the

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Coleman-Mazur eigencurve parametrising these families and their associated Galois representations. By way of background and motivation, this section explains how much of the progress achieved on the Birch and Swinnerton-Dyer conjecture, including the results of [DLR1], [BDR2] and [DR2], can be viewed as part of the larger program of understanding the modularity of (*non-semisimple*) p-adic Galois representations.

One of the most celebrated modularity results is the statement that all elliptic curves over $\mathbb{Q}$ arise as quotients of suitable modular curves: more precisely, that an elliptic curve E over $\mathbb{Q}$ of conductor N is equipped with a surjective parameterization

$$\pi_E : X_0(N) \longrightarrow E, \tag{1}$$

where $X_0(N)$ is the modular curve attached to Hecke's congruence subgroup $\Gamma_0(N)$. This was proved in [Wi95], [TW], and [BCDT] by showing that the p-adic representation

$$H^1(E) := H^1_{\text{et}}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(1) = (\lim_{\leftarrow, n} E[p^n]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

of $G_{\mathbb{Q}} := \text{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$ arises as a quotient of the étale cohomology group[1]

$$H^1(X_0(N)) := H^1_{\text{et}}(X_0(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(1)).$$

The existence of a Galois-equivariant projection

$$\pi_E : H^1(X_0(N)) \longrightarrow H^1(E) \tag{2}$$

is the real content of the breakthrough in [Wi95] and [TW], the ostensibly stronger geometric version (1) being deduced from it by invoking the Tate conjecture for curves[2].

Let E' be an open subvariety of E, i.e., the complement of a zero-dimensional subvariety Σ of E over $\mathbb{Q}$. The p-adic Galois representation $H^1(E')$ sits in the middle of the short exact excision sequence

$$0 \longrightarrow H^1(E) \longrightarrow H^1(E') \longrightarrow H^0(\Sigma)_0 \longrightarrow 0$$

of étale cohomology groups, where the subscript of 0 denotes the degree 0 elements of $H^0(\Sigma)$. By analogy with (2), the curve E' is (provisionally) said to be *modular* if $H^1(E')$ arises as a subquotient of $H^1(Y)$, where Y is an *open sub-Shimura variety* of $X_0(N)$—the latter being defined, in the style of La Palice, as the complement of a closed sub-Shimura variety.

To completely describe the open sub-Shimura varieties of the modular curve $X_0(N)$ over $\mathbb{Q}$, note that the latter is the coarse moduli space of elliptic curves A with a marked subgroup scheme of order N, and that its closed sub-Shimura varieties are obtained by imposing additional endomorphism rings, which can only be equal to orders in quadratic imaginary fields. Given such an order $\mathcal{O} \subset K$, the associated closed sub-Shimura variety $\Sigma_{\mathcal{O}} \subset X_0(N)$ consists of *CM* points for $\mathcal{O}$, and is the coarse moduli space of elliptic curves A with level N structure equipped with an optimal embedding $\iota : \mathcal{O} \longrightarrow \text{End}(A)$ (respecting the level structure) and acting in a prescribed way on the cotangent space of A. By the theory of complex multiplication, the 0-dimensional variety $\Sigma_{\mathcal{O}}$ is isomorphic over K (at least, when the discriminant of $\mathcal{O}$ is prime to N) to $\phi_K(N)$ copies of spec$(H_{\mathcal{O}})$, where $\phi_K(N)$ is the number of primitive ideals of K of norm N and $H_{\mathcal{O}}$ is the ring class field of K attached to $\mathcal{O}$, whose Galois group over K is canonically identified with the Picard group of $\mathcal{O}$ via global class field theory.

The complements

$$Y_{\mathcal{O}}(N) := X_0(N) - \Sigma_{\mathcal{O}}$$

thus provide an exhaustive list of the open sub-Shimura varieties of $X_0(N)$. Given the modularity of E, the modularity of E' amounts to the existence of a Galois-equivariant inclusion

$$i : H^0(\Sigma)_0 \longrightarrow H^0(\Sigma_{\mathcal{O}})_0$$

[1]The systematic shorthand $H^i(X) := H^i_{\text{et}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(i))$ for any variety X over $\mathbb{Q}$ is adopted henceforth to lighten the notations.

[2]Subsequently, (2) has been generalised to a host of other p-adic Galois representations, while analogues of (1) remain unavailable in all but the simplest geometric settings.

for suitable $\mathcal{O}$, realising $H^1(E')$ as a subquotient of $H^1(Y_{\mathcal{O}}(N))$ via the pushforward under π_E and the pullback under ι of the first row in the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(X_0(N)) & \longrightarrow & H^1(Y_{\mathcal{O}}(N)) & \longrightarrow & H^0(\Sigma_{\mathcal{O}})_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_E} & & \vdots{\scriptstyle ?} & & \uparrow{\scriptstyle i} & & \\ 0 & \longrightarrow & H^1(E) & \longrightarrow & H^1(E') & \longrightarrow & H^0(\Sigma)_0 & \longrightarrow & 0. \end{array} \tag{3}$$

Consider the simplest non-trivial setting where $\Sigma = \{P_1, P_2\} \subset E(\mathbb{Q})$ consists of two points defined over $\mathbb{Q}$, so that $H^0(\Sigma)_0 = \mathbb{Q}_p$ with trivial Galois action. The resulting extension

$$0 \longrightarrow H^1(E) \longrightarrow H^1(E') \longrightarrow \mathbb{Q}_p \longrightarrow 0 \tag{4}$$

encodes the image of the point $P_2 - P_1 \in E(\mathbb{Q})$ under the connecting homomorphism

$$\delta : E(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, H^1(E)) := \mathrm{Ext}^1_{G_{\mathbb{Q}}}(\mathbb{Q}_p, H^1(E))$$

of Kummer theory, where the Ext group is taken in the category of continuous p-adic representations of $G_{\mathbb{Q}}$. The following statement, which gives a "modularity criterion" for E' and encapsulates many of the deepest theorems on the Birch and Swinnerton-Dyer conjecture obtained in the last decades, is of course expected to hold for all elliptic curves E, but the reader is cautioned that the proof of the implication (d) $\Rightarrow$ (a) currently requires that E be a semistable elliptic curve having at least one odd prime of nonsplit multiplicative reduction or at least two odd primes of split multiplicative reduction.

Theorem 1.1. *Assume that the point $P_2 - P_1$ is of infinite order in $E(\mathbb{Q})$. Then the following are equivalent:*

(a) *The curve $E' = E \setminus \{P_1, P_2\}$ is modular;*
(b) *the Hasse-Weil L-series $L(E, s)$ has a simple zero at $s = 1$;*
(c) *the point $P_2 - P_1$ generates $E(\mathbb{Q}) \otimes \mathbb{Q}$ and Ш$(E/\mathbb{Q})$ is finite;*
(d) *for all primes p, the group $\mathrm{Ext}^1_{\mathrm{fin}}(\mathbb{Q}_p, H^1(E))$ of extensions of p-adic representations of the Galois group of $\mathbb{Q}$ that are cristalline at p is one-dimensional over $\mathbb{Q}_p$.*

Sketch of proof. The modularity of E' amounts to the statement that there exists an order $\mathcal{O}$ in an imaginary quadratic field K such that the extension (4) can be obtained as the pullback of (3) via an inclusion $i : \mathbb{Q}_p \longrightarrow H^0(\Sigma_{\mathcal{O}})^{G_{\mathbb{Q}}}$, whose image contains a degree 0 divisor

$$D_K \in \mathrm{Div}^0(\Sigma_{\mathcal{O}})^{G_{\mathbb{Q}}} \subset \mathrm{Div}^0(X_0(N))(\mathbb{Q}).$$

This means that the point $P_1 - P_2 \in E(\mathbb{Q})$ is a non-zero multiple of the *Heegner point* $P_{E,K} := \pi_E(D_K)$.

The implication (a)$\Rightarrow$(b) therefore follows from the Gross-Zagier formula [GZ86] expressing the height of $P_{E,K}$ as a non-zero multiple of

$$L'(E/K, 1) = L'(E, 1) \cdot L(E^K, 1),$$

where E^K is the quadratic twist of E by K. The existence of a suitable K for which $L(E^K, 1) \neq 0$ follows from a non-vanishing result of Waldspurger or can be deduced from analytic number theory techniques (cf. [MM]).

The implication (b) $\Rightarrow$ (c) was subsequently proved by Kolyvagin [Ko89], who parlayed the non-triviality of $P_{E,K}$ into a bound on the Mordell-Weil rank and the Selmer group of E over K.

The implication (c) $\Rightarrow$ (d) is a direct consequence of the definitions: in fact (d) is ostensibly weaker than (c), Selmer groups being less subtle to control than Mordell-Weil and Shafarevich-Tate groups.

The striking implication (d) $\Rightarrow$ (a) follows from Skinner's "converse of the Gross-Zagier-Kolyvagin Theorem" [Sk15]. This last step is the most recent, and combines several new ingredients: the powerful techniques developed by Skinner and Urban to prove the Iwasawa-Greenberg Main Conjecture for elliptic curves over $\mathbb{Q}$ [SU14], an important variant explored by Xin Wan in his PhD thesis [Wa], and the p-adic analogue of [GZ86] formulated and proved in [BDP13].

More precisely, choose a prime $p \geq 5$ of good ordinary reduction for E such that $E[p]$ is an irreducible $G_{\mathbb{Q}}$-representation and the image of the restriction map $\mathrm{Sel}_p(E) \longrightarrow E(\mathbb{Q}_p)/pE(\mathbb{Q}_p)$ does not lie in the image of $E(\mathbb{Q}_p)[p]$. A result of Waldspurger ensures the existence of an odd quadratic character χ such that $L(E, \chi, 1) \neq 0$, which can be chosen so that $\chi(2) = \chi(p) = 1$. Let K denote the imaginary quadratic field associated to χ. The p-adic Selmer group $\mathrm{Ext}^1_{K,\mathrm{fin}}(\mathbb{Q}_p, H^1(E))$ of E over K (defined as an Ext group in the category of cristalline representations of G_K) decomposes as a direct sum of eigenspaces

$$\mathrm{Ext}^1_{K,\mathrm{fin}}(\mathbb{Q}_p, H^1(E)) \simeq \mathrm{Ext}^1_{\mathrm{fin}}(\mathbb{Q}_p, H^1(E)) \oplus \mathrm{Ext}^1_{K,\mathrm{fin}}(\mathbb{Q}_p, H^1(E))^-$$

with respect to the action of complex conjugation. Because $L(E,\chi,1) \neq 0$, the results of Kolyvagin (or of Kato) imply the triviality of $\mathrm{Ext}^1_{K,\mathrm{fin}}(\mathbb{Q}_p, H^1(E))^-$. Assumption (d) therefore implies that $\mathrm{Ext}^1_{K,\mathrm{fin}}(\mathbb{Q}_p, H^1(E))$ is one-dimensional over $\mathbb{Q}_p$. One can then argue as in [Sk15]. Namely, the running hypotheses ensure that both Lemma 2.3.2 and Proposition 2.7.3 of loc.cit. apply, and hence, that a p-adic L-function of the type that occurs in [Wa] and [BDP13] (which interpolates critical values of the L-series of the Rankin convolution of the modular form f associated to E with suitable Hecke characters of K of higher infinity-type) does not vanish at the trivial point, which lies outside its region of classical interpolation. This in turn implies, in light of [Sk15, Corollary 2.6.2] resting on the variant of the Gross-Zagier formula of [BDP13], that the Heegner point $P_{E,K}$ has non-trivial p-adic formal group logarithm, and is therefore non-torsion. As already explained, the non-triviality of $P_{E,K}$ is equivalent to (a), and the implication $(d) \Rightarrow (a)$ follows. □

The Birch and Swinnerton-Dyer conjecture admits an extension to elliptic curves twisted by Artin representations which arises very naturally in the context of the modularity questions framed above. Let

$$\varrho : \mathrm{Gal}\,(H/\mathbb{Q}) \hookrightarrow \mathrm{Aut}(V_\varrho) \simeq \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$$

be an n-dimensional representation of the Galois group of a finite extension $H/\mathbb{Q}$, a so-called *Artin representation*, viewed as having coefficients in $\bar{\mathbb{Q}}_p$. The pair (E,ϱ) gives rise to the Hasse-Weil-Artin L-series

$$L(E,\varrho,s) := \prod_\ell \det(1 - \ell^{-s}(\mathrm{Fr}_\ell^{-1})_{(H^1(E)\otimes V_\varrho)^{I_\ell}})^{-1},$$

where the product is taken over the rational primes ℓ, the arithmetic frobenius element at ℓ is denoted by Fr_ℓ, and I_ℓ denotes the inertia group at ℓ. The equivariant Birch and Swinnerton-Dyer conjecture for E and ϱ, denoted $\mathrm{BSD}(E,\varrho)$, asserts that

$$\mathrm{ord}_{s=1} L(E,\varrho,s) = \dim_{\bar{\mathbb{Q}}_p} (E(H)\otimes V_\varrho)^{G_\mathbb{Q}}. \tag{5}$$

As a first step to understanding $\mathrm{BSD}(E,\varrho)$, it is natural to ask which $\kappa \in \mathrm{Ext}^1_{\mathrm{fin}}(V_\varrho, H^1(E))$ can be realised as a subquotient of a suitable $H^1(Y_\mathcal{O}(N))$. The Artin representation $H^0(\Sigma_\mathcal{O})_0$ which appears in the upper rightmost term of the diagram (3) is readily analyzed using the theory of complex multiplication. Namely, the slightly larger Artin representation $H^0(\Sigma_\mathcal{O})$ decomposes as a direct sum

$$H^0(\Sigma_\mathcal{O}) \otimes \bar{\mathbb{Q}}_p = \oplus_{j=1}^{\phi_K(N)} W_j, \qquad \text{where } W_j = \oplus_\psi V_j(\psi), \quad \text{with } V_j(\psi) \subset V_\psi := \mathrm{Ind}_K^\mathbb{Q}\psi.$$

In this equation, the second direct sum is taken over the non-trivial, $\bar{\mathbb{Q}}_p$-valued, finite order characters ψ of $\mathrm{Gal}\,(H_\mathcal{O}/K)$ modulo the involution $\psi \mapsto \psi^{-1}$, and $V_j(\psi)$ is a non-trivial irreducible constituent of the two-dimensional representation V_ψ obtained by inducing the Galois character ψ from G_K to $G_\mathbb{Q}$. The representation V_ψ is irreducible precisely when $\psi \neq \psi^{-1}$, and in this case a non-trivial class $\kappa \in \mathrm{Ext}^1_{\mathrm{fin}}(V_\psi, H^1(E)\otimes\bar{\mathbb{Q}}_p)$ is expected to be modular if and only if (any of) the analogues of conditions (b)-(d) of Theorem 1.1 are satisfied, namely:

- (b') The Hasse-Weil-Artin L-series $L(E,V_\psi,s)$ has a simple zero at $s=1$;
- (c') the representation V_ψ occurs with multiplicity one in $E(H)\otimes\bar{\mathbb{Q}}_p$, and the V_ψ-isotypic component of the $Ш(E/H)$ is finite;
- (d') the group $\mathrm{Ext}^1_{\mathrm{fin}}(V_\psi, H^1(E)\otimes\bar{\mathbb{Q}}_p)$ is one-dimensional over $\bar{\mathbb{Q}}_p$, and generated by κ.

Although such a precise result does not seem to appear in the literature, all the ingredients needed to prove it seem to be available in principle.

The rather narrow notion of modularity described above has a few visible drawbacks:

(1) Very few Artin representations arise in the cohomology of the 0-dimensional Shimura varieties $\Sigma_\mathcal{O}$, which are not even rich enough to capture all of the irreducible two-dimensional Artin representations of $\mathbb{Q}$. The open Shimura varieties $Y_\mathcal{O}(N)$ thus appear to give no purchase on $\mathrm{BSD}(E,\varrho)$ when ϱ is not induced from a ring class character of an imaginary quadratic field.

(2) Theorem 1.1 suggests that the modularity of elements of $\mathrm{Ext}^1_{\mathrm{fin}}(V_\psi, H^1(E))$ is purely a "rank one phenomenon": if this Ext group has dimension > 1, none of its elements are expected to be realised in subquotients of any $H^1(Y_{\mathcal{O}}(N))$.

In order to relate a larger class of non-semisimple Galois representations to modular forms, it becomes desirable to relax the notion of modularity. One way in which one might try to do this is by replacing the curves $Y_{\mathcal{O}}(N)$ with more general "open Shimura varieties". These should include all the varieties whose cohomology (at least, after semisimplification) is directly related to automorphic forms via a suitable generalisation of the Eichler-Shimura congruence, and would eventually encompass the complements of sub-Shimura varieties in larger Shimura varieties, as well as Kuga-Sato varieties and other natural varieties fibered over Shimura varieties, the complements of Heegner cycles in such varieties, and so on. With this expanded notion of modularity, the program of characterising the non-semisimple Galois representations that are modular becomes richer and more subtle. See [BDP14] for a fragment of experimental mathematics that might be viewed as fitting into this program. The following question seems like it might repay further investigation, given the paucity of evidence, both theoretical and experimental, that has been gathered around it so far:

Question 1.2. Let V_1 and V_2 be Galois representations for which $\hom(V_1, V_2)$ is irreducible. Suppose that there is a non-trivial $\kappa \in \mathrm{Ext}^1_{\mathrm{fin}}(V_1, V_2)$ arising as a subquotient of the cohomology of an open Shimura variety. Is $\mathrm{Ext}^1_{\mathrm{fin}}(V_1, V_2)$ necessarily one-dimensional?

If the answer to this question were "yes", it would imply that the open curve $E - \{P_1, P_2\}$ discussed in Theorem 1.1 is *never* modular when $\mathrm{rank}(E(\mathbb{Q})) > 1$. (But see the inspiring article [NS], as well as the striking ongoing work of Zhiwei Yun and Wei Zhang in the function field case, for some tantalizing ideas in the opposite, more optimistic direction.)

A second idea for enlarging the class of p-adic Galois representations deemed to be modular is to allow *p-adic limits* of Galois representations arising in the cohomology of (open) Shimura varieties. This idea is very natural in light of the classical work of Deligne-Serre on Artin representations attached to weight one forms, whereby such Artin representations are obtained by piecing together the Galois representations attached to modular forms of higher weights which are realised in the cohomology of Kuga-Sato varieties. It is via this broader notion of modularity that all odd, irreducible two-dimensional Artin representations of $\mathbb{Q}$ can be related to modular forms. The idea of realising automorphic Galois representations as p-adic limits has become pervasive in the subject, and led to important advances: for example, it plays a key role in the recent construction [HLTT] by Harris, Lan, Taylor, and Thorne of Galois representations attached to non-self-dual automorphic forms on GL_n. Even more germane to this article, p-adic limits of automorphic Galois representations appear to capture non-trivial extension classes going beyond settings of "multiplicity one", as is illustrated by the following theorem of Skinner and Urban [SU06, Thm. B]:

Theorem 1.3. *Let E be an elliptic curve over $\mathbb{Q}$. If $L(E, s)$ vanishes to even order ≥ 2 at $s = 1$, then the Selmer group $\mathrm{Ext}^1_{\mathrm{fin}}(\mathbb{Q}_p, H^1(E))$ of E contains at least two linearly independent modular classes.*

The modular classes in this theorem are constructed as p-adic limits of geometric Galois representations in the cohomology of Shimura varieties associated to the unitary group $U(2, 2)$. Although these geometric Galois representations are believed to be semisimple, Theorem 1.3 rests on the fact that this feature need not persist in the limit.

The primary goal of this article is to discuss a different approach for constructing *canonical* extension classes of ϱ by $H^1(E)$ for a large class of self-dual Artin representations ϱ of dimension 4 (and their lower-dimensional subrepresentations, in case ϱ is reducible) arising as the tensor product $\varrho = \varrho_1 \otimes \varrho_2$ of a pair of odd, two-dimensional Artin representations. The construction of these classes is one of the main results of [DR2] (resp. [BDR2]) when both ϱ_1 and ϱ_2 are irreducible (resp. when exactly one of ϱ_1 and ϱ_2 is irreducible), and is based on p-adic limits of *non-semisimple*, but "geometrically modular" Galois representations. These limit classes are referred to as *generalised Kato classes* because their construction is inspired by the seminal work [Ka98] of Kato (cf. also [Sc], [BD14]) on $\mathrm{BSD}(E, \chi)$ for χ a Dirichlet character. Like Heegner points in the setting of $\mathrm{BSD}(E, V_\psi)$, generalised Kato classes enjoy close relations to (p-adic) Hasse-Weil-Artin L-functions attached to E and ϱ, but unlike Heegner points, they are expected to generate a non-trivial subgroup of the Selmer group attached to E and

ϱ precisely when $\mathrm{ord}_{s=1} L(E, \varrho, s) = 2$. The formulae of [DR2] (cf. Corollary 3.6 below) relating the linear independence of two generalised Kato classes to the non-vanishing of certain p-adic L-series can thus be regarded as a p-adic Gross-Zagier formula "in analytic rank two".

The main new contribution of this article is a conjecture expressing the same generalised Kato classes as canonical elements in $(E(H) \otimes V_\varrho)^{G_\mathbb{Q}}$ when this latter space is two-dimensional. This conjecture strengthens and refines the "elliptic Stark conjecture" of [DLR1], and provides a framework for understanding the results of [DLR1], [BDR2] and [DR2]. The settings in which ϱ is reducible often take on special arithmetic interest and are described in detail in the last chapter.

2. Hida families and periods for weight one forms

This section provides background on certain canonical structures associated to a weight one form g, arising from the Hida families specialising in weight one to (a p-stabilisation of) g. These are important for the conjectures of Section 3.4, but Section 2 can be skipped on a first reading by the reader wishing to get a quick feeling for the generalised Kato classes described in Sections 3.1 and 3.2. On the other hand, it is also worth noting that Section 2 is entirely self-contained. Conjecture 2.1, which can be viewed as a p-adic analogue of the Stark conjecture for the adjoint of the Galois representation attached to a weight one form, appears to be new and may be of independent interest.

Let $g \in S_1(N, \chi)$ be a newform of weight one and level N with Fourier coefficients in a field L, and let

$$\varrho : G_\mathbb{Q} \longrightarrow \mathrm{Aut}(V) \simeq \mathrm{GL}_2(L)$$

be the Artin representation associated to it by the construction of Deligne and Serre. We view ϱ as acting on a two-dimensional L-vector space V, where $L \subset \mathbb{C}$ can be chosen to be contained in a cyclotomic field.

Let H be the number field cut out by ϱ, so that ϱ factors through $\mathrm{Gal}\,(H/\mathbb{Q})$. Fix a rational prime p and choose a prime $\mathfrak{p}$ of H above p. The latter determines a canonical inclusion

$$H \subset H_p \subset \bar{\mathbb{Q}}_p$$

of H in its completion H_p at $\mathfrak{p}$. Assume that the pair (ϱ, p) satisfies the following conditions:

- (I) The prime p splits completely in $L/\mathbb{Q}$, so that L is equipped with an embedding into $\mathbb{Q}_p$ which will be fixed from now on. This assumption, which is made solely to lighten the notations and could easily be dispensed with, allows ϱ to be viewed as a $\mathbb{Q}_p$-linear representation via the natural action of $G_\mathbb{Q}$ on the $\mathbb{Q}_p$-vector space $V \otimes_L \mathbb{Q}_p$.
- (II) The representation V is unramified at p. There is then a well defined arithmetic frobenius element
$$\mathrm{Fr}_p \in \mathrm{Gal}\,(H/\mathbb{Q})$$
acting canonically on V, and the characteristic polynomial of $\varrho(\mathrm{Fr}_p)$ is equal to the Hecke polynomial
$$x^2 - a_p(g)x + \chi(p) =: (x - \alpha_g)(x - \beta_g)$$
attached to g.
- (III) The modular form g is *regular* at p, i.e., $\alpha_g \neq \beta_g$. After possibly enlarging L, it may also be assumed that this coefficient field contains the roots of unity α_g and β_g.
- (IV) The representation ϱ_g is not induced from a character of a real quadratic field K in which the prime p splits. The rationale for this condition, which seems to be essential for a number of the constructions and conjectures proposed in this paper, is explained in [DLR1, §1.1].

The p-stabilisations of g at p are the normalised eigenforms of weight one with Fourier coefficients in L defined by

$$g_\alpha := g(z) - \beta_g g(pz), \qquad g_\beta := g(z) - \alpha_g g(pz).$$

They are eigenvectors for the U_p-operator satisfying

$$U_p g_\alpha = \alpha_g g_\alpha, \qquad U_p g_\beta = \beta_g g_\beta.$$

The Artin representation V decomposes naturally as a direct sum

$$V = V^\alpha \oplus V^\beta$$

into one-dimensional eigenspaces for Fr_p, with eigenvalues α_g and β_g respectively.

By a theorem of Hida, there exists a finite flat extension Λ_g of the Iwasawa algebra Λ and a Hida family $\underline{g} \in \Lambda_{\underline{g}}[[q]]$ of tame level N and tame character χ passing through the p-stabilised weight one eigenform g_α. When g is cuspidal, the regularity hypothesis imposed on g implies that such a Hida family is unique, thanks to a recent result of Bellaïche and Dimitrov [BeDi].

The Hida family $\underline{g}$ comes equipped with the following canonical structures:

(a) There is a locally free Λ_g-module $\mathbb{V}_g$ of rank two, affording Hida's ordinary Λ-adic Galois representation

$$\varrho_{\underline{g}} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\Lambda_g}(\mathbb{V}_g)$$

which is realised in the inverse limit of ordinary étale cohomology groups associated to the tower $X_1(Np^r)$ of modular curves. This representation interpolates the Galois representations associated by Deligne to the classical specializations of $\underline{g}$.

(b) The restriction of $\mathbb{V}_g$ to $G_{\mathbb{Q}_p}$ admits a stable filtration

$$0 \longrightarrow \mathbb{U}_g \longrightarrow \mathbb{V}_g \longrightarrow \mathbb{W}_g \longrightarrow 0,$$

where both $\mathbb{U}_g$ and $\mathbb{W}_g$ are flat $\Lambda_{\underline{g}}[G_{\mathbb{Q}_p}]$-modules that are locally free of rank one over $\Lambda_{\underline{g}}$, and the quotient $\mathbb{W}_g$ is unramified, with Fr_p acting on $\mathbb{W}_g$ as multiplication by the p-th Fourier coefficient $a_p(\underline{g})$.

(c) Let $\mathbb{Q}_p^{\mathrm{nr}}$ denote the maximal unramified extension of $\mathbb{Q}_p$ and let $\widehat{\mathbb{Q}}_p^{\mathrm{nr}}$ denote its p-adic completion. In [Oh95], Ohta constructs a canonical Λ_g-adic period

$$\omega_{\underline{g}} \in D(\mathbb{W}_g) := (\widehat{\mathbb{Q}}_p^{\mathrm{nr}} \hat{\otimes} \mathbb{W}_g)^{G_{\mathbb{Q}_p}},$$

corresponding to the normalised Λ-adic eigenform $\underline{g}$ under the isomorphism in Theorem (A) of the introduction of [Oh95].

(d) There is a natural perfect Galois-equivariant duality, given in Theorem (B) of the introduction of [Oh95],

$$\mathbb{U}_g \times \mathbb{W}_g \longrightarrow \Lambda_g(\det(\varrho_{\underline{g}})),$$

where $G_{\mathbb{Q}}$ acts on the module Λ_g of the right-hand side via the determinant of $\varrho_{\underline{g}}$.

Let

$$y_g : \Lambda_g \longrightarrow \mathbb{Q}_p$$

be the specialization map attached to the p-stabilised weight one form g_α. By specialising the structures above attached to $\underline{g}$ via the map y_g, we obtain

(a') A *non-canonical* isomorphism of $\mathbb{Q}_p[G_{\mathbb{Q}}]$-modules

$$\Phi_{g_\alpha} : V_g := \mathbb{V}_g \otimes_{y_g} \mathbb{Q}_p \xrightarrow{\sim} V \otimes_L \mathbb{Q}_p.$$

(b') A non-trivial $G_{\mathbb{Q}_p}$-stable filtration

$$0 \longrightarrow U_g \longrightarrow V_g \longrightarrow W_g \longrightarrow 0$$

of V_g by one-dimensional subspaces, where $U_g := \mathbb{U}_g \otimes_{y_g} \mathbb{Q}_p$ and $W_g := \mathbb{W}_g \otimes_{y_g} \mathbb{Q}_p$. The Frobenius element Fr_p acts on W_g and U_g as multiplication by α_g and β_g respectively. Since these eigenvalues are assumed to be distinct, the exact sequence above splits canonically, leading to the identifications

$$U_g = V_g^\beta, \qquad W_g = V_g^\alpha, \qquad V_g = U_g \oplus W_g = V_g^\beta \oplus V_g^\alpha.$$

(c') Specialising Ohta's period leads to a canonical element

$$\omega_{g_\alpha} := y_g(\omega_{\underline{g}}) \in D(V_g^\alpha) := (\mathbb{Q}_p^{\mathrm{nr}} \otimes V_g^\alpha)^{G_{\mathbb{Q}_p}} = (H_p \otimes V_g^\alpha)^{G_{\mathbb{Q}_p}}. \tag{6}$$

(d') The duality in (d) above specialises via y_g to a canonical pairing of $\mathbb{Q}_p$-vector spaces

$$\langle\ ,\rangle : V_g^\beta \times V_g^\alpha \longrightarrow \mathbb{Q}_p(\chi),$$

which induces a pairing by functoriality (denoted by the same symbol by a slight abuse of notation):

$$\langle\ ,\ \rangle : D(V_g^\beta) \times D(V_g^\alpha) \longrightarrow D(\mathbb{Q}_p(\chi)). \tag{7}$$

When this pairing is perfect, it can be used to define a period $\eta_{g_\alpha} \in D(V_g^\beta)$, as the unique element satisfying

$$\langle \eta_{g_\alpha}, \omega_{g_\alpha} \rangle = \mathfrak{g}(\chi) \otimes 1,$$

where

$$\mathfrak{g}(\chi) := \sum_{j=1}^{\mathfrak{f}_\chi)} \chi(j) e^{2\pi i j/\mathfrak{f}_\chi}$$

is the Gauss sum attached to the Dirichlet character χ, viewed as an element of H_p by assigning an $\mathfrak{f}_\chi$-th root of unity in H_p to the complex number $e^{2\pi i/\mathfrak{f}_\chi}$.

Making use of the above arsenal we now turn to introduce certain p-adic periods associated to g and the choice of a L-structure on V_g. We assume for simplicity in the sequel that g is a *cusp* form, and thus V_g is irreducible.

Fix a $G_{\mathbb{Q}}$-equivariant isomorphism $j_g : V \otimes_L \mathbb{Q}_p \longrightarrow V_g$ and let $V_g^L := j_g(V)$ denote the associated L-rational structure on V_g, which by Schur's lemma is well-defined up to scaling by $\mathbb{Q}_p^\times$. Since j_g induces isomorphisms $V_g^\alpha \simeq V^\alpha \otimes_L \mathbb{Q}_p$ and $V_g^\beta \simeq V^\beta \otimes_L \mathbb{Q}_p$, we may choose L-bases v_g^α and v_g^β for $V_g^L \cap V_g^\alpha$ and $V_g^L \cap V_g^\beta$ respectively, so that

$$V_g^L \cap V_g^\alpha = \langle v_g^\alpha \rangle_L \quad \text{and} \quad V_g^L \cap V_g^\beta = \langle v_g^\beta \rangle_L.$$

Define p-adic periods

$$\Omega_{g_\alpha} = \Omega_{g_\alpha}(V_g^L) \in H_p^{\mathrm{Fr}_p = \alpha_g^{-1}}, \qquad \Xi_{g_\alpha} = \Xi_{g_\alpha}(V_g^L) \in H_p^{\mathrm{Fr}_p = \beta_g^{-1}} \tag{8}$$

by setting

$$\Omega_{g_\alpha} \otimes v_g^\alpha = \omega_{g_\alpha}, \qquad \Xi_{g_\alpha} \otimes v_g^\beta = \eta_{g_\alpha}.$$

These periods depend on the choice of the basis (v_g^α, v_g^β) for V_g^L , but only up to multiplication by $L^\times$.

Furthermore, for all $\mu \in \mathbb{Q}_p^\times$,

$$\Omega_{g_\alpha}(\mu V_g^L) = \mu^{-1} \cdot \Omega_{g_\alpha}(V_g^L), \qquad \Xi_{g_\alpha}(\mu \cdot V_g^L) = \mu^{-1} \cdot \Xi_{g_\alpha}(V_g^L).$$

It follows that the ratio

$$\mathcal{L}_{g_\alpha} := \frac{\Omega_{g_\alpha}}{\Xi_{g_\alpha}} \in (H_p)^{\mathrm{Fr}_p = \frac{\beta_g}{\alpha_g}} \tag{9}$$

is a number in $H_p^\times$ that is well-defined up to multiplication by elements in $L^\times$.

This expression is a canonical p-adic period attached to the eigenform g_α, and can be viewed as a p-adic avatar of the Petersson norm of g.

Definition 1.8 of [DLR1, §1.2] associates in many cases a canonical *p-adic Stark unit* u_{g_α} attached to g_α as follows. Let $V_{\mathrm{ad}} := \mathrm{End}_0(V)$ be the three dimensional adjoint representation attached to V consisting of trace zero endomorphisms of V. Since complex conjugation acts with eigenvalues -1, -1 and 1 on V_{ad}, it follows that $\hom_{G_{\mathbb{Q}}}(V_{\mathrm{ad}}, (\mathcal{O}_H^\times)_L) = L \cdot \varphi_{\mathrm{ad}}$ for a suitable generator φ_{ad}. If one further assumes that $\alpha_g \neq \pm\beta_g$, then the subspace of V_{ad} on which Fr_p acts as multiplication by $\frac{\beta_g}{\alpha_g}$ is one-dimensional; after choosing an L-basis v_{g_α} for it, one lets

$$u_{g_\alpha} := \varphi_{\mathrm{ad}}(v_{g_\alpha}). \tag{10}$$

This element is well defined up to multiplication by $L^\times$, and hypothesis (IV) above guarantees that it is a non-zero vector of the V_{ad}-isotypic subspace of $(\mathcal{O}_H^\times)_L$. See [DLR1, §1.2] for further details.

Conjecture 2.1. *The period in* (9) *satisfies*

$$\mathcal{L}_{g_\alpha} = \log_p(u_{g_\alpha}) \pmod{L^\times}.$$

Remark 2.2. It would be interesting to test this conjecture numerically. To the extent that $\mathcal{L}_{g_\alpha}$ is a p-adic avatar of the Petersson norm of g, Conjecture 2.1 can be viewed as a p-adic analogue of the Stark conjecture for the L-function attached to the adjoint of g, in the form in which it is illustrated, for example, in the concluding paragraphs of [St75].

3. Generalized Kato classes

3.1. **Definition.** Let E be an elliptic curve over $\mathbb{Q}$ and let

$$\varrho_1, \varrho_2 : \mathrm{Gal}\,(H/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(L)$$

be odd, irreducible two-dimensional Artin representations of $G_{\mathbb{Q}}$ satisfying

$$\chi := \det(\varrho_1) = \det(\varrho_2)^{-1},$$

where L and H are finite extensions of $\mathbb{Q}$ (and L is chosen, as before, to be contained in a cyclotomic field). Let V_1 and V_2 be $L[\mathrm{Gal}\,(H/\mathbb{Q}]$-modules which are two-dimensional over L and realise ϱ_1 and ϱ_2 respectively. Observe that

$$V_{12} := V_1 \otimes_L V_2$$

is a four-dimensional L-linear representation of $\mathrm{Gal}\,(H/\mathbb{Q})$ with real traces, i.e., it is isomorphic to its contragredient representation.

Fix a rational prime p and continue to assume that hypotheses (I-IV) of the previous section hold for both the pairs (ϱ_1, p) and (ϱ_2, p).

The progress in modularity realised over the last two decades implies the existence of cusp forms f, g, and h attached to E, ϱ_1 and ϱ_2 respectively, whose associated p-adic representations, denoted V_f, V_g and V_h, satisfy

$$H^1(E) = V_f, \qquad V_1 \otimes_L \mathbb{Q}_p \simeq V_g, \qquad V_2 \otimes_L \mathbb{Q}_p \simeq V_h. \tag{11}$$

It is important to keep in mind that the last two isomorphisms, whose existence is proved by comparing traces on both sides, are only well-defined up to multiplication by a scalar in $\mathbb{Q}_p^\times$ (by Schur's lemma), and that the $\mathbb{Q}_p$-vector spaces V_g and V_h therefore admit no natural L-rational structure. Let

$$V_{gh} := V_g \otimes V_h, \qquad V_{fgh} := V_f \otimes V_{gh}$$

denote the tensor products of $\mathbb{Q}_p$-linear representations of $G_{\mathbb{Q}}$, and write

$$j_{gh} : V_{12} \otimes_L \mathbb{Q}_p \longrightarrow V_{gh} \tag{12}$$

for the isomorphism induced from (11). Let

$$V_{gh}^L := j_{gh}(V_{12}) \tag{13}$$

denote the resulting $G_{\mathbb{Q}}$-stable L-rational structure on V_{gh}, which is well-defined up to multiplication by a scalar in $\mathbb{Q}_p^\times$, even when V_{gh} is reducible, because V_g and V_h themselves are irreducible.

Because of (11), the Hasse-Weil and Artin L-functions attached to E, ϱ_1 and ϱ_2 are equal to the Hecke L-functions attached to f, g and h respectively:

$$L(E, s) = L(f, s), \qquad L(\varrho_1, s) = L(g, s), \qquad L(\varrho_2, s) = L(h, s),$$

and therefore admit functional equations and analytic continuations to the entire complex plane. By the theory of Rankin-Selberg and Garrett, the same is true of the degree 8 L-function $L(V_{fgh}, s)$ attached to the convolution of f, g and h.

Let $N = \mathrm{lcm}(N_f, N_g, N_h)$ denote the least common multiple of the conductors of E, ϱ_1 and ϱ_2 and assume further that p does not divide N. As in the previous section, let

$$x^2 - a_p(g)x + \chi(p) =: (x - \alpha_g)(x - \beta_g), \qquad x^2 - a_p(h)x + \chi^{-1}(p) =: (x - \alpha_h)(x - \beta_h)$$

be the Hecke polynomials at p attached to g and h respectively, and assume that the coefficient field L contains the roots of unity α_g, β_g, α_h and β_h. Denote as before by

$$g_\alpha := g(z) - \beta_g g(pz), \quad g_\beta := g(z) - \alpha_g g(pz), \qquad h_\alpha := h(z) - \beta_h h(pz), \quad h_\beta := h(z) - \alpha_h h(pz)$$

the relevant p-stabilisations of g and h.

One of the running assumptions of [DR2] that is also enforced in this article is that the Artin conductor of V_{gh} is relatively prime to the conductor of E. Under this assumption, [Pr, Theorem 1.4] implies that the local root numbers that govern the sign in the functional equation for $L(E, V_{gh}, s)$ are equal to 1 at all places of $\mathbb{Q}$, and the Hasse-Weil-Artin L-function attached to E and V_{gh} therefore vanishes to *even order* at the symmetry point $s = 1$ for its functional equation.

The article [DR2] describes the construction of four canonical (a priori non-trivial, and distinct) *generalised Kato classes*

$$\kappa(f,g_\alpha,h_\alpha),\ \kappa(f,g_\alpha,h_\beta),\ \kappa(f,g_\beta,h_\alpha),\ \kappa(f,g_\beta,h_\beta) \quad\in\quad H^1(\mathbb{Q},V_{fgh}). \tag{14}$$

These classes are essentially obtained as *p-adic limits*

$$\kappa(f,g_\alpha,h_\alpha) := \lim_{k\longrightarrow 1}\kappa(f,g_k,h_k), \tag{15}$$

as (g_k,h_k) range over the classical specialisations of weight $k\geq 2$ of Hida families $\underline{g}$ and $\underline{h}$ specialising to g_α and h_α respectively in weight one, and $\kappa(f,g_k,h_k)$ arises from a geometric construction whereby it is realised in the p-adic étale cohomology of some (open) variety over $\mathbb{Q}$.

More precisely, let $V_f(N)$ denote the f-isotypic component of $H^1(X_0(N))$, which is (non-canonically) isomorphic to a finite number of copies of V_f, indexed by the positive divisors of N/N_f. Let $V_g(N)$ and $V_h(N)$ denote the similar spaces occurring as the weight one specialisations of the $\underline{g}$ and $\underline{h}$-isotypic parts of the inverse limits of the ordinary quotients of $H^1(X_1(Np^s))$, which are abstractly isomorphic to a direct sum of finitely many copies of V_g and V_h respectively, endowed with all the structures described in (a')-(d') of Section 2. The classes in (14) of [DR2] take values in the Galois representation

$$V_{fgh}(N) = V_f(N)\otimes V_g(N)\otimes V_h(N)$$

and the classes of (14) are obtained by applying to them a suitable surjective $G_\mathbb{Q}$-equivariant projection

$$\pi : V_{fgh}(N)\longrightarrow V_{fgh} \tag{16}$$

compatible with the L-structure, filtration, Ohta periods and dualities described in (a')-(d'). The dependence of $\kappa(f,g_\alpha,h_\alpha)$ on the choice of π is supressed from the notations but should be kept in mind.

The generalised Kato classes belong to the global cohomology group $H^1(\mathbb{Q},V_{fgh}) = \mathrm{Ext}^1_{G_\mathbb{Q}}(\mathbb{Q}_p,V_{fgh})$, where $\mathbb{Q}_p$ stands for the one-dimensional p-adic representation of $G_\mathbb{Q}$ with trivial action and the Ext group is taken in the category of finite dimensional $\mathbb{Q}_p$-vector spaces equipped with a continuous $G_\mathbb{Q}$-action (whose restriction to $G_{\mathbb{Q}_p}$ need not be de Rham).

When $\underline{g}$ and $\underline{h}$ are cuspidal Hida families, the "weight two" classes $\kappa(f,g_2,h_2)$ attached to weight two specialisations g_2 and h_2 of $\underline{g}$ and $\underline{h}$ are obtained from the p-adic étale Abel-Jacobi image of a *Gross-Kudla-Schoen diagonal cycles* in the Chow group of null-homologous codimension two cycles in the triple product of the modular curve $X_1(Np^s)$. It is worth noting that when passing from $k=1$ to $k>1$, the local root number at ∞ attached to $L(V_{fg_kh_k},s)$ changes sign (while the other root numbers stay the same), so that this L-function vanishes to *odd order* at its center. The presence of Gross-Kudla-Schoen diagonal cycles in this range is consistent with the Beilinson-Bloch conjecture for $L(V_{fg_kh_k},s)$ and in fact provides evidence for it. (Cf. the preprint [YZZ] of Yuan-Zhang-Zhang, where the case $k=2$ is studied.) The fact that the extension $\kappa(f,g_\alpha,h_\alpha)$ does not arise directly in p-adic étale cohomology, but only as a p-adic limit of geometric Galois representations, explains why $\kappa(f,g_\alpha,h_\alpha)$ need not be cristalline at p in general.

The analogy with the work of Kato [Ka98], [Sc] arises when the cuspidal Hida families $\underline{g}$ and $\underline{h}$ are replaced by Hida families of Eisenstein series. A global class $\kappa_{BK}(f,g_\alpha,h_\alpha)$, designated as the *Beilinson-Kato class* attached to (f,g_α,h_α), is then defined as in (15), but replacing the étale Abel-Jacobi images $\kappa(f,g_2,h_2)$ by p-adic étale regulators of *Beilinson elements* in the higher Chow group $K_2(X_1(Np^s)) = \mathrm{CH}^2(X_1(Np^s),2)$ attached to a pair of modular units whose logarithmic derivatives give rise to g_2 and h_2. We refer the reader to [BD14] for more details in this setting.

In the intermediate setting where exactly one of $\underline{g}$ and $\underline{h}$ (say, $\underline{g}$) is cuspidal (and thus $\underline{h}$ is Eisenstein), global classes $\kappa(f,g_2,h_2)$ can be constructed geometrically as p-adic étale regulators of suitable *Beilinson-Flach elements* in the higher Chow group $K_1(X_1(Np^s)^2) = \mathrm{CH}^2(X_1(Np^s)^2,1)$ attached to a modular unit whose logarithmic derivative is h_2. The limit cohomology class arising in (15) is then denoted $\kappa_{BF}(f,g_\alpha,h_\alpha)$, and called the *Beilinson-Flach class* attached to the triple (f,g_α,h_α). The Beilinson-Flach classes in p-adic families were introduced and studied in [BCDDPR], [BDR1] and [BDR2]. See also [LLZ] and [KLZ] for more recent work leading to substantial extensions and refinements of the results of loc.cit. in the setting of Beilinson-Flach elements.

The p-adic Selmer group

$$H^1_{\mathrm{fin}}(\mathbb{Q},V_{fgh}) := \mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p,V_{fgh}) \tag{17}$$

attached to E and V_{gh} are the group of extensions of $\mathbb{Q}_p$ by V_{fgh} in the category of $\mathbb{Q}_p$-linear representations of $G_{\mathbb{Q}}$ that are cristalline at p. Let

$$(E(H)_L \otimes V_{12})^{\mathrm{Gal}\,(H/\mathbb{Q})}, \qquad \text{where } E(H)_L := E(H) \otimes_{\mathbb{Z}} L$$

denote the ϱ_{12}-isotypic part of the Mordell-Weil group of E. It is a finite-dimensional L-vector space by the Mordell-Weil theorem, and is equipped with a natural inclusion

$$(E(H)_L \otimes V_{12})^{\mathrm{Gal}\,(H/\mathbb{Q})} \subset H^1_{\mathrm{fin}}(H, V_p(E) \otimes_{\mathbb{Q}_p} V_{gh})^{\mathrm{Gal}\,(H/\mathbb{Q})} = H^1_{\mathrm{fin}}(\mathbb{Q}, V_{fgh}) \tag{18}$$

induced from the connecting homomorphism δ of Kummer theory for $E(H)$ and the map j_{gh} of (13).

When $L(E, \varrho_{gh}, s)$ has a double zero at $s = 1$, Conjecture BSD(E, ϱ_{gh}) described in the previous section predicts that the associated Mordell-Weil and Selmer group in (18) are 2-dimensional over L and $\mathbb{Q}_p$ respectively. The finiteness of the relevant Shafarevich-Tate group furnishes the Selmer group with a natural L-rational structure

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(V_{12}, E(H)_L) \subset H^1_{\mathrm{Sel}}(\mathbb{Q}, V_{fgh}).$$

As mentioned above, the vanishing of the central critical value $L(E, \varrho_{gh}, 1)$ implies that the generalised Kato classes in (14) are cristalline at p and thus belong to $H^1_{\mathrm{fin}}(\mathbb{Q}, V_{fgh})$. The main goal of this article is to give a precise conjectural description of the position of the generalised Kato classes in $H^1_{\mathrm{fin}}(\mathbb{Q}, V_{fgh})$ relative to the L-structure given by (18), in a way that recovers older conjectures of Kato and Perrin-Riou in the setting of Beilinson-Kato classes when g and h are Eisenstein series, and is consistent with the theorems and conjectures of [DR2] and [DLR1].

3.2. **Basic properties.** In this section we recall some of the main properties of the generalised Kato classes already established in [DR2] and [DLR1].

Restricting (17) to $G_{\mathbb{Q}_p}$, let

$$H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fgh}) := \mathrm{Ext}^1_{\mathrm{fin},\mathbb{Q}_p}(\mathbb{Q}_p, V_{fgh})$$

denote the group of cristalline extensions of $\mathbb{Q}_p$ by V_{fgh} in the category of $\mathbb{Q}_p$-linear representations of $G_{\mathbb{Q}_p}$, and let $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_{fgh}) := H^1(\mathbb{Q}_p, V_{fgh})/H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fgh})$ denote the "singular quotient" of the local cohomology at p. Recall that L has been chosen to be large enough to contain the frobenius eigenvalues α_g, β_g, α_h and β_h, which therefore belong to $\mathbb{Q}_p$. Let

$$V_g^{\alpha}, V_g^{\beta} \subset V_g, \qquad V_h^{\alpha}, V_h^{\beta} \subset V_h \tag{19}$$

be the eigenspaces in V_g and V_h respectively associated to these eigenvalues, and set

$$V_{gh}^{\alpha\alpha} := V_g^{\alpha} \otimes V_h^{\alpha}, \qquad V_{gh}^{\alpha\beta} := V_g^{\alpha} \otimes V_h^{\beta}, \qquad V_{gh}^{\beta\alpha} := V_g^{\beta} \otimes V_h^{\alpha}, \qquad V_{gh}^{\beta\beta} := V_g^{\beta} \otimes V_h^{\beta} \tag{20}$$

of V_{gh}. Even though V_g and V_h are both assumed to be regular at p, the same need not be true for V_{gh}, and in this case

$$V_{gh} = V_{gh}^{\alpha\alpha} \oplus V_{gh}^{\alpha\beta} \oplus V_{gh}^{\beta\alpha} \oplus V_{gh}^{\beta\beta}. \tag{21}$$

gives a strict refinement of the decomposition of V_{gh} into frobenius eigenspaces. In fact, some of the most interesting arithmetic applications of the generalised Kato classes (notably those spelled out in sections 4.3, 4.4, and 4.5) arise when V_{gh} is not regular at p.

The first basic result extends Kato's explicit reciprocity law (corresponding to the case where g and h are both Eisenstein series) to the setting where both g and h are cuspidal (Theorem C of [DR2]) as well as to the intermediate Beilinson-Flach setting (Theorem 3.10 of [BDR2]).

Theorem 3.1. *The natural image of $\kappa(f, g_\alpha, h_\alpha)$ in $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_{fgh})$ belongs to $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_f \otimes V_{gh}^{\beta\beta})$, and analogously for the remaining classes of* (14)*. Moreover, the following are equivalent:*

1. *For all choices of π in* (16)*, the generalised Kato classes of* (14) *belong to the Bloch-Kato Selmer group of V_{fgh}, i.e., their images in $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_{fgh})$ are trivial;*
2. *the central critical value $L(E, V_{gh}, 1)$ vanishes.*

Assume from now on that $L(E, V_{gh}, 1) = 0$, so that

1. the L-series $L(E, V_{gh}, s)$ has a zero of even order ≥ 2 at $s = 1$;
2. the generalised Kato classes of (14) belong to the Selmer group attached to E and V_{gh}.

One is then naturally interested in a formulating non-vanishing criterion for these Selmer classes:

Conjecture 3.2. *The generalised Kato classes in* (14) *generate a non-trivial subgroup of the Selmer group of* V_{fgh} *for a suitable choice of* π *in* (16), *if and only if the following equivalent conditions are satisfied:*

(a) *The L-series* $L(E, V_{gh}, s)$ *has a double zero at* $s = 1$*;*
(b) *the Mordell-Weil group* $(E(H)_L \otimes V_{12})^{G_{\mathbb{Q}}}$ *is two-dimensional over* L*;*
(c) *the Selmer group* $H^1_{\mathrm{fin}}(\mathbb{Q}, V_{fgh})$ *is two-dimensional over* $\mathbb{Q}_p$.

Remark 3.3. Although the equivalence of conditions (a), (b) and (c) certainly lies very deep, it is part of a well-established conjecture, namely $\mathrm{BSD}(E, V_{gh})$. The main novelty of Conjecture 3.2 is in providing a criterion for the non-triviality of the space generated by the generalised Kato classes. Note that Conjecture 3.2 does not predict that all four of the classes in (14) are non-trivial, nor even that these four classes generate the Selmer group, when (a), (b), and/or (c) are satisfied. These stronger conclusions are expected to be false in general, as illustrated by some of the examples in Chapter 4.

Let

$$\kappa_p(f, g_\alpha, h_\alpha) = \mathrm{res}_p(\kappa(f, g_\alpha, h_\alpha))$$

denote the image of the global class $\kappa(f, g_\alpha, h_\alpha)$ in the local cohomology group

$$H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fgh}) = (H^1_{\mathrm{fin}}(H_p, V_f) \otimes V_{gh})^{\mathrm{Gal}\,(H_p/\mathbb{Q}_p)} = (E(H_p) \otimes V_{gh})^{\mathrm{Gal}\,(H_p/\mathbb{Q}_p)}.$$

As we describe more explicitly below, Theorem D of [DR2] asserts that this image is controlled by suitable p-adic avatars of the *second derivative* of the classical L-series $L(f, V_{gh}, s)$ at the central critical point $s = 1$.

These p-adic values were defined and explored in [DR2] and [DLR1] and are denoted

$$\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}), \qquad \mathscr{L}_p{}^{g_\beta}(\breve{f}, \breve{g}^*, \breve{h}), \qquad \mathscr{L}_p{}^{h_\alpha}(\breve{f}, \breve{g}, \breve{h}^*), \qquad \mathscr{L}_p{}^{h_\beta}(\breve{f}, \breve{g}, \breve{h}^*). \tag{22}$$

They depend on the choice of certain test vectors

$$(\breve{f}, \breve{g}, \breve{h}) \in S_2(N; L) \times M_1(N, \chi; L) \times M_1(N, \chi^{-1}; L)$$

with the same system of Hecke eigenvalues as f, g and h respectively, and with fourier coefficients in L, and on the choice of dual test vectors

$$(\breve{g}^*, \breve{h}^*) \in \mathrm{Hom}(M_1(N, \chi^{-1}; L), L) \times \mathrm{Hom}(M_1(N, \chi; L), L)$$

with the same system of Hecke eigenvalues as g and h. We refer to the introduction of [DR2] for more details on their definition, contenting ourselves with remark that the p-adic L-value $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h})$ is defined essentially as the p-adic limit of central critical values

$$\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}) := \lim_{\ell \to 1} \mathcal{E}(f, g_\ell, h) \times C(\breve{f}, \breve{g}^*, \breve{h}) \times \frac{L(V_f \otimes V_{g_\ell} \otimes V_h, (\ell+1)/2)}{\langle g_\ell, g_\ell \rangle},$$

as g_ℓ ranges over the specialisations of (odd) weight $\ell \geq 3$ of the Hida family $\underline{g}$ specialising to g_α in weight one. Here $\mathcal{E}(f, g_\ell, h)$ is a p-adic multiplier arising from a recipe of Panciskin, whose presence allows the p-adic interpolation of the special values above, and $C(\breve{f}, \breve{g}^*, \breve{h})$ is a product over the primes dividing $N \cdot \infty$ of local terms which depend in a simple way on the choice of test vectors.

Choose a basis of V_{gh} (over $\mathbb{Q}_p$, for now) which is compatible with the decomposition (21), i.e., choose non-zero vectors

$$v_{gh}^{\alpha\alpha} \in V_{gh}^{\alpha\alpha}, \quad v_{gh}^{\alpha\beta} \in V_{gh}^{\alpha\beta}, \quad v_{gh}^{\beta\alpha} \in V_{gh}^{\beta\alpha}, \quad v_{gh}^{\beta\beta} \in V_{gh}^{\beta\beta}. \tag{23}$$

Write

$$\kappa_p(f, g_\alpha, h_\alpha) = R_{\alpha\alpha} \otimes v_{gh}^{\beta\beta} + R_{\alpha\beta} \otimes v_{gh}^{\beta\alpha} + R_{\beta\alpha} \otimes v_{gh}^{\alpha\beta} + R_{\beta\beta} \otimes v_{gh}^{\alpha\alpha}. \tag{24}$$

The coordinate R_ξ belongs to $E(H_p)_{\mathbb{Q}_p}^{\mathrm{Fr}_p = \xi}$, where ξ ranges over the index set

$$\{\alpha\alpha = \alpha_g \alpha_h, \quad \alpha\beta = \alpha_g \beta_h, \quad \beta\alpha = \beta_g \alpha_h, \quad \beta\beta = \beta_g \beta_h\}.$$

Note that R_ξ is even the image of a global point in $E(H)_{\mathbb{Q}_p}$, assuming the finiteness of the Shafarevich-Tate group of E over H. Let

$$\log_p : E(H_p)_{\mathbb{Q}_p} \longrightarrow H_p \tag{25}$$

denote the formal group logarithm attached to an invariant differential on $E/\mathbb{Q}$. The following theorem is stated in Section 6.4 of [DR2]:

Theorem 3.4. *When $L(E, V_{gh}, 1) = 0$, there exists a choice of π in (16) and of test vectors for f, g and h such that the coordinates in (24) satisfy*

$$\log_p(R_{\alpha\beta}) \sim \mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}), \qquad \log_p(R_{\beta\alpha}) \sim \mathscr{L}_p{}^{h_\alpha}(\breve{f}, \breve{g}, \breve{h}^*), \qquad \log_p(R_{\beta\beta}) = 0, \tag{26}$$

where $\sim$ denotes equality up to a non-zero p-adic period in $H_p^\times$.

Remark 3.5. This theorem says nothing about the quantity $\log_p(R_{\alpha\alpha})$, which does not bear any direct relationship with p-adic L-values introduced above. We expect that $\log_p(R_{\alpha\alpha})$ may rather be connected with the first derivative of a putative refinement of $\mathscr{L}_p{}^f(f, g_\alpha, h_\alpha)$ in which all three modular forms would be made to vary in a Hida family.

As explained in the introduction and in Section 6.3. of [DR2], Theorem 3.4 has the following corollary which can be viewed as a p-adic Gross-Zagier formula in "analytic rank two":

Corollary 3.6. *If $L(E, V_{gh}, 1) = 0$ and $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}) \neq 0$ for a suitable choice $(\breve{f}, \breve{g}^*, \breve{h})$ of test vectors, then the two global classes*

$$\kappa(f, g_\alpha, h_\alpha), \qquad \kappa(f, g_\alpha, h_\beta)$$

are linearly independent in the Selmer group $H^1_{\mathrm{fin}}(\mathbb{Q}, V_{fgh})$ attached to E and V_{gh}, for a suitable choice of π in (16).

Theorem 3.4 and its corollary motivated the experimental study undertaken in [DLR1] of the special values of p-adic L-functions appearing in (26). This led to a precise conjecture for these values up to a factor of $L^\times$ rather than $\mathbb{Q}_p^\times$.

To formulate this conjecture, recall that the class $\kappa(f, g_\alpha, h_\alpha)$ is expected to be trivial when $\mathrm{ord}_{s=1} L(E, V_{gh}, s) > 2$. Assume that this L-function has a *double* zero at the center, which implies, by Conjecture BSD(E, V_{gh}), that $(E(H)_L \otimes V_{12})^{G_\mathbb{Q}}$ is a two-dimensional L-vector space.

Fix vectors $v_{gh}^{\alpha\alpha}, \dots, v_{gh}^{\beta\beta}$ chosen as in (23), with the difference that they belong to L-vector space V_{12} rather than the $\mathbb{Q}_p$-vector space V_{gh}. Choose a basis (P, Q) for this L-vector space, and write

$$\begin{aligned} P &= P_{\alpha\alpha} \otimes v_{gh}^{\beta\beta} + P_{\alpha\beta} \otimes v_{gh}^{\beta\alpha} + P_{\beta\alpha} \otimes v_{gh}^{\alpha\beta} + P_{\beta\beta} \otimes v_{gh}^{\alpha\alpha}, \\ Q &= Q_{\alpha\alpha} \otimes v_{gh}^{\beta\beta} + Q_{\alpha\beta} \otimes v_{gh}^{\beta\alpha} + Q_{\beta\alpha} \otimes v_{gh}^{\alpha\beta} + Q_{\beta\beta} \otimes v_{gh}^{\alpha\alpha}, \end{aligned}$$

where P_ξ, Q_ξ are points in $E(H)_L^{\mathrm{Fr}_p = \xi}$ for every $\xi \in \{\alpha\alpha = \alpha_g\alpha_h, \alpha\beta = \alpha_g\beta_h, \beta\alpha = \beta_g\alpha_h, \beta\beta = \beta_g\beta_h\}$.

These points can be used to define a regulator attached to g_α, whose entries are the p-adic formal group logarithms of the coordinates attached to the vectors $v_{gh}^{\alpha\alpha}$ and $v_{gh}^{\alpha\beta}$ (and similarly for h_α):

Definition 3.7. The regulators attached to E and V_{12} are

$$\mathrm{Reg}_{g_\alpha}(E, V_{12}) = \det\begin{pmatrix} \log_p P_{\beta\beta} & \log_p P_{\beta\alpha} \\ \log_p Q_{\beta\beta} & \log_p Q_{\beta\alpha} \end{pmatrix} = \log_p P_{\beta\beta} \cdot \log_p Q_{\beta\alpha} - \log_p Q_{\beta\beta} \cdot \log_p P_{\beta\alpha},$$

$$\mathrm{Reg}_{h_\alpha}(E, V_{12}) = \det\begin{pmatrix} \log_p P_{\beta\beta} & \log_p P_{\alpha\beta} \\ \log_p Q_{\beta\beta} & \log_p Q_{\alpha\beta} \end{pmatrix} = \log_p P_{\beta\beta} \cdot \log_p Q_{\alpha\beta} - \log_p Q_{\beta\beta} \cdot \log_p P_{\alpha\beta}.$$

The main conjecture of [DLR1] is the following[3], assuming $\frac{\alpha_g}{\beta_g} \neq \pm 1$ (resp. $\frac{\alpha_h}{\beta_h} \neq \pm 1$) so that the Stark unit u_{g_α} (resp. u_{h_α}) is well-defined:

Conjecture 3.8. *Assume that $L(E, V_{gh}, s)$ vanishes to order 2 at $s = 1$. Then there exists a choice of test vectors $(\breve{f}, \breve{g}^*, \breve{h})$ and $(\breve{f}, \breve{g}, \breve{h}^*)$ such that*

$$\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}) = \frac{\mathrm{Reg}_{g_\alpha}(E, V_{12})}{\log_p u_{g_\alpha}}, \qquad \mathscr{L}_p{}^{h_\alpha}(\breve{f}, \breve{g}, \breve{h}^*) = \frac{\mathrm{Reg}_{h_\alpha}(E, V_{12})}{\log_p u_{h_\alpha}} \pmod{L^\times}.$$

[3] We warn the reader that here in this note we have chosen to state the main conjecture of [DLR1] in terms of the *arithmetic* frobenius Fr_p at p, while in [DLR1] we rather employ the *geometric* frobenius $\sigma_p = \mathrm{Fr}_p^{-1}$. It is for this reason that the roles of α and β are swapped in both formulations.

Remark 3.9. Conjecture 3.8 lends itself to numerical verification and has been extensively tested in [DLR1]. This is because the p-adic L-values $\mathscr{L}_p{}^{g_\alpha}(\breve{f},\breve{g}^*,\breve{h})$ and $\mathscr{L}_p{}^{h_\alpha}(\breve{f},\breve{g},\breve{h}^*)$ can be expressed in terms of the rather concrete *p-adic iterated integrals* of loc.cit., which can be computed efficiently using Alan Lauder's [La] fast ordinary projection algorithms on the space of overconvergent modular forms. In contrast, the generalised Kato classes themselves (like many objects constructed in étale cohomology) seem difficult to compute in practice, even though their theoretical usefulness is amply illustrated in [BDR2] and [DR2].

3.3. **Enhanced regulators.** The goal of this article is to combine the insights arising from Theorem 3.4 and Conjecture 3.8 to formulate a conjecture on the position of the generalised Kato classes themselves in $(E(H)\otimes V_{gh})^{G_\mathbb{Q}}$, specifying this position up to an ambiguity of $L^\times$ rather than the less precise $\mathbb{Q}_p^\times$ ambiguity of Theorem 3.4.

The most important ingredients in the formulation of this conjecture are the so-called *enhanced regulators*

$$\begin{aligned}\widetilde{\mathrm{Reg}}(E,V_{12}) &\in (E(H)_L\otimes V_{12})^{G_\mathbb{Q}}\otimes(E(H)_L\otimes V_{12})^{G_\mathbb{Q}},\\ \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12}) &\in (H_p)^{\mathrm{Fr}_p=\beta_g\beta_h}\otimes(E(H)_L\otimes V_{12})^{G_\mathbb{Q}},\\ \\ \widetilde{\mathrm{Reg}}(E,V_{gh}) &\in (E(H)_L\otimes V_{gh})^{G_\mathbb{Q}}\otimes(E(H)_L\otimes V_{gh})^{G_\mathbb{Q}},\\ \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{gh}) &\in D(V_{gh}^{\alpha\alpha})\otimes(E(H)_L\otimes V_{gh})^{G_\mathbb{Q}},\end{aligned}$$

whose definition is somewhat in the spirit of the regulator R_S defined in equation (2) of [Da92], and which we now proceed to describe. As in (6), here $D(V_{gh}^{\alpha\alpha}):=(\mathbb{Q}_p^{\mathrm{nr}}\otimes V_{gh}^{\alpha\alpha})^{G_{\mathbb{Q}_p}}=(H_p\otimes V_{gh}^{\alpha\alpha})^{G_{\mathbb{Q}_p}}$.

Definition 3.10. Choose an L-basis (P,Q) of the two-dimensional vector space $(E(H)\otimes V_{12})^{G_\mathbb{Q}}$, and set

$$\widetilde{\mathrm{Reg}}(E,V_{12}):=\det\begin{pmatrix} P & P\\ Q & Q\end{pmatrix}:=P\otimes Q-Q\otimes P. \tag{27}$$

It does not depend on the choice of basis that was made to define it, up to multiplication by $L^\times$.

The function $\log_{\alpha\alpha}:(E(H)_L\otimes V_{12})^{G_\mathbb{Q}}\longrightarrow(H_p)^{\mathrm{Fr}_p=\beta_g\beta_h}$ defined by

$$\log_{\alpha\alpha}(P):=\log_p(P_{\beta\beta})$$

induces a linear map

$$\log_{\alpha\alpha}\otimes 1:(E(H)_L\otimes V_{12})^{G_\mathbb{Q}}\otimes(E(H)_L\otimes V_{12})^{G_\mathbb{Q}}\longrightarrow(H_p)^{\mathrm{Fr}_p=\beta_g\beta_h}\otimes(E(H)_L\otimes V_{12})^{G_\mathbb{Q}},$$

and we set

$$\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12}):=(\log_{\alpha\alpha}\otimes 1)(\widetilde{\mathrm{Reg}}(E,V_{12}))=\log_p(P_{\beta\beta})\otimes Q-\log_p(Q_{\beta\beta})\otimes P. \tag{28}$$

Recall the embedding $j_{gh}:V_{12}\longrightarrow V_{gh}^L\subset V_{gh}$ of (12). Although this embedding is completely non-canonical and only defined up to scaling by $\mathbb{Q}_p^\times$, there is a canonical way of embedding $V_{12}^{\otimes 2}$ into $V_{gh}^{\otimes 2}$. This is done by exploiting the canonical dualities on V_g and V_h described in Section 2, which gives rise to perfect pairings

$$V_g\times V_g\longrightarrow\mathbb{Q}_p(\chi),\qquad V_h\times V_h\longrightarrow\mathbb{Q}_p(\chi^{-1}),\qquad V_{gh}\times V_{gh}\longrightarrow\mathbb{Q}_p.$$

These pairings allow us to define L-rational structures V_g^{L*}, V_h^{L*} and V_{gh}^{L*} which are dual to V_g^L, V_h^L and V_{gh}^L respectively, by letting V_g^{L*} be the L-dual of V_g^L in V_g, and likewise for V_h^{L*} and V_{gh}^{L*}. We may then choose $G_\mathbb{Q}$-equivariant embeddings

$$j_g^*:V_1\longrightarrow V_g^{L*},\qquad j_h^*:V_2\longrightarrow V_h^{L*},\qquad j_{gh}^*:=j_g^*\otimes j_h^*:V_{12}\longrightarrow V_{gh}^{L*},$$

which are well-defined up to scaling by $L^\times$. Replacing j_{gh} by $\mu\cdot j_{gh}$, for any $\mu\in\mathbb{Q}_p^\times$, has the effect of replacing j_{gh}^* by $\mu^{-1}\cdot j_{gh}^*$. Hence the map

$$j_{gh}\otimes j_{gh}^*:V_{12}\otimes V_{12}\longrightarrow V_{gh}\otimes V_{gh}$$

is well defined up to scaling by $L^\times$.

Definition 3.11. The enhanced regulator $\widetilde{\mathrm{Reg}}(E, V_{gh})$ associated to E and V_{gh} is

$$\widetilde{\mathrm{Reg}}(E, V_{gh}) := (j_{gh} \otimes j_{gh}^*)(\widetilde{\mathrm{Reg}}(E, V_{12})) \in (E(H) \otimes V_{gh})^{G_{\mathbb{Q}}} \otimes (E(H) \otimes V_{gh})^{G_{\mathbb{Q}}}. \tag{29}$$

Finally, let

$$\mathrm{Log}_p : (E(H) \otimes V_{gh})^{G_{\mathbb{Q}}} \longrightarrow (H_p \otimes V_{gh})^{G_{\mathbb{Q}_p}} = D(V_{gh})$$

be the canonical p-adic logarithm map induced from the p-adic logarithm of (25) via the fixed embedding $H \subset H_p$, and let

$$\mathrm{Log}_{\alpha\alpha} : (E(H) \otimes V_{gh})^{G_{\mathbb{Q}}} \longrightarrow D(V_{gh}^{\alpha\alpha})$$

be its composition with the functorial projection $D(V_{gh}) \longrightarrow D(V_{gh}^{\alpha\alpha})$. This logarithm map is just the more canonical counterpart of the map $\log_{\alpha\alpha}$: the latter depends on the choice of a basis vector $v_{gh}^{\alpha\alpha}$ for $V^{\alpha\alpha}$ and is related to $\mathrm{Log}_{\alpha\alpha}$ by the rule

$$\mathrm{Log}_{\alpha\alpha} := \log_{\alpha\alpha} \otimes v_{gh}^{\alpha\alpha}.$$

We set

$$\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{gh}) := (\mathrm{Log}_{\alpha\alpha} \otimes 1)(\widetilde{\mathrm{Reg}}(E, V_{gh})) = \mathrm{Log}_{\alpha\alpha}(P) \otimes Q - \mathrm{Log}_{\alpha\alpha}(Q) \otimes P. \tag{30}$$

It is worth noting that the enhanced regulator $\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{gh})$ is a canonical invariant associated to E and V_{gh}, i.e., it is well-defined up to multiplication by $L^\times$, while the less canonical $\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{12})$ depends on the choice of a basis $v_{gh}^{\alpha\alpha}$ for $V_{gh}^{\alpha\alpha}$. The two regulators are related by

$$\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{gh}) = \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{12}) \otimes v_{gh}^{\alpha\alpha}. \tag{31}$$

3.4. **The conjecture.** Recall the periods

$$\omega_{g_\alpha} \in D(V_g^\alpha), \qquad \omega_{h_\alpha} \in D(V_h^\alpha)$$

constructed in (6). The main conjecture of this note is:

Conjecture 3.12. *Assume that $r(E, V_{gh}) = 2$. The generalised Kato class $\kappa(f, g_\alpha, h_\alpha)$ belongs to $(E(H) \otimes V_{gh})^{G_{\mathbb{Q}}}$ and satisfies the relation*

$$\omega_{g_\alpha}\omega_{h_\alpha} \otimes \kappa(f, g_\alpha, h_\alpha) \sim_L \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{gh})$$

in $D(V_{gh}^{\alpha\alpha}) \otimes (E(H) \otimes V_{gh})^{G_{\mathbb{Q}}}$, where $\sim_L$ denotes an equality up to scaling by a factor in L which is non-zero for a suitable choice of π in (16).

The following proposition shows that, under Conjecture 2.1 (relating the canonical period attached to g to the Stark unit u_{g_α}) and Conjecture 3.2 (a mild strengthening of $\mathrm{BSD}(E, \varrho_{gh})$), Conjecture 3.12 implies the main conjecture of [DLR1]. Before dismissing this proposition as mere conjectural relations between conjectures, the reader is reminded that Conjecture 3.8 lends itself to experiment and has been extensively tested numerically in [DLR1], while the strengthening described in Conjecture 3.12 lies for the moment beyond the range of explicit calculations (cf. Remark 3.9).

Proposition 3.13. *Assume Conjectures 2.1 and 3.2. Then Conjecture 3.12 implies Conjecture 3.8.*

Proof. Consider the product of periods

$$\eta_{g_\alpha}\omega_{h_\alpha} = (\Xi_{g_\alpha} \otimes v_g^\beta) \cdot (\Omega_{h_\alpha} \otimes v_h^\alpha) = \Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \otimes v_{gh}^{\beta\alpha} \in D(V_{gh}^{\beta\alpha})$$

defined in Section 2.

The pairing introduced in (7) gives rise to a pairing

$$\langle\, , \rangle : D(V_{gh}^{\alpha\beta}) \times D(V_{gh}^{\beta\alpha}) \longrightarrow D(\mathbb{Q}_p) = \mathbb{Q}_p.$$

As shown in the proof of [DR2, Theorem 6.10 (ii)],

$$\left\langle \mathrm{Log}_{\alpha\beta}\, \kappa(f, g_\alpha, h_\alpha), \eta_{g_\alpha}\omega_{h_\alpha} \right\rangle = \mathscr{L}_p^{g_\alpha}(f, g, h) \pmod{L^\times}. \tag{32}$$

On the other hand, by the definition of the enhanced regulator,

$$\begin{aligned} \mathrm{Log}_{\alpha\beta} \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{gh}) &= (\log_p P_{\beta\beta} \log_p Q_{\beta\alpha} - \log_p Q_{\beta\beta} \log_p P_{\beta\alpha}) \otimes v_{gh}^{\alpha\alpha} \otimes v_{gh}^{*\alpha\beta} \\ &= \mathrm{Reg}_{g_\alpha}(E, V_{12}) \otimes v_{gh}^{\alpha\alpha} \otimes v_{gh}^{*\alpha\beta} \pmod{L^\times}. \end{aligned}$$

Hence the following equality holds in $D(V_{gh}^{\alpha\alpha})$:

$$\langle \mathrm{Log}_{\alpha\beta}\, \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{gh}), \eta_{g_\alpha}\omega_{h_\alpha}\rangle = \Xi_{g_\alpha}\cdot \Omega_{h_\alpha}\cdot \mathrm{Reg}_{g_\alpha}(E, V_{gh}) \otimes v_{gh}^{\alpha\alpha} \pmod{L^\times}. \tag{33}$$

By pairing the value of $\mathrm{Log}_{\alpha\beta}$ at both sides of the displayed identity in Conjecture 3.12 with the class $\eta_{g_\alpha}\omega_{h_\alpha}$ and invoking (32) and (33), we obtain

$$\omega_{g_\alpha}\omega_{h_\alpha} \otimes \mathscr{L}_p{}^{g_\alpha}(f,g,h) = \Xi_{g_\alpha}\cdot \Omega_{h_\alpha}\cdot \mathrm{Reg}_{g_\alpha}(E, V_{12}) \otimes v_{gh}^{\alpha\alpha} \in D(V_{gh}^{\alpha\alpha}) \pmod{L^\times}.$$

Since

$$\omega_{g_\alpha}\omega_{h_\alpha} = \Omega_{g_\alpha}\cdot\Omega_{h_\alpha}\cdot v_{gh}^{\alpha\alpha} \pmod{L^\times},$$

it follows that

$$\Omega_{g_\alpha}\mathscr{L}_p{}^{g_\alpha}(f,g,h) = \Xi_{g_\alpha}\,\mathrm{Reg}_{g_\alpha}(E, V_{12}) \pmod{L^\times},$$

and therefore that

$$\mathscr{L}_p{}^{g_\alpha}(f,g,h) = \frac{\mathrm{Reg}_{g_\alpha}(E, V_{12})}{\mathcal{L}_{g_\alpha}} \pmod{L^\times}.$$

Conjecture 3.8 now follows directly from this equality after invoking Conjecture 2.1. □

Remark 3.14. As explained in a number of the examples covered in Chapter 4 below, it may happen that all four of the p-adic iterated integrals in (22) are equal to zero even when some of the generalised Kato classes are non-trivial. This suggests that Conjecture 3.12 is a genuine strengthening of Conjecture 3.8.

4. Special cases

This section examines Conjecture 3.12, and the special forms taken by the enhanced regulators

$$\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E;V_{12}), \qquad \widetilde{\mathrm{Reg}}_{\alpha\beta}(E;V_{12}), \qquad \widetilde{\mathrm{Reg}}_{\beta\alpha}(E;V_{12}), \qquad \widetilde{\mathrm{Reg}}_{\beta\beta}(E;V_{12}),$$

in the arithmetically interesting cases where V_{gh} is *reducible*. According to [DLR2, §2], the following is a complete list of scenarios where this occurs:

1. The original *Beilinson-Kato setting* where V_g and V_h are both reducible, i.e., where g and h are both Eisenstein series of weight one;
2. the *Beilinson-Flach setting* where exactly one of V_g or V_h is reducible, i.e., where exactly one of g or h is cuspidal;
3. the *complex multiplication* case where V_g and V_h are both induced from characters of a common imaginary quadratic field;
4. the *real multiplication* case where V_g and V_h are induced from characters of mixed signature of a common real quadratic field;
5. the *adjoint case* where h is (a twist of) the dual of g, so that V_{gh} is the direct sum of a one-dimensional representation and a twist of the adjoint of V_g.

The reader will notice that some of the above settings arise when ϱ_g and/or ϱ_h are *reducible*, while in §2 and §3 these representations were assumed to be irreducible. This assumption was imposed to a large extent for the sake of simplicity of the exposition, and the statement (and presumed validity) of Conjecture 3.12 does not rely on it. For completeness, we have therefore described the enhanced regulators that appear in Conjecture 3.12 in all of the above cases.

4.1. Beilinson-Kato classes.

Assume that g and h are both Eisenstein series. After possibly twisting g or h, there is no real loss of generality in assuming that there exist Dirichlet characters χ_1, χ_2 such that g and h are given by

$$g = E_1(\chi_1,\chi_2), \qquad h = E_1(1,\chi_{12}^{-1}), \qquad \text{where} \qquad \chi_{12} = \chi_1\chi_2.$$

We refer to e.g. [BDR1, §2.1.2] for the definition of these weight one Eisenstein series in terms of their q-expansions. The Galois representations attached to g and h are reducible, namely,

$$V_1 = L(\chi_1)\oplus L(\chi_2), \qquad V_2 = L\oplus L(\chi_{12}^{-1}), \qquad V_{12} = L(\chi_1)\oplus L(\chi_1^{-1})\oplus L(\chi_2)\oplus L(\chi_2^{-1}), \tag{34}$$

where the coefficient field L is the cyclotomic field generated by the images of χ_1 and χ_2. These representations factor through the Galois group $\mathrm{Gal}\,(H/\mathbb{Q})$ of an abelian extension H of $\mathbb{Q}$. We may set

$$\alpha_g = \chi_1(p), \quad \beta_g = \chi_2(p), \qquad \alpha_h = 1, \quad \beta_h = \chi_{12}^{-1}(p).$$

The regularity asumption implies that V_1 and V_2 decompose uniquely as a direct sum of two $G_{\mathbb{Q}_p}$-stable lines, which are also stable under $G_{\mathbb{Q}}$. More precisely,

$$V_{12}^{\alpha\alpha} = L \cdot v_{\chi_1}, \qquad V_{12}^{\beta\beta} = L \cdot v_{\bar{\chi}_1}, \qquad V_{12}^{\alpha\beta} = L \cdot v_{\bar{\chi}_2}, \qquad V_{12}^{\beta\alpha} = L \cdot v_{\chi_2},$$

where $(v_{\chi_1}, v_{\bar{\chi}_1}, v_{\bar{\chi}_2}, v_{\chi_2})$ is a basis for V_{12} on which $G_{\mathbb{Q}}$ acts via the characters χ_1, $\bar{\chi}_1$, $\bar{\chi}_2$, and χ_2 respectively.

The class $\kappa(f, g_\alpha, h_\alpha) = \kappa_{\mathrm{BK}}(f, g_\alpha, h_\alpha)$ was constructed by Kato as a p-adic limit of Beilinson elements attached to pairs of modular units whose logarithmic derivatives are weight two Eisenstein series. Theorem 3.1 in this case boils down to Kato's reciprocity law, which asserts that $\kappa(f, g_\alpha, h_\alpha)$ belongs to the Selmer group of E over H if and only if the L-function

$$L(E, V_{gh}, s) = L(E, \chi_1, s)L(E, \bar{\chi}_1, s)L(E, \chi_2, s)L(E, \bar{\chi}_2, s)$$

vanishes at $s = 1$. In this case, it clearly vanishes to even order, and vanishes to order two if and only if (after eventually interchanging the characters χ_1 and χ_2)

$$\mathrm{ord}_{s=1} L(E, \chi_1, s) = \mathrm{ord}_{s=1} L(E, \bar{\chi}_1, s) = 1, \qquad L(E, \chi_2, 1), L(E, \bar{\chi}_2, 1) \neq 0.$$

Assuming that this is the case, Conjectures $\mathrm{BSD}(E, \chi_1)$ and $\mathrm{BSD}(E, \chi_2)$ predict that $(E(H)_L \otimes V_{12})^{G_{\mathbb{Q}}}$ is two-dimensional over L and that a basis for it can be chosen to be

$$P := P_{\bar{\chi}_1} \otimes v_{\chi_1}, \qquad Q := Q_{\chi_1} \otimes v_{\bar{\chi}_1},$$

where $P_{\bar{\chi}_1}$ and Q_{χ_1} are global points in $E(H)_L$ generating the $\bar{\chi}_1$ and χ_1 eigenspaces respectively for the natural action of $G_{\mathbb{Q}}$. With these notations, we have

$$P_{\alpha\alpha} = P_{\alpha\beta} = P_{\beta\alpha} = 0, \qquad P_{\beta\beta} = P_{\bar{\chi}_1},$$
$$Q_{\alpha\beta} = Q_{\beta\alpha} = Q_{\beta\beta} = 0, \qquad Q_{\alpha\alpha} = Q_{\chi_1}.$$

This immediately implies that

$$\begin{array}{ll} \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{12}) = \log_p(P_{\bar{\chi}_1}) \cdot Q, & \widetilde{\mathrm{Reg}}_{\alpha\beta}(E, V_{12}) = 0, \\ \widetilde{\mathrm{Reg}}_{\beta\alpha}(E, V_{12}) = 0, & \widetilde{\mathrm{Reg}}_{\beta\beta}(E; V_{12}) = \log_p(Q_{\chi_1}) \cdot P. \end{array}$$

It follows that

$$\mathrm{Reg}_{g_\alpha}(E; V_{12}) = \mathrm{Reg}_{g_\beta}(E; V_{12}) = \mathrm{Reg}_{h_\alpha}(E; V_{12}) = \mathrm{Reg}_{h_\beta}(E; V_{12}) = 0.$$

This accounts for the fact that the p-adic iterated integrals

$$\mathscr{L}_p{}^{g_\alpha}(f, g, h), \quad \mathscr{L}_p{}^{g_\beta}(f, g, h), \quad \mathscr{L}_p{}^{h_\alpha}(f, g, h), \quad \mathscr{L}_p{}^{h_\beta}(f, g, h)$$

systematically vanish[4] when g and h are Eisenstein series that are regular at p. Conjecture 3.12 makes the stronger prediction that the generalised Kato classes themselves are non-trivial, and is consistent with a Conjecture of Perrin-Riou, since it predicts that

$$\log_{\beta\beta}(\kappa(f, g_\alpha, h_\alpha)) = \log_{\alpha\alpha}(\kappa(f, g_\beta, h_\beta)) = \log_p(P_{\bar{\chi}_1}) \log_p(Q_{\chi_1}) \pmod{L^\times}.$$

4.2. **Belinson-Flach classes.** In the Beilinson-Flach setting, it can be assumed without loss of generality that g is a weight one cusp form with nebentypus character χ and Galois representation $V_g = V_1 \otimes_L \mathbb{Q}_p$, and that $h := E_1(1, \chi^{-1})$ is the weight one Eisenstein series attached to the pair $(1, \chi^{-1})$ of Dirichlet characters. The relevant four-dimensional representations are then equal to

$$V_{gh} = V_g \oplus V_{\bar{g}}; \qquad V_{12} = V_1 \oplus \bar{V}_1,$$

and the Hasse-Weil-Artin L-series

$$L(E, V_{gh}, s) = L(E, V_g, s)L(E, \bar{V}_g, s)$$

has a double zero at $s = 1$ precisely when each of the primitive L-series $L(E, V_g, s)$ and $L(E, V_{\bar{g}}, s)$ have a simple zero at $s = 1$. Conjecture $\mathrm{BSD}(E, V_g)$ then implies that each of the L-vector spaces on the right-hand side of

$$(E(H)_L \otimes V_{12})^{G_{\mathbb{Q}}}) = (E(H)_L \otimes V_1)^{G_{\mathbb{Q}}} \oplus (E(H)_L \otimes \bar{V}_1)^{G_{\mathbb{Q}}},$$

[4]But see the experiments described in [DLR1, §7] in the case where g is *irregular* at p, which suggest that the irregular setting of Conjecture 3.12 would merit further investigation.

is one-dimensional. Let P be an L-basis for $(E(H)_L \otimes V_1)^{G_{\mathbb{Q}}}$ and let $\bar{P}$ be the associated L-basis for $(E(H)_L \otimes \bar{V}_1)^{G_{\mathbb{Q}}}$, obtained by applying complex conjugation to the coefficients in L.

After fixing an ordering $\alpha_g, \beta_g \in L$ for the eigenvalues of Fr_p on V_1, and setting

$$\alpha_h = 1, \qquad \beta_h = \chi^{-1}(p) = (\alpha_g \beta_g)^{-1},$$

we have

$$V_{12}^{\alpha\alpha} = V_1^{\alpha_g}, \quad V_{12}^{\alpha\beta} = \bar{V}_1^{\beta_g^{-1}}, \quad V_{12}^{\beta\alpha} = V_1^{\beta_g}, \quad V_{12}^{\beta\beta} = \bar{V}_1^{\alpha_g^{-1}},$$

and hence

$$\begin{array}{llll} P_{\alpha\alpha} = P_{\alpha_g}, & P_{\beta\alpha} = P_{\beta_g}, & P_{\alpha\beta} = 0 & P_{\beta\beta} = 0, \\ \bar{P}_{\alpha\alpha} = 0 & \bar{P}_{\beta\alpha} = 0, & \bar{P}_{\alpha\beta} = \bar{P}_{\beta_g^{-1}}, & \bar{P}_{\beta\beta} = \bar{P}_{\alpha_g^{-1}}. \end{array}$$

A direct calculation reveals that, up to multiplication by $L^\times$,

$$\begin{array}{ll} \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{12}) = \log_p(\bar{P}_{\alpha_g^{-1}}) \cdot P, & \widetilde{\mathrm{Reg}}_{\alpha\beta}(E, V_{12}) = \log_p(P_{\beta_g}) \cdot \bar{P}, \\ \widetilde{\mathrm{Reg}}_{\beta\alpha}(E, V_{12}) = \log_p(\bar{P}_{\beta_g^{-1}}) \cdot P, & \widetilde{\mathrm{Reg}}_{\beta\beta}(E; V_{12}) = \log_p(P_{\alpha_g}) \cdot \bar{P}. \end{array}$$

It follows that

$$\begin{array}{ll} \mathrm{Reg}_{g_\alpha}(E, V_{12}) = \log_p(\bar{P}_{\alpha_g^{-1}}) \cdot \log_p(P_{\beta_g}), & \mathrm{Reg}_{g_\beta}(E, V_{12}) = \log_p(\bar{P}_{\beta_g^{-1}}) \cdot \log_p(P_{\alpha_g}), \\ \mathrm{Reg}_{h_\alpha}(E, V_{12}) = 0, & \mathrm{Reg}_{h_\beta}(E, V_{12}) = 0. \end{array}$$

as described in [DLR1, §6].

4.3. **Complex multiplication classes and Heegner points.** In this chapter we consider the setting where g and h are theta series attached to characters ψ_g and ψ_h of the same imaginary quadratic field K, and with inverse nebentypus character. Given any character ψ of G_K, let ψ' denote the character obtained by conjugating it with the involution in $\mathrm{Gal}(K/\mathbb{Q})$. Then

$$V_g = \mathrm{Ind}_K^{\mathbb{Q}} \psi_g = \mathrm{Ind}_K^{\mathbb{Q}} \psi'_g, \qquad V_h = \mathrm{Ind}_K^{\mathbb{Q}} \psi_h = \mathrm{Ind}_K^{\mathbb{Q}} \psi'_h,$$

and therefore

$$V_{gh} = \mathrm{Ind}_K^{Q} \psi_\bullet \oplus \mathrm{Ind}_K^{\mathbb{Q}} \psi_\circ, \qquad \text{where } \psi_\bullet = \psi_g \psi_h, \qquad \psi_\circ = \psi_g \psi'_h.$$

The self-duality assumption implies that $\psi_\bullet$ and $\psi_\circ$ are are *ring class characters*, i.e., they satisfy

$$\psi'_\bullet = \psi_\bullet^{-1}, \qquad \psi'_\circ = \psi_\circ^{-1}.$$

Assume that the induced representations

$$V_\bullet := \mathrm{Ind}_K^{\mathbb{Q}} \psi_\bullet, \qquad V_\circ := \mathrm{Ind}_K^{\mathbb{Q}} \psi_\circ$$

appearing in the decomposition

$$V_{12} = V_\bullet \oplus V_\circ \tag{35}$$

(viewed as representations with coefficients in the number field L) are *irreducible*, which is always the case unless $\psi_\bullet$ or $\psi_\circ$ is a quadratic, i.e., a genus character. (The more degenerate case where this arises can be subsumed under the "adjoint setting" considered in Section 4.5.)

The Hasse-Weil-Artin L-series

$$L(E, V_{gh}, s) = L(E, V_\bullet, s) L(E, V_\circ, s) = L(E/K, \psi_\bullet, s) L(E/K, \psi_\circ, s)$$

has a double zero at $s = 1$ in one of the following two cases:

(1) the primitive L-series $L(E, V_\bullet, s)$ and $L(E, V_\circ, s)$ each have a simple zero at $s = 1$. This setting, which resembles more closely the phenomena described in the previous two sections on Beilinson-Kato and Beilinson-Flach elements, will be referred to as the *rank (1,1) setting* of Conjecture 3.12.

(2) Exactly one of the primitive L-series $L(E, V_\bullet, s)$ or $L(E, V_\circ, s)$ has a double zero at $s = 1$, and the other is non-vanishing at the center. This case shall be referred to as the *rank (2,0) setting* of Conjecture 3.12. The possible non-triviality of the generalised Kato classes in the presence of a "genuine" double zero of a primitive Hasse-Weil-Artin L-function represents a novel feature that did not arise in the setting of Beilinson-Kato or Beilinson-Flach elements.

4.3.1. *The rank* $(1,1)$ *setting.* In this case, Conjectures $\mathrm{BSD}(E,V_\bullet)$ and $\mathrm{BSD}(E,V_\circ)$ predict that the Mordell-Weil groups $(E(H)_L\otimes V_\bullet)^{G_\mathbb{Q}}$ and $(E(H)_L\otimes V_\circ)^{G_\mathbb{Q}}$ are both one-dimensional L-vector spaces, with generators $P_\bullet$ and $P_\circ$ respectively. It is natural to write

$$P_\bullet = P_{\psi_\bullet}\otimes v_{\psi'_\bullet} + P_{\psi'_\bullet}\otimes v_{\psi_\bullet}, \qquad P_\circ = P_{\psi_\circ}\otimes v_{\psi'_\circ} + P_{\psi'_\circ}\otimes v_{\psi_\circ}, \tag{36}$$

where $P_{\psi_\bullet}$, $P_{\psi'_\bullet}$, $P_{\psi_\circ}$, and $P_{\psi'_\circ}$ are generators for the one-dimensional subspaces of $E(H)_L$ on which G_K acts via the characters $\psi_\bullet$, $\psi'_\bullet$, $\psi_\circ$, and $\psi'_\circ$ respectively.

The description of the enhanced regulators attached to V_{12} and to $(P_\bullet,P_\circ)$ can be further subdivided into two cases, with markedly different features: the case where the prime p is split in K, and the case where it is inert in K.

a) The case where p is split in K. In this case, let $p=\mathfrak{p}\,\mathfrak{p}'$ be the factorisation of p into distinct primes of K. We can then set

$$\alpha_g=\psi_g(\mathfrak{p}),\quad \beta_g=\psi_g(\mathfrak{p}'),\qquad \alpha_h=\psi_h(\mathfrak{p}),\quad \beta_h=\psi_h(\mathfrak{p}'),$$

so that

$$\alpha_g\alpha_h=\psi_\bullet(\mathfrak{p}),\quad \alpha_g\beta_h=\psi_\circ(\mathfrak{p}),\quad \beta_g\alpha_h=\psi_\circ(\mathfrak{p}'),\quad \beta_g\beta_h=\psi_\bullet(\mathfrak{p}').$$

The decomposition of the $G_{K_\mathfrak{p}}=G_{\mathbb{Q}_p}$ representations attached to (35) into Fr_p-eigenspaces is also stable under the action of the global Galois group G_K, and is described by:

$$V_{12}^{\alpha\alpha}=V_\bullet^{\psi_\bullet},\quad V_{12}^{\alpha\beta}=V_\circ^{\psi_\circ},\quad V_{12}^{\beta\alpha}=V_\circ^{\psi'_\circ},\quad V_{12}^{\beta\beta}=V_\bullet^{\psi'_\bullet}.$$

It follows that, up to multiplication by $L^\times$,

$$\begin{aligned}
\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12})&=\log_\mathfrak{p}(P_{\psi'_\bullet})\cdot P_\circ, &\qquad \widetilde{\mathrm{Reg}}_{\alpha\beta}(E,V_{12})&=\log_\mathfrak{p}(P_{\psi'_\circ})\cdot P_\bullet,\\
\widetilde{\mathrm{Reg}}_{\beta\alpha}(E,V_{12})&=\log_\mathfrak{p}(P_{\psi_\circ})\cdot P_\bullet, &\qquad \widetilde{\mathrm{Reg}}_{\beta\beta}(E,V_{12})&=\log_\mathfrak{p}(P_{\psi_\bullet})\cdot P_\circ,
\end{aligned}$$

and therefore that

$$\begin{aligned}
\mathrm{Reg}_{g_\alpha}(E,V_{12})&=\log_\mathfrak{p}(P_{\psi'_\bullet})\cdot\log_\mathfrak{p}(P_{\psi'_\circ}), &\quad \mathrm{Reg}_{g_\beta}(E,V_{12})&=\log_\mathfrak{p}(P_{\psi_\bullet})\cdot\log_\mathfrak{p}(P_{\psi_\circ}),\\
\mathrm{Reg}_{h_\alpha}(E,V_{12})&=\log_\mathfrak{p}(P_{\psi'_\bullet})\cdot\log_\mathfrak{p}(P_{\psi_\circ}), &\quad \mathrm{Reg}_{h_\beta}(E,V_{12})&=\log_\mathfrak{p}(P_{\psi_\bullet})\cdot\log_\mathfrak{p}(P_{\psi'_\circ}).
\end{aligned}$$

The corresponding formulae for the p-adic iterated integrals ${\mathscr{L}_p}^{g_\alpha}(f,g,h)$, ${\mathscr{L}_p}^{g_\beta}(f,g,h)$, ${\mathscr{L}_p}^{h_\alpha}(f,g,h)$, and ${\mathscr{L}_p}^{h_\beta}(f,g,h)$ were proved in [DLR1, §3], by using the p-adic Gross-Zagier formula of [BDP13] to express these L-values in terms of products of p-adic logarithms of Heegner points. Theorem 3.3 of loc.cit. is one of the few pieces of theoretical evidence in support of Conjecture 3.12.

b) The case where p is inert in K. In this case, the eigenvalues of the Frobenius automorphism Fr_p acting on V_g and V_h are of the form

$$\alpha_g,\qquad \beta_g=-\alpha_g,\qquad \alpha_h=\alpha_g^{-1},\qquad \beta_h=-\alpha_g^{-1}.$$

Let $(v_{\psi_g},v_{\psi'_g})$ be a eigenbasis of V_g for the action of G_K relative to the distinct characters ψ_g and ψ'_g, and let $(v_{\psi_h},v_{\psi'_h})$ be a similar basis for V_h. These vectors can be scaled so that Fr_p acts on them as

$$\mathrm{Fr}_p(v_{\psi_g})=\alpha_g\cdot v_{\psi'_g},\qquad \mathrm{Fr}_p(v_{\psi'_g})=\alpha_g\cdot v_{\psi_g},\qquad \mathrm{Fr}_p(v_{\psi_h})=\alpha_g^{-1}\cdot v_{\psi'_h},\qquad \mathrm{Fr}_p(v_{\psi'_h})=\alpha_g^{-1}\cdot v_{\psi_h},$$

and therefore we may set

$$V_g^\alpha=L\cdot(v_{\psi_g}+v_{\psi'_g}),\qquad V_g^\beta=L\cdot(v_{\psi_g}-v_{\psi'_g}),\qquad V_h^\alpha=L\cdot(v_{\psi_h}+v_{\psi'_h}),\qquad V_h^\beta=L\cdot(v_{\psi_h}-v_{\psi'_h}).$$

After setting

$$v_{\psi_\bullet}:=v_{\psi_g}\otimes v_{\psi_h},\qquad v_{\psi'_\bullet}:=v_{\psi'_g}\otimes v_{\psi'_h},\qquad v_{\psi_\circ}:=v_{\psi_g}\otimes v_{\psi'_h},\qquad v_{\psi'_\circ}:=v_{\psi'_g}\otimes v_{\psi_h},$$

and letting

$$v_\bullet^+:=v_{\psi_\bullet}+v_{\psi'_\bullet},\qquad v_\bullet^-:=v_{\psi_\bullet}-v_{\psi'_\bullet},\qquad v_\circ^+:=v_{\psi_\circ}+v_{\psi'_\circ},\qquad v_\bullet^-:=v_{\psi_\circ}-v_{\psi'_\circ},$$

it is easy to see that

$$\begin{aligned}
V_{12}^{\alpha\alpha}&=L\cdot(v_\bullet^+ + v_\circ^+), &\qquad V_{12}^{\alpha\beta}&=L\cdot(v_\bullet^- + v_\circ^-),\\
V_{12}^{\beta\alpha}&=L\cdot(v_\bullet^- - v_\circ^-), &\qquad V_{12}^{\beta\beta}&=L\cdot(v_\bullet^+ - v_\circ^+).
\end{aligned}\tag{37}$$

Note that Fr_p acts on both $V_{12}^{\alpha\alpha}$ and $V_{12}^{\beta\beta}$ with the eigenvalue 1, and on $V_{12}^{\alpha\beta}$ and $V_{12}^{\beta\alpha}$ with the eigenvalue -1. In particular, while V_g and V_h are *always* regular at p, the tensor product $V_{gh} \simeq V_{12}$ never enjoys this property, even though the vector spaces described in (37) are one dimensional.

Keeping the same notations as in (36), let

$$P_\bullet^+ := P_{\psi_\bullet} + P_{\psi'_\bullet}, \qquad P_\bullet^- := P_{\psi_\bullet} - P_{\psi'_\bullet}, \qquad P_\circ^+ := P_{\psi_\circ} + P_{\psi'_\circ}, \qquad P_\circ^- := P_{\psi_\circ} - P_{\psi'_\circ}.$$

With these notations, the enhanced regulators describing the associated generalised Kato classes are given by

$$\begin{aligned} \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12}) &= \log_p(P_\bullet^+)\cdot P_\circ - \log_p(P_\circ^+)\cdot P_\bullet, & \widetilde{\mathrm{Reg}}_{\alpha\beta}(E,V_{12}) &= \log_p(P_\bullet^-)\cdot P_\circ - \log_p(P_\circ^-)\cdot P_\bullet, \\ \widetilde{\mathrm{Reg}}_{\beta\alpha}(E,V_{12}) &= \log_p(P_\bullet^-)\cdot P_\circ + \log_p(P_\circ^-)\cdot P_\bullet, & \widetilde{\mathrm{Reg}}_{\beta\beta}(E,V_{12}) &= \log_p(P_\bullet^+)\cdot P_\circ + \log_p(P_\circ^+)\cdot P_\bullet. \end{aligned}$$

The four regulators $\mathrm{Reg}_{g_\alpha}(E,V_{12})$, $\mathrm{Reg}_{g_\beta}(E,V_{12})$, $\mathrm{Reg}_{h_\alpha}(E,V_{12})$, and $\mathrm{Reg}_{h_\beta}(E,V_{12})$ are all seen to be simple $L^\times$-multiples of the expression

$$\log_p(P_\bullet^+)\cdot\log_p(P_\circ^-) - \log_p(P_\circ^+)\cdot\log_p(P_\bullet^-). \tag{38}$$

The resulting formula for $\mathscr{L}_p{}^{g_\alpha}(f,g,h)$ predicted by Conjecture 3.8 has been extensively tested numerically in [DLR1, §3.3].

Remark 4.1. Even though the points $P_\bullet^+$, $P_\bullet^-$, $P_\circ^+$, and $P_\circ^-$ that figure in the generalised Kato classes are in principle expressed as linear combinations of Heegner points, the methods used to prove Conjecture 3.8 when p is split in K, which are based on the p-adic Gross-Zagier formula of [BDP13] and on properties of the Katz p-adic L-function attached to K, seem to break down completely when p is inert in K. A theoretical understanding of the p-adic iterated integrals of [DLR1] in this setting would seem to require a new idea.

Remark 4.2. It is worth contrasting the expressions arising in (38) with the simpler formulae

$$\begin{aligned} \log_{\alpha\alpha}\widetilde{\mathrm{Reg}}_{\beta\beta}(E,V_{12}) = \log_{\beta\beta}\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12}) &= \log_p(P_\bullet^+)\cdot\log_p(P_\circ^+), \\ \log_{\alpha\beta}\widetilde{\mathrm{Reg}}_{\beta\alpha}(E,V_{12}) = \log_{\beta\alpha}\widetilde{\mathrm{Reg}}_{\alpha\beta}(E,V_{12}) &= \log_p(P_\bullet^-)\cdot\log_p(P_\circ^-). \end{aligned}$$

In certain very special settings–notably, when the elliptic curve E has multiplicative reduction at p–these expressions arise as the leading terms of the p-adic L-series

$$\mathscr{L}_p{}^f(f,g_\alpha,h_\alpha), \quad \ldots, \quad \mathscr{L}_p{}^f(f,g_\beta,h_\beta).$$

Methods based on the Cerednik-Drinfeld theory of p-adic uniformisation of Shimura curves make it possible to relate these leading terms to the p-adic logarithms of Heegner points, leading to some indirect theoretical evidence for Conjecture 3.12 in the setting where p is inert in K. See [DR3] for a description of this approach.

4.3.2. *The rank* $(2,0)$ *setting.* Assume, after possibly interchanging $V_\bullet$ and $V_\circ$, that $L(E,V_\bullet,s)$ has a double zero at $s=1$ and that $L(E,V_\circ,1)\neq 0$. In this case, Conjectures $\mathrm{BSD}(E,V_\bullet)$ and $\mathrm{BSD}(E,V_\circ)$ predict that

$$(E(H)_L\otimes V_{12})^{G_\mathbb{Q}} = (E(H)_L\otimes V_\bullet)^{G_\mathbb{Q}}$$

is two-dimensional over L. Choose a basis (P,Q) for this vector space. It is natural to write

$$P = P_{\psi_\bullet}\otimes v_{\psi'_\bullet} + P_{\psi'_\bullet}\otimes v_{\psi_\bullet}, \qquad Q = Q_{\psi_\bullet}\otimes v_{\psi'_\bullet} + Q_{\psi'_\bullet}\otimes v_{\psi_\bullet},$$

where $(P_{\psi_\bullet}, Q_{\psi_\bullet})$ and $(P_{\psi'_\bullet}, Q_{\psi'_\bullet})$ are bases for the two-dimensional subspaces of $E(H)_L$ on which G_K acts via the characters $\psi_\bullet$ and $\psi'_\bullet$ respectively.

As in the rank $(1,1)$ setting, the shape of the enhanced regulators attached to V_{12} and to the basis (P,Q) depend very much on whether the prime p is split or inert in K.

a) The case where p is split in K. In this case, letting $p=\mathfrak{p}\,\mathfrak{p}'$ be the factorisation of p into distinct primes of K, we can set

$$\alpha_g = \psi_g(\mathfrak{p}), \quad \beta_g = \psi_g(\mathfrak{p}'), \qquad \alpha_h = \psi_h(\mathfrak{p}), \quad \beta_h = \psi_h(\mathfrak{p}'),$$

so that

$$\alpha_g\alpha_h = \psi_\bullet(\mathfrak{p}), \quad \alpha_g\beta_h = \psi_\circ(\mathfrak{p}), \quad \beta_g\alpha_h = \psi_\circ(\mathfrak{p}'), \quad \beta_g\beta_h = \psi_\bullet(\mathfrak{p}').$$

The decomposition of the $G_{K_\mathfrak{p}} = G_{\mathbb{Q}_p}$ representations attached to (35) into Fr_p-eigenspaces is also stable under the action of the global Galois group G_K, and is described by:

$$V_{12}^{\alpha\alpha} = V_\bullet^{\psi_\bullet}, \quad V_{12}^{\alpha\beta} = V_\circ^{\psi_\circ}, \quad V_{12}^{\beta\alpha} = V_\circ^{\psi'_\circ}, \quad V_{12}^{\beta\beta} = V_\bullet^{\psi'_\bullet}.$$

It follows that, up to multiplication by $L^\times$,

(39)
$$\begin{aligned} &\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{12}) = \log_\mathfrak{p}(P_{\psi'_\bullet}) \cdot Q - \log_\mathfrak{p}(Q_{\psi'_\bullet}) \cdot P, && \widetilde{\mathrm{Reg}}_{\alpha\beta}(E, V_{12}) = 0, \\ &\widetilde{\mathrm{Reg}}_{\beta\alpha}(E, V_{12}) = 0, && \widetilde{\mathrm{Reg}}_{\beta\beta}(E, V_{12}) = \log_\mathfrak{p}(P_{\psi_\bullet}) \cdot Q - \log_\mathfrak{p}(Q_{\psi_\bullet}) \cdot P. \end{aligned}$$

This suggests that the generalised Kato classes $\kappa(f, g_\alpha, h_\alpha)$ and $\kappa(f, g_\beta, h_\beta)$ give non-trivial elements of the two-dimensional vector space $(E(H)_L \otimes V_\bullet)^{G_\mathbb{Q}}$, while the generalised Kato classes $\kappa(f, g_\alpha, h_\beta)$ and $\kappa(f, g_\beta, h_\alpha)$ should vanish. Furthermore, a direct calculation shows that

$$\mathrm{Reg}_{g_\alpha}(E, V_{12}) = \mathrm{Reg}_{g_\beta}(E, V_{12}) = \mathrm{Reg}_{h_\alpha}(E, V_{12}) = \mathrm{Reg}_{h_\beta}(E, V_{12}) = 0,$$

which is consistent with the fact, proved in [DLR1, §3.2], that all the p-adic iterated integrals attached to (f, g, h) vanish in the rank $(2, 0)$ setting when p is split in K. In this case the generalised Kato classes carry more arithmetic information that the p-adic iterated integrals which describe (certain of) their p-adic logarithms. This represents yet another setting where Conjecture 3.12 is a genuine strengthening of Conjecture 3.8 of [DLR1].

b) The case where p is inert in K. After scaling the points $P_{\psi_\bullet}$, and $P_{\psi'_\bullet}$, $Q_{\psi_\bullet}$ and $Q_{\psi'_\bullet}$ in such a way that

$$\mathrm{Fr}_p(P_{\psi_\bullet}) = P_{\psi'_\bullet}, \qquad \mathrm{Fr}_p(P_{\psi'_\bullet}) = P_{\psi_\bullet}, \qquad \mathrm{Fr}_p(Q_{\psi_\bullet}) = Q_{\psi'_\bullet}, \qquad \mathrm{Fr}_p(Q_{\psi'_\bullet}) = Q_{\psi_\bullet}$$

and letting

$$P^\pm := P_{\psi_\bullet} \pm P_{\psi'_\bullet}, \qquad Q^\pm := Q_{\psi_\bullet} \pm Q_{\psi'_\bullet},$$

the enhanced regulators are given by

(40)
$$\begin{aligned} &\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E, V_{12}) = \log_p(P^+) \cdot Q - \log_p(Q^+) \cdot P, && \widetilde{\mathrm{Reg}}_{\alpha\beta}(E, V_{12}) = \log_p(P^-) \cdot Q - \log_p(Q^-) \cdot P, \\ &\widetilde{\mathrm{Reg}}_{\beta\alpha}(E, V_{12}) = \log_p(P^-) \cdot Q - \log_p(Q^-) \cdot P, && \widetilde{\mathrm{Reg}}_{\beta\beta}(E, V_{12}) = \log_p(P^+) \cdot Q - \log_p(Q^+) \cdot P. \end{aligned}$$

In this case, the four regulators $\mathrm{Reg}_{g_\alpha}(E, V_{12}), \ldots, \mathrm{Reg}_{h_\beta}(E, V_{12})$ attached to (f, g, h) are explicit multiples of the expression

$$\log_p(P^+) \cdot \log_p(Q^-) - \log_p(Q^+) \cdot \log_p(P^-). \tag{41}$$

See [DLR1, Ex. 3.14] for some numerical verifications of the agreement between this value and the p-adic iterated integrals attached to (f, g, h).

Equations (39) and (40) combined with Conjecture 3.12 suggest that the generalised Kato classes always generate the Mordell-Weil group $(E(H)_L \otimes V_\bullet)^{G_\mathbb{Q}}$ (tensored over L with $\mathbb{Q}_p$) in the rank $(2, 0)$ setting. Since the irreducible representation $V_\bullet$ occurs with multiplicity two in $E(H)_L$, none of the $V_\bullet$-isotypic part of the Mordell-Weil group is expected to be accounted for by Heegner points, as discussed in the introduction.

4.4. **Real multiplication classes and Stark-Heegner points.** In this chapter we consider the setting where g and h are theta series attached to characters ψ_g and ψ_h of mixed signature of the same *real* quadratic field K. In that case, we have, exactly as in Section 4.3,

$$V_g = \mathrm{Ind}_K^\mathbb{Q} \psi_g, \quad V_h = \mathrm{Ind}_K^\mathbb{Q} \psi_h,$$

and

$$V_{gh} = V_\bullet \oplus V_\circ := \mathrm{Ind}_K^\mathbb{Q} \psi_\bullet \oplus \mathrm{Ind}_K^\mathbb{Q} \psi_\circ, \qquad \text{where } \psi_\bullet = \psi_g \psi_h, \qquad \psi_\circ = \psi_g \psi'_h.$$

The characters $\psi_\bullet$ and $\psi_\circ$ are also ring class characters of K, with one totally even, and the other totally odd. Once again, it is convenient to assume that $V_\bullet$ and $V_\circ$ are both irreducible, i.e., that neither $\psi_\bullet$ nor $\psi_\circ$ is a genus character of K.

As in the case where K is imaginary, the study of the generalised Kato classes divides naturally into the rank $(1, 1)$ and rank $(2, 0)$ settings, depending on the orders of vanishing of $L(E/K, \psi_\bullet, s)$ and $L(E/K, \psi_\circ, s)$ (or, alternately, on the dimensions of $(E(H)_L \otimes V_\bullet)^{G_\mathbb{Q}}$ and $(E(H)_L \otimes V_\circ)^{G_\mathbb{Q}}$), and continue to depend in a crucial way on whether p is split or inert in K. In all four cases, the formulae for the enhanced regulators are identical to those obtained in Section 4.3, so it is unecessary to reproduce them here, contenting ourselves with the following comments in connection with the rank $(1,1)$ setting.

a) The case where p is split in K. This setting, where the greatest amount of theoretical evidence was available when K is imaginary quadratic, thanks to the theory of Heegner points, is a lot more mysterious when K is real quadratic. With notations being the same as in Section 4.3.1, we have

$$\begin{array}{ll} \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12}) = \log_{\mathfrak{p}}(P_{\psi'_\bullet}) \cdot P_\circ, & \widetilde{\mathrm{Reg}}_{\alpha\beta}(E,V_{12}) = \log_{\mathfrak{p}}(P_{\psi'_\circ}) \cdot P_\bullet, \\ \widetilde{\mathrm{Reg}}_{\beta\alpha}(E,V_{12}) = \log_{\mathfrak{p}}(P_{\psi_\circ}) \cdot P_\bullet, & \widetilde{\mathrm{Reg}}_{\beta\beta}(E,V_{12}) = \log_{\mathfrak{p}}(P_{\psi_\bullet}) \cdot P_\circ, \end{array} \tag{42}$$

and

$$\begin{array}{ll} \mathrm{Reg}_{g_\alpha}(E,V_{12}) = \log_{\mathfrak{p}}(P_{\psi'_\bullet}) \cdot \log_{\mathfrak{p}}(P_{\psi'_\circ}), & \mathrm{Reg}_{g_\beta}(E,V_{12}) = \log_{\mathfrak{p}}(P_{\psi_\bullet}) \cdot \log_{\mathfrak{p}}(P_{\psi_\circ}), \\ \mathrm{Reg}_{h_\alpha}(E,V_{12}) = \log_{\mathfrak{p}}(P_{\psi'_\bullet}) \cdot \log_{\mathfrak{p}}(P_{\psi_\circ}), & \mathrm{Reg}_{h_\beta}(E,V_{12}) = \log_{\mathfrak{p}}(P_{\psi_\bullet}) \cdot \log_{\mathfrak{p}}(P_{\psi'_\circ}). \end{array} \tag{43}$$

This setting has special appeal in connection with an eventual (for now, highly conjectural, and not even clearly fomulated) theory of *Stark-Heegner points* over ring class fields of real quadratic fields. It would be of great interest to relate (conjecturally, at least) the regulators in (42) and in (43) to generalised Kato classes and p-adic iterated integrals, respectively.

The obstruction to doing this is that the modular forms g and h (more precisely, their stabilisations) fail to obey Hypothesis IV in Section 2. When g is a modular form of RM type which is regular at a prime p which splits in K, the Stark unit u_{g_α} is also unavailable, and an analogue of Conjecture 3.8 has yet to be formulated precisely in this setting. Because of the tantalising connection with Stark-Heegner points defined over ring class fields of K, it would be of great interest to extend the Conjectures of [DLR1], as well as Conjecture 3.12, to the real quadratic context. A first step has been made in [DLR3] towards understanding the periods of §2 in this setting.

b) The case where p is inert in K. The formulae for the enhanced regulators are identical to those in Part b) of Section 4.3.1, namely:

$$\begin{array}{ll} \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12}) = \log_p(P_\bullet^+) \cdot P_\circ - \log_p(P_\circ^+) \cdot P_\bullet, & \widetilde{\mathrm{Reg}}_{\alpha\beta}(E,V_{12}) = \log_p(P_\bullet^-) \cdot P_\circ - \log_p(P_\circ^-) \cdot P_\bullet, \\ \widetilde{\mathrm{Reg}}_{\beta\alpha}(E,V_{12}) = \log_p(P_\bullet^-) \cdot P_\circ + \log_p(P_\circ^-) \cdot P_\bullet, & \widetilde{\mathrm{Reg}}_{\beta\beta}(E,V_{12}) = \log_p(P_\bullet^+) \cdot P_\circ + \log_p(P_\circ^+) \cdot P_\bullet. \end{array}$$

The p-adic logarithms of these enhanced regulators ought to involve linear combinations of products of logarithms of so-called *Stark-Heegner points*. This prediction has been extensively tested numerically in [DLR1, §4.2].

Remark 4.3. The logarithms of the generalised Kato classes that are not amenable to expressions in terms of p-adic iterated integrals are expected to admit particularly simple expressions, as suggested by the formulae

$$\begin{aligned} \log_{\alpha\alpha} \widetilde{\mathrm{Reg}}_{\beta\beta}(E,V_{12}) = \log_{\beta\beta} \widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12}) &= \log_p(P_\bullet^+) \cdot \log_p(P_\circ^+), \\ \log_{\alpha\beta} \widetilde{\mathrm{Reg}}_{\beta\alpha}(E,V_{12}) = \log_{\beta\alpha} \widetilde{\mathrm{Reg}}_{\alpha\beta}(E,V_{12}) &= \log_p(P_\bullet^-) \cdot \log_p(P_\circ^-). \end{aligned}$$

In the special case where E has *multiplicative* reduction at p, the article [DR3] in progress proves a formula of the shape

$$\log_{\alpha\alpha} \kappa(f, g_\alpha, h_\alpha) = \log_p(P_\bullet^{?\pm}) \cdot \log_p(P_\circ^{?\pm}),$$

where $P_\bullet^{?\pm}$ and $P_\circ^{?\pm}$ are the Stark-Heegner points of [Da01], and the sign that arises depends on whether E has split or non-split multiplicative reduction at p.

4.5. **Adjoint classes.** The case where h is dual to g is of considerable arithmetic interest, since in that case the representation

$$V_{gh} = \mathbb{Q}_p \oplus M_g, \qquad (\text{where } M_g := \mathrm{Ad}^0(V_g)),$$

admits the trivial representation as a constituent. The generalised Kato classes attached to g and h may then, in appropriate circumstances, contribute to the Mordell-Weil group $E(\mathbb{Q})$, and it is interesting to understand when this occurs.

The Hasse-Weil-Artin L-series

$$L(E, V_{gh}, s) = L(E, s) L(E, M_g, s)$$

has a double zero at $s = 1$ in one of the following three cases:

(1) the rank $(0,2)$ setting where $L(E,1)\neq 0$ and $L(E,M_g,s)$ has a double zero at $s=1$;
(2) the rank $(1,1)$ case where $L(E,s)$ and $L(E,M_g,s)$ each vanish to order 1 at $s=1$;
(3) the rank $(2,0)$ setting where $L(E,s)$ has a double zero at the center and $L(E,M_g,1)\neq 0$. This case is particularly intriguing for its direct connection with the arithmetic of elliptic curves of rank two over $\mathbb{Q}$.

In all the examples that will be treated below, we always set

$$\alpha_h=\alpha_g^{-1},\qquad \beta_h=\beta_g^{-1},$$

so that

$$\alpha_g\alpha_h=1,\quad \alpha_g\beta_h=\alpha_g/\beta_g,\quad \beta_g\alpha_h=\frac{\beta_g}{\alpha_g},\quad \beta_g\beta_h=1.$$

4.5.1. *The rank* $(0,2)$ *setting.* Let (P,Q) be an L-basis for $(E(H)_L\otimes V_{12})^{G_{\mathbb{Q}}}=(E(H)_L\otimes M_g)^{G_{\mathbb{Q}}}$, and write

$$\begin{aligned} P &= P_{\frac{\alpha}{\beta}}\otimes v_{\beta/\alpha}+P_1\otimes v_1+P_{\frac{\beta}{\alpha}}\otimes v_{\alpha/\beta},\\ Q &= Q_{\frac{\alpha}{\beta}}\otimes v_{\beta/\alpha}+Q_1\otimes v_1+Q_{\frac{\beta}{\alpha}}\otimes v_{\alpha/\beta}, \end{aligned}$$

where $v_{\beta/\alpha}$, v_1, and $v_{\alpha/\beta}$ are bases for the Fr_p-eigenspaces of W_g attached to the eigenvalues $\frac{\beta_g}{\alpha_g}$, 1, and α_g/β_g respectively. Then the four enhanced regulators are given, up to $L^\times$, as follows:

$$\begin{aligned} &\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12})=\log_p(P_1)\cdot Q-\log_p(Q_1)\cdot P, &&\widetilde{\mathrm{Reg}}_{\alpha\beta}(E,V_{12})=\log_p(P_{\frac{\beta}{\alpha}})\cdot Q-\log_p(Q_{\frac{\beta}{\alpha}})\cdot P,\\ &\widetilde{\mathrm{Reg}}_{\beta\alpha}(E,V_{12})=\log_p(P_{\frac{\alpha}{\beta}})\cdot Q-\log_p(Q_{\frac{\alpha}{\beta}})\cdot P, &&\widetilde{\mathrm{Reg}}_{\beta\beta}(E,V_{12})=\log_p(P_1)\cdot Q-\log_p(Q_1)\cdot P. \end{aligned}$$

Conjecture 3.12 suggests in this case that the generalised Kato class $\kappa(f,g_\alpha,h_\beta)$ generates the kernel of the map $\log_{\alpha\beta}$ in the two dimensional $\mathbb{Q}_p$-vector space $(E(H)_L\otimes M_g)^{G_{\mathbb{Q}}}\otimes_L\mathbb{Q}_p$.

We refer the reader to [DLR1, Example 5.4] for the numerical verification of Conjecture 3.8 for two different instances in this setting.

4.5.2. *The rank* $(1,1)$ *setting.* Let P be a generator of $E(\mathbb{Q})_L$ and let Q be a generator of $(E(H)_L\otimes M_g)^{G_{\mathbb{Q}}}$. With the same notational conventions as before, we find:

$$\begin{aligned} &\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12})=\log_p(P)\cdot Q-\log_p(Q_1)\cdot P, &&\widetilde{\mathrm{Reg}}_{\alpha\beta}(E,V_{12})=\log_p(Q_{\frac{\beta}{\alpha}})\cdot P,\\ &\widetilde{\mathrm{Reg}}_{\beta\alpha}(E,V_{12})=\log_p(Q_{\frac{\alpha}{\beta}})\cdot P, &&\widetilde{\mathrm{Reg}}_{\beta\beta}(E,V_{12})=\log_p(P)\cdot Q-\log_p(Q_1)\cdot P. \end{aligned} \tag{44}$$

In contrast, we have

$$\begin{aligned} &\mathrm{Reg}_{g_\alpha}(E,V_{12})=\log_p(P)\cdot\log_p(Q_{\frac{\beta}{\alpha}}) &&\mathrm{Reg}_{g_\beta}(E,V_{12})=\log_p(P)\cdot\log_p(Q_{\frac{\alpha}{\beta}}),\\ &\mathrm{Reg}_{h_\alpha}(E,V_{12})=\log_p(P)\log_p(Q_{\frac{\alpha}{\beta}}), &&\mathrm{Reg}_{h_\beta}(E,V_{12})=\log_p(P)\cdot\log_p(Q_{\frac{\beta}{\alpha}}). \end{aligned} \tag{45}$$

Many numerical examples where the p-adic iterated integrals atttached to (f,g,h) are seen to agree with these regulators are described in in [DLR1, §5]. It is worth noting that the expression $\log_p(Q_1)$ that appears in the enhanced regulators of (44) disappears from the regulators (45) that arose in [DLR1].

4.5.3. *The rank* $(2,0)$ *setting.* Let (P,Q) be an L-basis of the two-dimensional L-vector space $E(\mathbb{Q})_L$. With the same notational conventions as before, we find:

$$\begin{aligned} &\widetilde{\mathrm{Reg}}_{\alpha\alpha}(E,V_{12})=\log_p(P)\cdot Q-\log_p(Q)\cdot P, &&\widetilde{\mathrm{Reg}}_{\alpha\beta}(E,V_{12})=0,\\ &\widetilde{\mathrm{Reg}}_{\beta\alpha}(E,V_{12})=0, &&\widetilde{\mathrm{Reg}}_{\beta\beta}(E,V_{12})=\log_p(P)\cdot Q-\log_p(Q)\cdot P. \end{aligned}$$

In other words, the generalised Kato classes $\kappa(f,g_\alpha,h_\alpha)$ and $\kappa(f,g_\beta,h_\beta)$ give (essentially, the same, up to $L^\times$-multiples) canonical element of $E(\mathbb{Q})_{\mathbb{Q}_p}$, which is expected to be non-trivial precisely when

$$L''(E,1)\neq 0,\qquad L(E,M_g,1)\neq 0.$$

Note that on the other hand

$$\mathrm{Reg}_{g_\alpha}(E,V_{12})=\mathrm{Reg}_{g_\beta}(E,V_{12})=\mathrm{Reg}_{h_\alpha}(E,V_{12})=\mathrm{Reg}_{h_\beta}(E,V_{12})=0.$$

This last example gives yet another instance where Conjecture 3.12 represents a genuine strengthening of the elliptic Stark conjectures of [DLR1]. It predicts that that generalised Kato classes of the form

$\kappa(f, g_\alpha, \bar{g}_{1/\alpha})$ ought to give non-trivial elements in the pro-p-Selmer groups of elliptic curves of rank two over $\mathbb{Q}$, when the auxiliary L-value $L(E, M_g, 1)$ is non-zero. Testing this prediction experimentally seems to present an interesting challenge.

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