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KATO'S EULER SYSTEM AND RATIONAL POINTS ON ELLIPTIC CURVES I: A p-ADIC BEILINSON FORMULA

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ABSTRACT

This article is the first in a series devoted to Kato's Euler system arising from p-adic families of Beilinson elements in the K-theory of modular curves. It proves a p-adic Beilinson formula relating the syntomic regulator (in the sense of Coleman–de Shalit and Besser) of certain distinguished elements in the K-theory of modular curves to the special values at integer points ≥ 2 of the Mazur–Swinnerton–Dyer p-adic L-function attached to cusp forms of weight 2. When combined with the explicit relation between syntomic regulators and p-adic étale cohomology, this leads to an alternate proof of the main results of [Br2] and [Ge] which is independent of Kato's explicit reciprocity law.

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1. Introduction

This article is the first in a series devoted to Kato's Euler system arising from p-adic families of Beilinson elements in the K-theory of modular curves. In a simple but prototypical setting, Kato's construction [Kato], [Colz2] yields a global class κ in $H^1(\mathbb{Q}, V_p(E))$, where $V_p(E)$ is the p-adic Galois representation attached to a modular elliptic curve $E/\mathbb{Q}$. Kato's reciprocity law implies that κ is crystalline, and hence belongs to the p-adic Selmer group of E, precisely when the Hasse–Weil L-series $L(E,s)$ vanishes at $s=1$. In this case, Perrin-Riou [PR1] conjectures that the image $\mathrm{res}_p(\kappa)$ in $H^1_f(\mathbb{Q}_p, V_p(E))$ is non-zero if and only if $L'(E,1)$ is non-zero, and predicts a precise relation between the logarithm of $\mathrm{res}_p(\kappa)$ and the formal group logarithm of a global point in $E(\mathbb{Q})$.

The ultimate goal of this series is the proof of Perrin-Riou's conjecture, which will be described in [BD3]. One of the cornerstones of our strategy is a proof of a p-adic Beilinson formula relating the syntomic regulators (in the sense of Coleman–de Shalit and Besser) of certain distinguished elements in the K-theory of modular curves to the special values at integer points ≥ 2 of the Mazur–Swinnerton–Dyer p-adic L-function attached to a cusp form f of weight 2. This proof is independent of Kato's reciprocity law, and will in fact be used in [BD2] to re-derive it. It is based instead on the direct evaluation (Theorems 4.4 and 5.1) of the p-adic Rankin L-function attached to a Hida family interpolating f introduced in Section 3.1. The p-adic Beilinson formula then follows from the factorisation of this p-adic Rankin L-function into a product of two Mazur–Kitagawa p-adic L-functions (Theorem 3.4).

In the complex setting, the connection between regulators and values of $L(E,s)$ at integers $\ell \geq 2$ was described in the work of Beilinson [Bei], and in prior work of Bloch [Bl] for elliptic curves with complex multiplication. The first p-adic avatar of this formula was obtained by Coleman and de Shalit [CodS] in the CM setting considered by Bloch. The work of Brunault [Br2] for $\ell = 2$ and Gealy [Ge] for $\ell \geq 2$ extended this p-adic Beilinson formula to all (modular) elliptic curves as a consequence of Kato's general machinery. Our approach, which is somewhat more direct, relies instead on Besser's description ([Bes1] and [Bes2]) of the Coleman–de Shalit p-adic regulator and on the techniques developed in [DR] for relating p-adic Abel–Jacobi images of diagonal cycles to values of Garrett–Rankin triple product p-adic L-functions. The results of

[DR]—and, by extension, of the present work as well as of [BDR]—were inspired by the study undertaken earlier in [BDP], which the reader may consult for an analogous formula in the setting of Heegner points (resp. "generalised Heegner cycles") on modular curves (resp. on products of Kuga–Sato varieties with powers of CM elliptic curves).

After this paper was submitted, the authors' attention was drawn to the earlier work of Maximilian Niklas [Nik] which also provides a direct proof of the p-adic Beilinson formula based on a description of the rigid syntomic regulator given in [BK]. The principal novelty of the present work—and, arguably, its main interest—lies in the explicit connection that it draws with

(1) the results of [DR] relating the p-adic Abel–Jacobi images of diagonal cycles on triple products of Kuga–Sato varieties to special values of Harris–Tilouine's p-adic L-functions attached to the Garrett–Rankin convolution of three Hida families of cusp forms;
(2) the results in [BDR] relating the syntomic regulators of Beilinson–Flach elements in higher Chow groups of products of two modular curves to special values of Hida's p-adic L-functions attached to the Rankin–Selberg convolution of two cusp forms;
(3) the results in [BDP] relating the p-adic logarithms of Heegner points on modular curves to special values of the p-adic L-functions attached to the Rankin convolution of a weight two cusp form and a theta series of an imaginary quadratic field, based on a formula of Waldspurger.

The authors' strategy for proving Perrin–Riou's conjecture is based on a comparison between Kato's Euler system and those arising in the above settings.

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2. Complex L-series

This section provides explicit formulae (see equations (20) and (23)) for the special values of the complex Rankin L-functions associated to the convolution of cusp forms and Eisenstein series, based on Rankin's method and the reducibility of the Galois representations of Eisenstein series. These formulae are crucial in the definition and study of the Rankin p-adic L-functions of Section 3. Along the way, we briefly recall the application of Rankin's method to the proof of the original complex Beilinson formula (cf. Proposition 2.3).

The Poincaré upper half plane of complex numbers with strictly positive imaginary part is denoted $\mathcal{H}$, and the variable on $\mathcal{H}$ is written as $z = x + iy$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$.

A Dirichlet character of **modulus** N is a homomorphism $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$, extended to a function on $\mathbb{Z}$ by the convention that $\psi(n) = 0$ whenever $\gcd(n, N) \neq 1$. The **conductor** of ψ is the smallest positive integer M for which there is a Dirichlet character of modulus M agreeing with ψ on the integers that are relatively prime to N. A Dirichlet character is said to be **primitive** if its conductor is equal to its modulus. If $N = N_1 N_2$ is a factorisation of N into co-prime positive integers N_1 and N_2, frequent use will be made of the fact (following from the Chinese remainder theorem) that a character ψ of modulus (resp. conductor) N can be uniquely expressed as $\psi = \psi_1\psi_2$, where ψ_j is of modulus (resp. of conductor) N_j.

We denote by $S_k(N, \chi) \subset M_k(N, \chi)$ the spaces of holomorphic cusp forms and modular forms of weight k, level N and character χ, and by $S_k^{\mathrm{an}}(N, \chi) \subset M_k^{\mathrm{an}}(N, \chi)$ their real analytic counterparts consisting of real analytic functions on $\mathcal{H}$ with the same transformation properties under $\Gamma_0(N)$, and having bounded growth (resp. rapid decay) at the cusps for elements of $M_k^{\mathrm{an}}(N, \chi)$ (resp. $S_k^{\mathrm{an}}(N, \chi)$). Likewise, for any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, the spaces $S_k(\Gamma) \subset M_k(\Gamma)$ and $S_k^{\mathrm{an}}(\Gamma) \subset M_k^{\mathrm{an}}(\Gamma)$ are given their obvious meanings. The only cases arising in this article are where Γ is one of the standard Hecke congruence groups $\Gamma_0(N)$ or $\Gamma_1(N)$.

2.1. Eisenstein series. The **non-holomorphic Eisenstein series** of weight k, level N attached to the primitive character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is the function on $\mathcal{H} \times \mathbb{C}$ defined by

$$\widetilde{E}_{k,\chi}(z, s) = \sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}}{}' \frac{\chi^{-1}(n)}{(mz+n)^k} \cdot \frac{y^s}{|mz+n|^{2s}}, \tag{1}$$

where the superscript $'$ indicates that the sum is taken over the non-zero lattice vectors $(m, n) \in N\mathbb{Z} \times \mathbb{Z}$.

The series defining $\widetilde{E}_{k,\chi}(z,s)$ converges for $\Re(s) > 1-k/2$ but admits a meromorphic continuation to all $s \in \mathbb{C}$. A direct calculation shows that

$$\widetilde{E}_{k,\chi}\left(\frac{az+b}{cz+d}, s\right) = \chi(d)(cz+d)^k \widetilde{E}_{k,\chi}(z,s), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

i.e., $\widetilde{E}_{k,\chi}(z,s)$ transforms like a modular form of weight k and character χ on $\Gamma_0(N)$ when viewed as a function of z, and hence belongs to $M_k^{\mathrm{an}}(N,\chi)$. In particular, if $k > 2$

$$\widetilde{E}_{k,\chi}(z) := \widetilde{E}_{k,\chi}(z,0) \text{ belongs to } M_k(N,\chi).$$

(The same conclusion holds for $k \geq 1$, provided that χ is non-trivial.) Assume from now on that χ satisfies the parity condition

$$\chi(-1) = (-1)^k, \tag{2}$$

which guarantees that $\widetilde{E}_{k,\chi}(z)$ is non-zero. We introduce (cf. [Hi93], p. 128) the normalised Eisenstein series $E_{k,\chi}(z)$, related to $\widetilde{E}_{k,\chi}(z)$ by the equation

$$\widetilde{E}_{k,\chi}(z) = 2N^{-k}\tau(\chi^{-1})\frac{(-2\pi i)^k}{(k-1)!}E_{k,\chi}(z), \tag{3}$$

where

$$\tau(\chi) = \sum_{a=1}^{N_\chi} \chi(a) e^{2\pi i a/N_\chi}, \quad N_\chi = \mathrm{cond}(\chi)$$

is the Gauss sum attached to χ, and the q-expansion of $E_{k,\chi}(z)$ is given by

$$E_{k,\chi}(z) = 2^{-1}L(\chi, 1-k) + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n, \quad \sigma_{k-1,\chi}(n) = \sum_{d|n} \chi(d) d^{k-1}. \tag{4}$$

The Shimura–Maass derivative operator

$$\delta_k := \frac{1}{2\pi i}\left(\frac{d}{dz} + \frac{ik}{2y}\right)$$

sends $M_k^{\mathrm{an}}(N,\chi)$ to $M_{k+2}^{\mathrm{an}}(N,\chi)$. A direct calculation (see also loc. cit., page 317, formula (13)) reveals that

$$\delta_k \widetilde{E}_{k,\chi}(z,s) = -\frac{(s+k)}{4\pi}\widetilde{E}_{k+2,\chi}(z,s-1). \tag{5}$$

Denoting by $\delta_k^t := \delta_{k+2t-2} \cdots \delta_{k+2}\delta_k$ the t-fold iterate of the Shimura–Maass derivative, it follows that

$$\delta_k^t \widetilde{E}_{k,\chi}(z,s) = \frac{(-1)^t}{(4\pi)^t}(s+k)\cdots(s+k+t-1)\widetilde{E}_{k+2t,\chi}(z,s-t).$$

Replacing k by $k-2t$ and setting $s=0$ in the above equation, we find

$$\widetilde{E}_{k,\chi}(z,-t)=\frac{(k-2t-1)!}{(k-t-1)!}(-4\pi)^t\delta_{k-2t}^t\widetilde{E}_{k-2t,\chi}(z). \tag{6}$$

In particular, if $0\le t\le k/2-1$, then the Eisenstein series $\widetilde{E}_{k,\chi}(z,-t)$, while not holomorphic, is an example of a **nearly holomorphic modular form** in the sense of [Sh2].

We will also have a need for the more general Eisenstein series $E_k(\chi_1,\chi_2)\in M_k(N,\chi_1\chi_2)$, attached to a pair χ_1 and χ_2 of Dirichlet characters of modulus N_1 and N_2, respectively, with $N_1N_2=N$. Recall that condition (2) is in force, i.e., $\chi_1\chi_2(-1)=(-1)^k$. Then, for $k\ge 1$ and $(\chi_1,\chi_2)\ne(\mathbf{1},\mathbf{1})$, the q-expansion of $E_k(\chi_1,\chi_2)$ is given by

$$E_k(\chi_1,\chi_2)(z)=\delta_{\chi_1}L(\chi_1^{-1}\chi_2,1-k)+\sum_{n=1}^{\infty}\sigma_{k-1}(\chi_1,\chi_2)(n)q^n, \tag{7}$$

where $\delta_{\chi_1}=1/2$ if $N_1=1$ and 0 otherwise, and

$$\sigma_{k-1}(\chi_1,\chi_2)(n)=\sum_{d|n}\chi_1(n/d)\chi_2(d)d^{k-1}.$$

Thus, $E_k(\mathbf{1},\chi)$ is equal to $E_{k,\chi}$. Note that $E_k(\chi_1,\chi_2)$ is a simultaneous eigenvector for all the Hecke operators, and satisfies

$$L(E_k(\chi_1,\chi_2),s)=L(\chi_1,s)L(\chi_2,s-k+1). \tag{8}$$

2.2. Rankin's method. Let

$$f:=\sum_{n=1}^{\infty}a_n(f)q^n\in S_k(N,\chi_f)$$

be a cusp form of weight k, level N and character χ_f, and let

$$g:=\sum_{n=0}^{\infty}a_n(g)q^n\in M_\ell(N,\chi_g)$$

be a modular form of weight $\ell<k$ and character χ_g. We do not assume for now that f or g are eigenforms. Let

$$D(f,g,s):=\sum a_n(f)a_n(g)n^{-s}$$

denote the Rankin L-series attached to f and g. Recall the Petersson scalar product defined on $S_k^{\mathrm{an}}(N,\chi)\times M_k^{\mathrm{an}}(N,\chi)$. It is given by the formula

$$\langle f_1, f_2\rangle_{k,N} := \int_{\Gamma_0(N)\backslash\mathcal{H}} y^k\overline{f_1(z)}f_2(z)\frac{dxdy}{y^2}, \tag{9}$$

and is hermitian linear in the first argument and $\mathbb{C}$-linear in the second. Let $\chi := \chi_f^{-1}\chi_g^{-1}$, and denote by $f^*\in S_k(N,\chi_f^{-1})$ the modular form satisfying $a_n(f^*)=\overline{a_n}$. Since the forms $f^*(z)$ and $\widetilde{E}_{k-\ell,\chi}(z,s)g(z)$ belong to $S_k(N,\chi_f^{-1})$ and $M_k^{\mathrm{an}}(N,\chi_f^{-1})$ respectively, it is natural to consider their Petersson scalar product.

Proposition 2.1 (Shimura): *For all $s\in\mathbb{C}$ with $\Re(s)\gg 0$,*

$$\left\langle f^*(z)\ ,\ \widetilde{E}_{k-\ell,\chi}(z,s)\cdot g(z)\right\rangle_{k,N} = 2\frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}L(\chi^{-1},2s+k-\ell)\times D(f,g,s+k-1).$$

Proof. See formula (2.4) of [Sh1], where this result is proved by an application of Rankin's method. See also [Hi93], page 317, formula (1), for a statement in the form given here. ∎

Replacing s by $s-k+1$ in Proposition 2.1 and rearranging the factors, we obtain

(10)

$$L(\chi^{-1},2s-k-\ell+2)D(f,g,s) = \frac{1}{2}\frac{(4\pi)^s}{\Gamma(s)}\left\langle f^*(z),\widetilde{E}_{k-\ell,\chi}(z,s-k+1)\cdot g(z)\right\rangle_{k,N}.$$

Assume now that the modular forms f and g are normalised eigenforms of level N. We do not assume that they are new of this level, but we do assume that they are simultaneous eigenvectors for the Hecke operators T_r with $\gcd(r,N)=1$ as well as the operators U_r attached to the primes r dividing N. For each prime p, let $\alpha_p(f)$ and $\beta_p(f)$ be the roots of the Hecke polynomial $x^2-a_p(f)x+\chi_f(p)p^{k-1}$, choosing $(\alpha_p(f),\beta_p(f))=(a_p(f),0)$ when $p|N$. Similarly, we let $\alpha_p(g)$ and $\beta_p(g)$ denote the roots of the polynomial $x^2-a_p(g)x+\chi_g(p)p^{\ell-1}$.

Then the coefficients of the L-series $D(f,g,s)$ are weakly multiplicative and hence $D(f,g,s)$ has an Euler product factorisation over the rational primes p:

(11)
$$D(f,g,s) = \prod_p D_{(p)}(f,g,s), \quad \text{where } D_{(p)}(f,g,s) = \sum_{n=0}^{\infty} a_{p^n}(f) a_{p^n}(g) p^{-s}.$$

Let

$$\begin{aligned} L(f\otimes g, s) &:= \prod_p L_{(p)}(f\otimes g, s), \quad \text{where} \\ L_{(p)}(f\otimes g,s) &:= (1-\alpha_p(f)\alpha_p(g)p^{-s})^{-1}(1-\alpha_p(f)\beta_p(g)p^{-s})^{-1} \\ &\quad \times (1-\beta_p(f)\alpha_p(g)p^{-s})^{-1}(1-\beta_p(f)\beta_p(g)p^{-s})^{-1}. \end{aligned}$$

The calculation of the Euler factors $D_{(p)}(f,g,s)$ — a mildly tedious exercise in manipulation and rearranging of infinite series — shows that, for all primes p,

$$D_{(p)}(f,g,s) = (1-\chi^{-1}(p)p^{k+\ell-2-2s})L_{(p)}(f\otimes g,s), \tag{12}$$

so that

$$L(f\otimes g,s) = L(\chi^{-1}, 2s-k-\ell+2)D(f,g,s). \tag{13}$$

By combining (10) and (13), we find that

$$L(f\otimes g,s) = \frac{1}{2}\frac{(4\pi)^s}{\Gamma(s)}\left\langle f^*(z), \widetilde{E}_{k-\ell,\chi}(z,s-k+1)\cdot g(z)\right\rangle_{k,N}. \tag{14}$$

Choose integers m and t satisfying

$$k = \ell + m + 2t,$$

and set

$$c := \frac{k+\ell+m-2}{2} = k-t-1.$$

By specialising equation (14) at $s=c$, we find

$$L(f\otimes g,c) = \frac{1}{2}\frac{(4\pi)^c}{\Gamma(c)}\left\langle f^*(z), \widetilde{E}_{k-\ell,\chi}(z,-t)\cdot g(z)\right\rangle_{k,N}. \tag{15}$$

If $m\geq 1$ and $t\geq 0$, then replacing k by $k-\ell$ in equation (6), we obtain

$$\widetilde{E}_{k-\ell,\chi}(z,-t) = \frac{(m-1)!}{(m+t-1)!}(-4\pi)^t\delta_m^t \widetilde{E}_{m,\chi}(z). \tag{16}$$

Combining (15) with (16) gives

(17)
$$L(f\otimes g,c)=\frac{1}{2}(-1)^t(4\pi)^{c+t}\frac{(m-1)!}{(m+t-1)!(c-1)!}\left\langle f^*(z),\delta_m^t\widetilde{E}_{m,\chi}(z)\cdot g(z)\right\rangle_{k,N}.$$

Furthermore, in light of equation (3), we have

$$(18)\quad L(f\otimes g,c) = \frac{(-1)^t 2^{k-1}(2\pi)^{k+m-1}(iN)^{-m}\tau(\chi^{-1})}{(m+t-1)!(c-1)!}\left\langle f^*(z),\delta_m^t E_{m,\chi}(z)\cdot g(z)\right\rangle_{k,N}.$$

Equation (18) is equivalent to Theorem 2 of [Sh1]. Note that the normalisations of Eisenstein series used in loc. cit. are different from those adopted here.

2.3. A factorisation of critical values. Let now g be the Eisenstein series $E_\ell(\chi_1,\chi_2)$ of equation (7), and assume that

$$\chi_g\ \ (=\chi_1\chi_2)=\chi_f^{-1}\chi^{-1}.$$

In light of (8), the left hand side of (18) becomes

$$(19)\qquad L(f\otimes E_\ell(\chi_1,\chi_2),c)=L(f,\chi_1,c)\cdot L(f,\chi_2,c-\ell+1).$$

Assumption 2.2: The following assumptions on (k,ℓ,m) and (f,χ_1,χ_2) will be enforced for the rest of the paper:

(1) $\ell=m$,
(2) $\chi_f=\mathbf{1}$, so that $f^*=f$ and $\chi=\bar\chi_1\bar\chi_2$,
(3) χ is primitive, so that $|\tau(\chi)|^2=N$,
(4) $(N_1,N_2)=1$, so that $N_{\chi_1}=N_1$ and $N_{\chi_2}=N_2$, and

$$\tau(\chi)=\tau(\bar\chi_1)\tau(\bar\chi_2)\chi_1(N_2)\chi_2(N_1)=\overline{\tau(\chi_1)}\ \overline{\tau(\chi_2)}\chi(-1)\chi_1(N_2)\chi_2(N_1).$$

Under the above assumptions, f is an eigenform on $\Gamma_0(N)$ of even weight $k=2\ell+2t$. If furthermore $t\geq 0$, then $c=k/2+\ell-1$ is a critical point for the L-functions $L(f\otimes E_\ell(\chi_1,\chi_2),s)$ and $L(f,\chi_1,s)$, and (19) becomes

$$(20)\quad L(f\otimes E_\ell(\chi_1,\chi_2),k/2+\ell-1)=L(f,\chi_1,k/2+\ell-1)\cdot L(f,\chi_2,k/2).$$

Note that $k/2$ is the central critical point for $L(f,\chi_2,s)$. We choose complex periods Ω_f^+ and Ω_f^- as in Proposition 1.1 of [BD1]. These periods satisfy

$$(21)\qquad \Omega_f^+\Omega_f^-=(2\pi)^2\langle f,f\rangle_{k,N},$$

and, for $1 \leq j \leq k-1$,

$$L^*(f,\psi,j) := \frac{(j-1)!\tau(\bar{\psi})}{(-2\pi i)^{j-1}\Omega_f^{\varepsilon}} L(f,\psi,j) \quad \text{belongs to } \mathbb{Q}_{f,\psi}, \tag{22}$$

where ψ is any Dirichlet character, $\varepsilon = \psi(-1)(-1)^{j-1}$ and $\mathbb{Q}_{f,\psi}$ is the field generated by the Fourier coefficients of f and the values of ψ. (See Proposition 1.3 of loc. cit., where $\psi = \bar{\psi}$.) By combining equations (18), (20) and (22), we find

$$L^*(f,\chi_1,k/2+\ell-1)\cdot L^*(f,\chi_2,k/2) = C_{f,\chi_1,\chi_2}\cdot \frac{\left\langle f,\, (\delta_\ell^{k/2-\ell}E_{\ell,\chi})\cdot E_\ell(\chi_1,\chi_2)\right\rangle_{k,N}}{\langle f,f\rangle_{k,N}}, \tag{23}$$

where

$$C_{f,\chi_1,\chi_2} := \frac{i2^{k-1}}{N^{\ell-1}}\chi_1(N_2)\chi_2(N_1). \tag{24}$$

2.4. Beilinson's formula. We now focus on the case $k=\ell=2$ (so that $c=2$ and $t=-1$) and deduce a complex Beilinson formula for the non critical value of $L(f,s)$ at $s=2$.

By equation (15) with $g = E_2(\chi_1,\chi_2)$,

$$L(f\otimes E_2(\chi_1,\chi_2),2) = \frac{1}{2}(4\pi)^2\left\langle f(z), \widetilde{E}_{0,\chi}(z,1)\cdot E_2(\chi_1,\chi_2)(z)\right\rangle_{2,N}. \tag{25}$$

By specialising equation (5) to the case $k=0$ and $s=1$, and invoking (3), we obtain

$$\frac{1}{2\pi i}\frac{d}{dz}\widetilde{E}_{0,\chi}(z,1) = -\frac{1}{4\pi}\widetilde{E}_{2,\chi}(z) = 2\pi N^{-2}\tau(\chi^{-1})E_{2,\chi}(z). \tag{26}$$

Given a field F, let $\mathrm{Eis}_\ell(\Gamma_1(N),F)$ denote the subspace of $M_\ell(\Gamma_1(N),F)$ spanned by the weight ℓ Eisenstein series with coefficients in F. Let $Y_1 = Y_1(N)$ denote the usual open modular curve of level N over $\mathbb{Q}$ whose complex points are identified with $\Gamma_1(N)\backslash\mathcal{H}$, and let $\bar{Y}_1 = \bar{Y}_1(N)$ denote its extension to $\bar{\mathbb{Q}}$. The logarithmic derivative

$$\mathrm{dlog}(u) := \frac{1}{2\pi i}\frac{u'(z)}{u(z)}$$

gives a surjective homomorphism

$$\mathcal{O}(\bar{Y}_1(N))^\times \otimes F \xrightarrow{\mathrm{dlog}} \mathrm{Eis}_2(\Gamma_1(N),F). \tag{27}$$

Take F to be a finite extension of $\mathbb{Q}$ containing the values of all characters of conductor dividing N. Let u_χ and $u(\chi_1, \chi_2)$ be units satisfying

$$\mathrm{dlog}(u_\chi) = E_{2,\chi}, \quad \mathrm{dlog}(u(\chi_1, \chi_2)) = E_2(\chi_1, \chi_2). \tag{28}$$

It can be shown that $u(\chi_1, \chi_2)$ belongs to the χ_1-eigenspace $(\mathcal{O}(\bar{Y}_1)^\times \otimes F)^{\chi_1}$ for the natural action of $G_\mathbb{Q}$ on the space of modular units. The unit u_χ is only determined up to a multiplicative constant. It can be shown (see for example [Br1], Section 5) that u_χ can be normalised in such a way that the equality

$$\widetilde{E}_{0,\chi}(z, 1) = 2\pi N^{-2}\tau(\chi^{-1}) \log |u_\chi(z)| \tag{29}$$

holds. Note that equation (29) is consistent with (26). By combining (25) with (29) and (28), we obtain

(30)

$$L(f \otimes E_2(\chi_1, \chi_2), 2) = 16\pi^3 N^{-2}\tau(\chi^{-1}) \langle f(z), \log |u_\chi(z)| \cdot \mathrm{dlog}(u(\chi_1, \chi_2)(z))\rangle_{2,N} .$$

Extend definition (22) of $L^*(f, \psi, j)$ to integers j lying outside of the critical range. By the formulae in Section 2.3, we may rewrite (30) as

(31)

$$L^*(f, \chi_1, 2) \cdot L^*(f, \chi_2, 1) = \frac{C_{f,\chi_1,\chi_2}}{\langle f, f\rangle_{2,N}} \cdot \int_{\Gamma_0(N)\backslash\mathcal{H}} \bar{f} \cdot \log |u_\chi| \cdot \mathrm{dlog}(u(\chi_1, \chi_2))dxdy.$$

Define the anti-holomorphic differential attached to f to be

$$\eta_f^{\mathrm{ah}} := \frac{\bar{f}(z)d\bar{z}}{\langle f, f\rangle_{2,N}} . \tag{32}$$

Up to the constant C_{f,χ_1,χ_2}, the right-hand side of (31) is the value on the class of η_f^{ah} of the complex regulator $\mathbf{reg}_\mathbb{C}\{u_\chi, u(\chi_1, \chi_2)\}$ attached to the symbol

$$\{u_\chi, u(\chi_1, \chi_2)\} \in K_2(\mathbb{C}(\bar{Y}_1(N)).$$

We obtain the following proposition, which generalizes slightly the explicit version of Beilinson's theorem proved in [Br1].

Proposition 2.3:

$$L^*(f, \chi_1, 2) \cdot L^*(f, \chi_2, 1) = C_{f,\chi_1,\chi_2} \cdot \mathbf{reg}_\mathbb{C}\{u_\chi, u(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ah}}).$$

This is the formula whose precise p-adic counterpart is obtained in Corollary 5.2 below.

2.5. Algebraicity. The modular form

$$\Xi_{k,\ell}(\chi_1,\chi_2) := (\delta_\ell^{k/2-\ell} E_{\ell,\chi}) \cdot E_\ell(\chi_1,\chi_2) \tag{33}$$

belongs to the space $M_k^{\mathrm{nh}}(N, \mathbb{Q}_{\chi_1\chi_2})$ of nearly-holomorphic modular forms defined over $\mathbb{Q}_{\chi_1\chi_2}$ in the sense of Shimura (cf. Section 2.3 of [DR]). Hence, its image

$$\Xi_{k,\ell}^{\mathrm{hol}}(\chi_1,\chi_2) := \Pi_N^{\mathrm{hol}}(\Xi_{k,\ell}(\chi_1,\chi_2)) \tag{34}$$

under the holomorphic projection Π_N^{hol} of loc. cit. belongs to $M_k(N, \mathbb{Q}_{\chi_1\chi_2})$. The ratio appearing in the right-hand side of (23) can then be re-written as

$$\frac{\left\langle f, \Xi_{k,\ell}(\chi_1,\chi_2)\right\rangle_{k,N}}{\langle f, f\rangle_{k,N}} = \frac{\left\langle f, \Xi_{k,\ell}^{\mathrm{hol}}(\chi_1,\chi_2)\right\rangle_{k,N}}{\langle f, f\rangle_{k,N}}, \tag{35}$$

and hence belongs to $\mathbb{Q}_{f,\chi_1\chi_2}$. One recovers Shimura's approach [Sh1] to the algebraicity of $L^*(f,\psi,j)$, which differs from the approach based on modular symbols followed in [BD1].

For the purposes of making the connection with p-adic regulators, it is useful to describe the right-hand side of (35) more algebraically, in terms of the Poincaré duality on the de Rham cohomology of the open modular curve with values in appropriate sheaves with connection (as described in Section 2.2 of [DR]). More precisely, let Y, resp. X be the open modular curve $Y_0(N)$, resp. the complete modular curve $X_0(N)$, and let K be any field containing $\mathbb{Q}_{f,\chi_1\chi_2}$. Denote by $\mathcal{E} \longrightarrow Y$ the universal elliptic curve over Y, and by ω the sheaf of relative differentials on $\mathcal{E}$ over Y, extended to $X = X_0(N)$ as in Section 1.1 of [BDP]. Recall the Kodaira–Spencer isomorphism $\omega^2 = \Omega^1_X(\log \mathrm{cusps})$, where $\Omega^1_X(\log \mathrm{cusps})$ is the sheaf of regular differentials on Y with log poles at the cusps. A modular form ϕ on $\Gamma_0(N)$ of weight $k = r+2$ with Fourier coefficients in K corresponds to a global section of the sheaf $\omega^{r+2} = \omega^r \otimes \Omega^1_X(\log \mathrm{cusps})$ over X_K.

The sheaf ω^r can be viewed as a subsheaf of $\mathcal{L}_r := \mathrm{Sym}^r \mathcal{L}$, where

$$\mathcal{L} := R^1\pi_*(\mathcal{E} \longrightarrow Y)$$

is the relative de Rham cohomology sheaf on Y, suitably extended to X, equipped with the filtration

$$0 \longrightarrow \omega \longrightarrow \mathcal{L} \longrightarrow \omega^{-1} \longrightarrow 0 \tag{36}$$

arising from the Hodge filtration on the fibers. The sheaf $\mathcal{L}_r$ is a coherent sheaf over X of rank $r+1$, endowed with the Gauss–Manin connection

$$\nabla : \mathcal{L}_r \longrightarrow \mathcal{L}_r \otimes \Omega^1_X(\log \mathrm{cusps}).$$

Let $H^1_{\mathrm{dR}}(Y_K, \mathcal{L}_r, \nabla)$, resp. $H^1_{\mathrm{dR},c}(Y_K, \mathcal{L}_r, \nabla)$ be the de Rham cohomology, resp. the de Rham cohomology with compact support of $\mathcal{L}_r$. These two groups are related by the perfect Poincaré pairing

$$\langle\ ,\ \rangle_{k,Y} : H^1_{\mathrm{dR},c}(Y_K, \mathcal{L}_r, \nabla) \times H^1_{\mathrm{dR}}(Y_K, \mathcal{L}_r, \nabla) \longrightarrow K. \tag{37}$$

There are exact sequences

$$0 \longrightarrow H^0(X_K, \omega^r \otimes \Omega^1_X) \longrightarrow H^1_{\mathrm{dR},c}(Y_K, \mathcal{L}_r, \nabla) \longrightarrow H^1(X_K, \omega^{-r} \otimes \mathcal{I}) \longrightarrow 0, \tag{38}$$

$$\begin{aligned} 0 \longrightarrow H^0(X_K, \omega^r \otimes \Omega^1_X(\log \mathrm{cusps})) &\longrightarrow H^1_{\mathrm{dR}}(Y_K, \mathcal{L}_r, \nabla) \\ &\longrightarrow H^1(X_K, \omega^{-r}) \longrightarrow 0, \end{aligned} \tag{39}$$

where $\mathcal{I}$ is the ideal sheaf of the cusps (cf. Sections 2 and 3 of [Col94]). The left-most terms of these two sequences are mutually isotropic, and hence (37) induces a perfect pairing

$$\langle\ ,\ \rangle_{k,Y} : H^1(X_K, \omega^{-r} \otimes \mathcal{I}) \times H^0(X_K, \omega^r \otimes \Omega^1_X(\log \mathrm{cusps})) \longrightarrow K, \tag{40}$$

which is denoted by the same symbol by a slight abuse of notation.

The antiholomorphic differential η_f^{ah} of equation (32) gives rise to a class in $H^1_{\mathrm{dR},c}(Y_{\mathbb{C}}, \mathcal{L}_r, \nabla)$, whose image η_f in $H^1(X_{\mathbb{C}}, \omega^{-r} \otimes \mathcal{I})$ belongs to $H^1(X_K, \omega^{-r} \otimes \mathcal{I})$ (cf. Corollary 2.13 of [DR]). Recalling the concrete definition of (40) via complex integration given in (9), we find that the right-hand side of (35) is equal to

$$\frac{\left\langle f, \Xi^{\mathrm{hol}}_{k,\ell}(\chi_1, \chi_2)\right\rangle_{k,N}}{\langle f, f\rangle_{k,N}} = \left\langle \eta_f, \Xi^{\mathrm{hol}}_{k,\ell}(\chi_1, \chi_2)\right\rangle_{k,Y}. \tag{41}$$

3. p-adic L-functions

This section defines the Rankin p-adic L-function associated to the convolution of two Hida families of cusp forms and Eisenstein series (cf. equation (46)). Furthermore, it shows that this p-adic L-function factors as a product of two Mazur–Kitagawa p-adic L-functions (cf. Theorem 3.4).

A similar p-adic L-function, associated to the convolution of two Hida families of cusp forms, has been constructed by Hida [Hi93]. For the sake of brevity, we follow here the approach of [DR], which constructs the p-adic L-function associated to a triple product of Hida families of cusp forms, referring to the calculations of this article whenever possible.

3.1. Rankin's p-adic L-functions. Let $p \geq 3$ be a prime, and fix an embedding of K into $\mathbb{C}_p$. From now on we will be working under the following

Assumption 3.1: The eigenform f is ordinary at p, and $p \nmid N$.

The f-isotypic part of the exact sequence (38) with $K = \mathbb{C}_p$ then admits a canonical unit root splitting, arising from the action of Frobenius on the de Rham cohomology. Let η_f^{ur} be the lift of η_f to the unit root subspace $H^1_{\mathrm{dR},c}(Y_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)^{f,\mathrm{ur}}$. The right-hand side of (41) is then equal to

$$(42)\qquad \left\langle \eta_f, \Xi^{\mathrm{hol}}_{k,\ell}(\chi_1,\chi_2)\right\rangle_{k,Y} = \left\langle \eta_f^{\mathrm{ur}}, \Xi^{\mathrm{hol}}_{k,\ell}(\chi_1,\chi_2)\right\rangle_{k,Y}.$$

After viewing $\Xi^{\mathrm{hol}}_{k,\ell}(\chi_1,\chi_2)$ as an overconvergent p-adic modular form, Lemma 2.10 of [DR] identifies its ordinary projection $e_{\mathrm{ord}}\Xi^{\mathrm{hol}}_{k,\ell}(\chi_1,\chi_2)$ with a cohomology class in $H^1_{\mathrm{dR}}(Y_K, \mathcal{L}_r, \nabla)^{\mathrm{ord}}$. By Proposition 2.11 of loc. cit., the right-hand side of (42) can be re-written as

$$(43)\qquad \left\langle \eta_f^{\mathrm{ur}}, \Xi^{\mathrm{hol}}_{k,\ell}(\chi_1,\chi_2)\right\rangle_{k,Y} = \left\langle \eta_f^{\mathrm{ur}}, e_{\mathrm{ord}}\Xi^{\mathrm{hol}}_{k,\ell}(\chi_1,\chi_2)\right\rangle_{k,Y}.$$

By Proposition 2.8 of loc. cit.,

$$(44)\qquad \Xi^{\mathrm{ord}}_{k,\ell}(\chi_1,\chi_2) := e_{\mathrm{ord}}\Xi^{\mathrm{hol}}_{k,\ell}(\chi_1,\chi_2) = e_{\mathrm{ord}}((d^{k/2-\ell}E_{\ell,\chi})\cdot E_\ell(\chi_1,\chi_2)),$$

where $d = q\frac{d}{dq}$ is Serre's derivative operator on p-adic modular forms.

Note that the ordinary p-stabilisation of $E_\ell(\chi_1,\chi_2)$ is the weight ℓ specialisation of a Hida family of Eisenstein series denoted $\mathbf{E}(\chi_1,\chi_2)$. Likewise, let $\mathbf{f}$ be a Hida family of eigenforms on $\Gamma_0(N)$, indexed by a weight variable k in a suitable neighborhood $U_{\mathbf{f}}$ of $(\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$, which is contained in a single residue class modulo $p-1$. For $k \in U_{\mathbf{f}} \cap \mathbb{Z}_{\geq 2}$, let $f_k \in S_k(N) = S_k(N,\mathbf{1})$ be the classical modular form whose p-stabilisation is the weight k specialisation of $\mathbf{f}$.

Given a p-adic modular form $g = \sum b_n q^n$, let

$$g^{[p]} := \sum_{p\nmid n} b_n q^n$$

denote its "p-depletion". The family of p-adic modular forms

$$(45)\qquad \Xi^{\mathrm{ord},p}_{k,\ell}(\chi_1,\chi_2) := e_{\mathrm{ord}}((d^{k/2-\ell}E^{[p]}_{\ell,\chi})\cdot E_\ell(\chi_1,\chi_2))$$

has Fourier coefficients which extend analytically to $U_{\mathbf{f}} \times (\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p)$, as functions in k and ℓ. See for example [Hi93] and [DR], Section 2.6.

PROPOSITION 3.2: *Let e_{f_k} be the projector to the f_k-isotypic subspace $H^1_{\mathrm{dR}}(Y_K, \mathcal{L}_r, \nabla)^{f_k}$. For all $k \geq 2$ and $2 \leq \ell \leq k/2$,*

$$e_{f_k} \Xi^{\mathrm{ord},p}_{k,\ell}(\chi_1, \chi_2) = \frac{\mathcal{E}(f_k, \chi_1, \chi_2, \ell)}{\mathcal{E}(f_k)} \cdot e_{f_k} \Xi^{\mathrm{ord}}_{k,\ell}(\chi_1, \chi_2),$$

where

$$\begin{aligned}\mathcal{E}(f_k, \chi_1, \chi_2, \ell) =&(1 - \beta_p(f_k)\chi_1(p)p^{-k/2-(\ell-1)})(1 - \beta_p(f_k)\bar{\chi_1}(p)p^{-k/2+(\ell-1)}) \\ &\times (1 - \beta_p(f_k)\chi_2(p)p^{-k/2})(1 - \beta_p(f_k)\bar{\chi_2}(p)p^{-k/2}), \\ \mathcal{E}(f_k) =&1 - \beta_p(f_k)^2 p^{-k}.\end{aligned}$$

Proof. This follows from Corollary 4.13 of [DR], in light of Proposition 2.8 of loc. cit. ∎

Set

$$\mathcal{E}^*(f_k) := 1 - \beta_p(f_k)^2 p^{1-k}.$$

Proposition 4.6 of loc. cit. shows that the expression

$$L_p(\mathbf{f}, \mathbf{E}(\chi_1, \chi_2))(k, \ell) := \frac{1}{\mathcal{E}^*(f_k)} \left\langle \eta^{\mathrm{ur}}_{f_k}, \Xi^{\mathrm{ord},p}_{k,\ell}(\chi_1, \chi_2) \right\rangle_{k,Y}, \tag{46}$$

defined for k in $U_{\mathbf{f}} \cap \mathbb{Z}_{\geq 2}$ and $2 \leq \ell \leq k/2$ extends to an analytic function $L_p(\mathbf{f}, \mathbf{E}(\chi_1, \chi_2))$ on $U_{\mathbf{f}} \times (\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p)$, which we refer to as the **p-adic Rankin L-function** attached to $\mathbf{f}$ and $\mathbf{E}(\chi_1, \chi_2)$.

3.2. Factorisation of p-adic L-functions.

Let $L_p(f_k, \psi, s)$ be the Mazur–Swinnerton–Dyer p-adic L-function attached to (f_k, ψ), with ψ equal to χ_1 or χ_2 (cf. Section 14 of [MTT]). We normalise $L_p(f_k, \psi, s)$ so that it satisfies the interpolation property for $1 \leq j \leq k-1$:

$$L_p(f_k, \psi, j) = (1-\overline{\psi}(p)\beta_p(f_k)p^{-j}) \times (1-\psi(p)\beta_p(f_k)p^{-(k-j)}) \times L^*(f_k, \psi, j). \tag{47}$$

Note that the values $L_p(f_k, \chi_1, j)$ and $L_p(f_k, \chi_2, j+\ell-1)$ depend on the choice of periods $\Omega^{\pm}_f$ and $\Omega^{\mp}_f$ that was made in Section 2.3, but their product does not, in light of the normalising condition imposed in (21), since $\chi_1\chi_2(-1) = (-1)^{\ell}$.

PROPOSITION 3.3: *For all $k \in U_{\mathbf{f}} \cap \mathbb{Z}_{\geq 2}$ and for all $2 \leq \ell \leq k/2$, we have*

$$L_p(\mathbf{f}, \mathbf{E}(\chi_1, \chi_2))(k, \ell) = (C_{f_k,\chi_1,\chi_2} \mathcal{E}(f_k)\mathcal{E}^*(f_k))^{-1} \times L_p(f_k, \chi_1, k/2+\ell-1) \times L_p(f_k, \chi_2, k/2).$$

Proof. We have the sequence of equalities:

$$\begin{aligned} L_p(\mathbf{f}, \mathbf{E}(\chi_1, \chi_2))(k, \ell) &= \frac{1}{\mathcal{E}^*(f_k)} \left\langle \eta^{\mathrm{ur}}_{f_k}, \Xi^{\mathrm{ord},p}_{k,\ell}(\chi_1, \chi_2) \right\rangle_{k,Y} && \text{by (46)} \\ &= \frac{\mathcal{E}(f_k, \chi_1, \chi_2, \ell)}{\mathcal{E}(f_k)\mathcal{E}^*(f_k)} \langle \eta^{\mathrm{ur}}_{f_k}, \Xi^{\mathrm{ord}}_{k,\ell}(\chi_1, \chi_2) \rangle_{k,Y} && \text{by Prop. 3.2} \\ &= \frac{\mathcal{E}(f_k, \chi_1, \chi_2, \ell)}{\mathcal{E}(f_k)\mathcal{E}^*(f_k)} \langle \eta_{f_k}, \Xi^{\mathrm{hol}}_{k,\ell}(\chi_1, \chi_2) \rangle_{k,Y} && \text{by (43) and (42)} \\ &= \frac{\mathcal{E}(f_k, \chi_1, \chi_2, \ell)}{\mathcal{E}(f_k)\mathcal{E}^*(f_k)} \frac{\left\langle f_k, \Xi_{k,\ell}(\chi_1, \chi_2) \right\rangle_{k,N}}{\langle f_k, f_k \rangle_{k,N}} && \text{by (41) and (35).} \end{aligned}$$

By (23), the last term can be re-written as

$$\frac{\mathcal{E}(f_k, \chi_1, \chi_2, \ell)}{\mathcal{E}(f_k)\mathcal{E}^*(f_k) C_{f_k,\chi_1,\chi_2}} L^*(f_k, \chi_1, k/2+\ell-1) \cdot L^*(f_k, \chi_2, k/2),$$

so that Theorem 3.3 follows by combining (47) with the exact shape of $\mathcal{E}(f_k, \chi_1, \chi_2, \ell)$ stated in Proposition 3.2. ∎

Fix $k_0 \in U_{\mathbf{f}} \cap \mathbb{Z}_{\geq 2}$. Recall the Mazur–Kitagawa two-variable p-adic L-function $L_p(\mathbf{f}, \psi)(k, s)$ defined in [Ki]. It is related to $L_p(f_k, \psi, s)$ by the equation

(48) $$L_p(\mathbf{f}, \psi)(k, s) = \lambda^{\pm}(k) \cdot L_p(f_k, \psi, s), \qquad k \in U_{\mathbf{f}} \cap \mathbb{Z}_{\geq 2},$$

where $\lambda^{\pm}(k) \in \mathbb{C}_p$ is a p-adic period, arising from the p-adic interpolation of modular symbols, such that $\lambda^{\pm}(k_0) = 1$ (cf. Section 1.4 of [BD1]). In particular, while we will see that $\lambda^{\pm}(k)$ need not extend to a p-adically continuous function of $k \in U_{\mathbf{f}}$, we nonetheless know that $\lambda^{\pm}(k) \neq 0$ for k in a neighborhood of k_0.

By combining (48) with Proposition 3.3, we obtain:

THEOREM 3.4: *There exists an analytic function $\eta(k)$ on a neighborhood $U_{\mathbf{f},k_0}$ of k_0, such that for all (k, ℓ) in $U_{\mathbf{f},k_0} \times (\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p)$,*

$$L_p(\mathbf{f}, \mathbf{E}(\chi_1, \chi_2))(k, \ell) = \eta(k) \times L_p(\mathbf{f}, \chi_1)(k, k/2+\ell-1) \times L_p(\mathbf{f}, \chi_2)(k, k/2).$$

The function $\eta(k)$ satisfies

(49) $$\eta(k) = (C_{f_k,\chi_1,\chi_2} \mathcal{E}(f_k)\mathcal{E}^*(f_k) \lambda^+(k)\lambda^-(k))^{-1}$$

for (k,ℓ) *in* $(U_{\mathbf{f},k_0}\cap\mathbb{Z}_{\geq 2})\times(\mathbb{Z}/(p-1)\mathbb{Z}\times\mathbb{Z}_p)$*, and in particular*

$$\eta(k_0)=(C_{f_{k_0},\chi_1,\chi_2}\mathcal{E}(f_{k_0})\mathcal{E}^*(f_{k_0}))^{-1}.$$

This is crucial for the calculations of the next sections, where both sides of (20) will be evaluated at points outside the range of classical interpolation.

Remark 3.5: Equation (49) gives insight into the question raised in Remark 1.6 of [BD1] about the behavior of the periods $\lambda^{\pm}$. See also Proposition 5.2 of loc. cit., where the product $\lambda^+\lambda^-$ is compared with a less explicit p-adic period arising from the Jacquet–Langlands correspondence to forms on definite quaternion algebras.

4. p-adic regulators

We will now describe the values of $L_p(\mathbf{f},\mathbf{E}(\chi_1,\chi_2))$ at integer points (k_0,ℓ_0) outside the range of classical interpolation in terms of certain p-adic regulators in K-theory. Recall that the range of classical interpolation defining $L_p(\mathbf{f},\mathbf{E}(\chi_1,\chi_2))$ is $k_0\in U_{\mathbf{f}}\cap\mathbb{Z}_{\geq 2}$ and $2\leq\ell_0\leq k_0/2$.

Proposition 4.1: *For all* $k_0\in U_{\mathbf{f}}\cap\mathbb{Z}_{\geq 2}$ *and* $\ell_0>k_0/2$*,*

$$L_p(\mathbf{f},\mathbf{E}(\chi_1,\chi_2))(k_0,\ell_0)=\frac{1}{\mathcal{E}^*(f_{k_0})}\left\langle\eta^{\mathrm{ur}}_{f_{k_0}},\Xi^{\mathrm{ord},p}_{k_0,\ell_0}(\chi_1,\chi_2)\right\rangle_{k_0,Y}. \tag{50}$$

Proof. The terms in the defining expression (46) for $L_p(\mathbf{f},\mathbf{E}(\chi_1,\chi_2))$ vary analytically with $\ell\in(\mathbb{Z}/(p-1)\mathbb{Z})\times\mathbb{Z}_p$. ∎

We will be particularly interested in the case where $k_0=2$ and $\mathbf{f}_{k_0}$ corresponds to an elliptic curve $A/\mathbb{Q}$. Let $X_1=X_1(N)$ denote the complete modular curve over $\mathbb{Q}$ of level N obtained by adjoining to $Y_1=Y_1(N)$ the finite set of cusps, and write $\bar{X}_1=\bar{X}_1(N)$ for its extension to an algebraic closure of $\mathbb{Q}$.

4.1. The regulator on K_2. Given modular units u_1 and u_2 in $\mathcal{O}(\bar{Y}_1)^\times$, let

$$\{u_1,u_2\}\in K_2(\bar{Y}_1)$$

be the associated Steinberg symbol.

We recall Besser's description [Bes2] of the **p-adic regulator** $\mathbf{reg}_p\{u_1,u_2\}\in H^1_{\mathrm{dR}}(Y_1)$ of Coleman–de Shalit [CodS]. Let Φ_{Y_1} denote the canonical lift of Frobenius on Y_1. It is a rigid morphism on a system $\{\mathcal{W}_\epsilon\}$ of wide open neighborhoods of the ordinary locus $\mathcal{A}\subset Y_1$ obtained by deleting from Y_1 both the supersingular and the cuspidal residue discs. (See Section 4.5 of [BDP], for instance, for a brief review of the relevant definitions.) Let $\Phi_{Y_1\times Y_1} = (\Phi_{Y_1}, \Phi_{Y_1})$ be the corresponding lift of Frobenius on $Y_1 \times Y_1$, and let $P \in \mathbb{Q}[x]$ be any polynomial satisfying

(1) $P(\Phi_{Y_1\times Y_1})$ annihilates the class of $\frac{du_1}{u_1} \otimes \frac{du_2}{u_2}$ in $H^2_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon)$,
(2) $P(\Phi_{Y_1})$ acts invertibly on $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$.

The choice of P gives rise to a rigid 1-form $\rho_{_P}(u_1, u_2)$ on $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$ satisfying

$$d\rho_{_P}(u_1, u_2) = P(\Phi_{Y_1\times Y_1})\left(\frac{du_1}{u_1} \otimes \frac{du_2}{u_2}\right), \tag{51}$$

which is well-defined up to closed rigid 1-forms on $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$. After choosing a base point $x \in \mathcal{W}_\epsilon$, let

$$\delta : \mathcal{W}_\epsilon \hookrightarrow \mathcal{W}_\epsilon \times \mathcal{W}_\epsilon, \quad i_x : \mathcal{W}_\epsilon = \mathcal{W}_\epsilon \times \{x\} \hookrightarrow \mathcal{W}_\epsilon \times \mathcal{W}_\epsilon,$$

$$j_x : \mathcal{W}_\epsilon = \{x\} \times \mathcal{W}_\epsilon \hookrightarrow \mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$$

denote the diagonal, horizontal, and vertical inclusions respectively, and set

$$\tilde{\xi}_{_{P,x}}(u_1, u_2) := (\delta^* - i_x^* - j_x^*)(\rho_{_P}(u_1, u_2)) \in \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon).$$

It follows from the Künneth formula (cf. also the argument in the proof of Lemma 3.5 of [DR]) that $\delta^* - i_x^* - j_x^*$ induces the zero map from $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon)$ to $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$, and therefore that it sends closed one-forms to exact one-forms. In particular, the natural image of $\tilde{\xi}_{_{P,x}}(u_1, u_2)$ in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$, denoted $\xi_{_{P,x}}(u_1, u_2)$, does *not depend* on the choice of one-form $\rho_{_P}(u_1, u_2)$ satisfying equation (51). Condition (2) imposed in the choice of the polynomial P then allows us to define the class

$$\xi_x(u_1, u_2) := P(\Phi_{Y_1})^{-1}\xi_{_{P,x}}(u_1, u_2) \in H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon),$$

which is independent of the choice of P as above.

The exact sequence

$$0 \longrightarrow H^1_{\mathrm{dR}}(X_1) \longrightarrow H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon) \longrightarrow \mathbb{C}_p(-1)^{\sigma-1} \longrightarrow 0, \tag{52}$$

admits a canonical splitting that respects the Frobenius action. Let $\xi(u_1, u_2) \in H^1_{\mathrm{dR}}(X_1)$ denote the image of $\xi_x(u_1, u_2)$ under this splitting.

Lemma 4.2: *The class $\xi(u_1, u_2)$ does not depend on the choice of base point x.*

Proof. Write $\Phi_{Y_1\times Y_1} = \Phi_h \times \Phi_v$, where Φ_h (resp. Φ_v) is the rigid endomorphisms of $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$ acting as the canonical lift of Frobenius on the horizontal (resp. vertical) factor of the product and trivially on the other factor. Following the proof of Proposition 3.3 of [Bes2], let $t \geq 1$ be such that $(\Phi/p)^t$ fixes the classes of $\frac{du_1}{u_1}$ and $\frac{du_2}{u_2}$, and choose

$$P(x) = (1 - x^t/p^{2t}), \tag{53}$$

so that, after setting $q := p^t$,

$$P(\Phi_{Y_1\times Y_1}) = P(\Phi_h^t\Phi_v^t) = \left(1 - \frac{\Phi_h^t}{q}\right)\frac{\Phi_v^t}{q} + \left(1 - \frac{\Phi_v^t}{q}\right).$$

After writing

$$u_j^{(0)} := u_j^q/\Phi^{t*}(u_j),$$

we find that the rigid one-form on $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$ defined by

$$\rho_P(u_1,u_2) = \frac{1}{q^2}\log u_1^{(0)}\Phi^{*t}\left(\frac{du_2}{u_2}\right) - \frac{1}{q}\frac{du_1}{u_1}\log u_2^{(0)}$$

satisfies equation (51). (Cf., for instance, equation (3.2) of loc. cit.) With this choice of primitive, we observe that

$$i_x^*(\rho_P(u_1,u_2)) = -\frac{1}{q}\frac{du_1}{u_1}\log u_2^{(0)}(x), \quad j_x^*(\rho_P(u_1,u_2)) = \frac{1}{q^2}\log u_1^{(0)}(x)\Phi^{*t}\left(\frac{du_2}{u_2}\right).$$

The cohomology classes of these one-forms, being multiples of the classes attached to the logarithmic derivatives of modular units, are in the kernel of the Frobenius splitting of (52) used to define $\xi(u_1,u_2)$, and the result follows. ∎

It follows from the proof of Lemma 4.2 above that the differential $\eta_0(f,g)$ (with $f = u_1$, $g = u_2$) that appears in Proposition 3.3 of [Bes2] represents the class $\xi_{P,x}(u_1,u_2)$, up to the addition of logarithmic derivatives of modular units which are in the kernel of the splitting (52), and therefore that the image of the class $\eta(f,g)$ appearing in Proposition 3.3 of [Bes2] in $H^1_{\mathrm{dR}}(X_1)$ agrees with the class of $\xi(u_1,u_2)$. We can therefore define, following [Bes2],

$$\mathbf{reg}_p\{u_1,u_2\} := \xi(u_1,u_2). \tag{54}$$

In parallel with the complex notation of Proposition 2.3, we therefore have

$$\mathbf{reg}_p\{u_\chi, u(\chi_1,\chi_2)\}(\eta_f^{\mathrm{ur}}) := \langle \eta_f^{\mathrm{ur}}, \mathbf{reg}_p\{u_\chi, u(\chi_1,\chi_2)\}\rangle_{2,Y}$$

(where $\mathbf{reg}_p\{u_\chi, u(\chi_1,\chi_2)\}$ is viewed as a class in the de Rham cohomology of Y). We are now ready to state one of the main results of our paper.

Theorem 4.3: *For all χ_1 and χ_2 as in Assumption 2.2, and setting $f = \mathbf{f}_2$,*

$$L_p(\mathbf{f}, \mathbf{E}(\chi_1, \chi_2))(2,2) = \frac{\mathcal{E}(f, \chi_1, \chi_2, 2)}{\mathcal{E}(f)\mathcal{E}^*(f)} \mathbf{reg}_p\{u_\chi, u(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ur}}).$$

Proof. By the recipe for the p-adic regulator described before, we have

$$\begin{aligned} \mathbf{reg}_p\{u_\chi, u(\chi_1,\chi_2)\}(\eta_f^{\mathrm{ur}}) &= \left\langle \eta_f^{\mathrm{ur}},\ \xi(u_\chi, u(\chi_1,\chi_2))\right\rangle_{2,Y} \\ &= \left\langle \eta_f^{\mathrm{ur}},\ P(\Phi_Y)^{-1}\xi_{P,x}(u_\chi, u(\chi_1,\chi_2))\right\rangle_{2,Y}. \end{aligned} \tag{55}$$

For any $\xi \in H^1_{\mathrm{dR}}(Y)$, note that

$$\begin{aligned} \left\langle \eta_f^{\mathrm{ur}}, \Phi_Y \xi\right\rangle_{Y,2} &= \alpha_p(f)^{-1}\left\langle \Phi_Y \eta_f^{\mathrm{ur}}, \Phi_Y\xi\right\rangle_{Y,2} \\ &= p\alpha_p(f)^{-1}\left\langle \eta_f^{\mathrm{ur}}, \xi\right\rangle_{Y,2} = \beta_p(f)\cdot\left\langle \eta_f^{\mathrm{ur}}, \xi\right\rangle_{Y,2}. \end{aligned} \tag{56}$$

Combining (55) and (56), we find

$$\begin{aligned} \left\langle \eta_f^{\mathrm{ur}}, \mathbf{reg}_p\{u_\chi, u(\chi_1,\chi_2)\}\right\rangle_{2,Y} &= P(\beta_p(f))^{-1}\left\langle \eta_f^{\mathrm{ur}},\ \xi_{P,x}(u_\chi, u(\chi_1,\chi_2))\right\rangle_{2,Y} \\ &= P(\beta_p(f))^{-1}\left\langle \eta_f^{\mathrm{ur}},\ e_f e_{\mathrm{ord}}\xi_{P,x}(u_\chi, u(\chi_1,\chi_2))\right\rangle_{2,Y}. \end{aligned} \tag{57}$$

Set

$$P(x) := p^{-4}(x - \chi_1(p))(x - \bar{\chi}_1(p)p^2)(x - \chi_2(p)p)(x - \bar{\chi}_2(p)p). \tag{58}$$

Following Besser as in the proof of Lemma 4.2 above, a more optimal choice of P would have been to take $(x - \bar{\chi}_1(p)p^2)$, as $(\Phi_Y - \bar{\chi}_1(p)p^2)$ already annihilates the class of $E_{2,\chi} \otimes E_2(\chi_1,\chi_2)$ in cohomology. However, the above choice of P allows us to directly invoke the calculations that were already carried out in Section 3.4 of [DR], in the setting where $E_{2,\chi}$ and $E_2(\chi_1,\chi_2)$ are replaced by cusp forms. (In such a setting, it became necessary to work with a degree 4 polynomial.) With our choice of P, and setting $x = \infty$, we have

$$\begin{aligned} e_f e_{\mathrm{ord}}\xi_{P,x}(u_\chi, u(\chi_1,\chi_2)) &= \mathcal{E}^*(f) e_f e_{\mathrm{ord}}(d^{-1}E_{2,\chi}^{[p]} \times E_2(\chi_1,\chi_2)) \\ &= \mathcal{E}^*(f) e_f \Xi^{\mathrm{ord},p}(\chi_1,\chi_2). \end{aligned} \tag{59}$$

This follows by replacing the cusp forms $\breve{g}$ and $\breve{h}$ by the Eisenstein series $E_{2,\chi}$ and $E_2(\chi_1,\chi_2)$ in Theorem 3.12 of loc. cit. Combining (57) and (59), and observing that $P(\beta_p(f)) = \mathcal{E}(f,\chi_1,\chi_2,2)$, we obtain

$$\begin{aligned} &\left\langle \eta_f^{\mathrm{ur}}, \mathbf{reg}_p\{u_\chi, u(\chi_1,\chi_2)\}\right\rangle_{2,Y} \\ &\qquad = \mathcal{E}(f,\chi_1,\chi_2,2)^{-1}\mathcal{E}^*(f)\left\langle \eta_f^{\mathrm{ur}},\ e_f\Xi^{\mathrm{ord},p}(\chi_1,\chi_2)\right\rangle_{2,Y}. \end{aligned} \tag{60}$$

Theorem 4.3 now follows from Proposition 4.1. ∎

4.2. The general case. Set $\ell_0 = r+2$, and assume in this section that $r > 0$. Let E_1, E_2 be weight ℓ_0 Eisenstein series in $\mathrm{Eis}_{\ell_0}(\Gamma_1(N), \mathbb{Q})$. Besser's description [Bes2] of the p-adic regulator $\mathbf{reg}_p\{u_1, u_2\} \in H^1_{\mathrm{dR}}(Y_1)$ admits a natural generalisation to the setting in which the logarithmic derivatives of u_1 and u_2 are replaced by the Eisenstein series E_1 and E_2. More precisely, note that $(\mathcal{L}_r, \nabla)$ is equipped with the structure of an overconvergent Frobenius isocrystal in the sense of Definition 4.15 of [BDP]. In particular, the cohomology groups $H^1_{\mathrm{dR}}(Y_1, \mathcal{L}_r, \nabla)$ are endowed with an action of the Frobenius lift Φ_{Y_1}. Let $P \in \mathbb{Q}[x]$ be any polynomial satisfying

(1) $P(\Phi_{Y_1\times Y_1})$ annihilates the class of $E_1 \otimes E_2$ in $H^2_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon, \mathcal{L}_r \otimes \mathcal{L}_r, \nabla)$,
(2) $P(\Phi_{Y_1})$ acts invertibly on $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)(r)$.

The choice of P gives rise to a $\mathcal{L}_r^{\otimes 2}$-valued rigid 1-form $\rho_P(E_1, E_2)$ on $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$ satisfying

$$\rho_P(E_1, E_2) = P(\Phi_{Y_1\times Y_1})(E_1 \otimes E_2).$$

Note that $\rho_P(E_1, E_2)$ is well-defined only up to closed forms on $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$. Let $\xi_P(E_1, E_2) \in H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)(r)$ denote the class of the restriction of $\rho_P(E_1, E_2)$ to the diagonal composed with the pairing

$$\mathcal{L}_r \times \mathcal{L}_r \longrightarrow \mathcal{O}_{Y_1}(r),$$

and set

$$\xi_0(E_1, E_2) := P(\Phi_{Y_1})^{-1}\xi_P(E_1, E_2) \in H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)(r).$$

As before, let $\xi(E_1, E_2)$ denote the natural image of $\xi_0(E_1, E_2)$ in $H^1_{\mathrm{dR}}(X_1)(r)$ under the Frobenius-equivariant splitting of the exact sequence (52). The p-adic regulator attached to (E_1, E_2) is then defined to be

$$\mathbf{reg}_p\{E_1, E_2\} = \xi(E_1, E_2). \tag{61}$$

The following extends Theorem 4.3 to general $\ell = \ell_0$, where as before we have set

$$\mathbf{reg}_p\{E_{\ell,\chi}, E_\ell(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ur}}) := \left\langle \eta_f^{\mathrm{ur}}, \mathbf{reg}_p\{E_{\ell,\chi}, E_\ell(\chi_1, \chi_2)\}\right\rangle_{2,Y}.$$

Theorem 4.4: *For all χ_1 and χ_2 as in Assumption 2.2, and setting $f = \mathbf{f}_2$,*

$$L_p(\mathbf{f}, \mathbf{E}(\chi_1, \chi_2))(2, \ell) = \frac{\mathcal{E}(f, \chi_1, \chi_2, \ell)}{\mathcal{E}(f)\mathcal{E}^*(f)} \mathbf{reg}_p\{E_{\ell,\chi}, E_\ell(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ur}}).$$

Proof. The proof is the same as the proof of Theorem 4.3, using the calculations of Section 3.4 of [DR]. ∎

5. The p-adic Beilinson formula

5.1. The main results. We can now state the main results of this article, which apply to any pair (χ_1, χ_2) of primitive Dirichlet characters with relatively prime conductors N_1 and N_2, satisfying

$$N = N_1 N_2, \qquad \chi^{-1} := \chi_1 \chi_2 \text{ is even.}$$

Theorem 5.1: *For all $\ell \geq 2$,*

$$L_p(f, \chi_1, \ell) \cdot L^*(f, \chi_2, 1) = C_{f,\chi_1,\chi_2}(1 - \beta_p(f)\chi_1(p)p^{-\ell})(1 - \beta_p(f)\bar{\chi}_1(p)p^{-(2-\ell)}) \times \mathbf{reg}_p\{E_{\ell,\chi}, E_\ell(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ur}}).$$

Proof. By comparing Theorem 3.4 with $k_0 = 2$ and Theorem 4.4, we obtain

$$L_p(f, \chi_1, \ell) \times L_p(f, \chi_2, 1) = C_{f,\chi_1,\chi_2}\mathcal{E}(f, \chi_1, \chi_2, \ell) \times \mathbf{reg}_p\{E_{\ell,\chi}, E_\ell(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ur}}).$$

The theorem now follows from equation (47), giving the interpolation properties of the Mazur–Swinnerton–Dyer p-adic L-function, and the definition of $\mathcal{E}(f, \chi_1, \chi_2, \ell)$ given in Proposition 3.2. ∎

For $\ell = 2$, Theorem 5.1 relates the value $L_p(f, \chi_1, 2)$ to the p-adic regulator

$$\mathbf{reg}_p\{E_{2,\chi}, E_2(\chi_1, \chi_2)\} := \mathbf{reg}_p\{u_\chi, u(\chi_1, \chi_2)\}$$

previously defined in terms of modular units.

Corollary 5.2:

$$L_p(f, \chi_1, 2) \cdot L^*(f, \chi_2, 1) = C_{f,\chi_1,\chi_2}(1 - \beta_p(f)\chi_1(p)p^{-2})(1 - \beta_p(f)\bar{\chi}_1(p)) \times \mathbf{reg}_p\{u_\chi, u(\chi_1, \chi_2)\}(\eta_f^{\mathrm{ur}}).$$

Note the strong analogy between Corollary 5.2 and the complex Beilinson formula, as stated in Proposition 2.3. The factor $L^*(f, \chi_2, 1)$ belongs to the field $\mathbb{Q}(f, \chi_2)$, and χ_2 can be chosen so that this factor does not vanish. Theorem 5.1 then expresses the value $L_p(f, \chi_1, \ell)$ of the Mazur–Swinnerton–Dyer p-adic L-function at a point outside the range of classical interpolation as the p-adic regulator attached to two Eisenstein series of weight ℓ, times a non-zero algebraic number.

5.2. Relation with the work of Brunault and Gealy. We conclude by explaining the relation between the main results of this paper and the p-adic Beilinson formulae proved in [Br2] (for the value at $\ell = 2$) and [Ge] (for the value at general $\ell \geq 2$).

The syntomic regulators of Coleman–de Shalit and Besser have counterparts in p-adic étale cohomology, whose definition we first recall for $K_2(Y_1)$ and for the eigenspaces $(K_2(\bar{Y}_1)\otimes F)^{\chi_1}$ under the action of $G_{\mathbb{Q}}$, where F is an extension of $\mathbb{Q}_p$ large enough to contain the values of χ_1 and χ_2. Kummer theory gives connecting homomorphisms

$$\delta : \mathcal{O}(Y_1)^\times \longrightarrow H^1_{\mathrm{et}}(Y_1, \mathbb{Q}_p(1)), \qquad \delta : (\mathcal{O}(\bar{Y}_1)^\times \otimes F)^{\chi_1} \longrightarrow H^1_{\mathrm{et}}(Y_1, F(1)(\chi_1)).$$

The **p-adic étale regulator** of $\{u_1, u_2\}$ is defined to be

$$\mathbf{reg}_{\mathrm{et}}\{u_1, u_2\} := \delta(u_1) \cup \delta(u_2) \in H^2_{\mathrm{et}}(Y_1, F(2)(\chi_1)) = H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\bar{Y}_1, F(2)(\chi_1))),$$

where the last identification follows from the Hochschild–Serre spectral sequence (cf. equation (28) of [Br2]). The restriction of $\mathbf{reg}_{\mathrm{et}}\{u_1, u_2\}$ to $G_{\mathbb{Q}_p} = \mathrm{Gal}\,(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ yields an element $\mathrm{res}_p(\mathbf{reg}_{\mathrm{et}}\{u_1, u_2\})$ of

$$H^1(\mathbb{Q}_p, H^1_{\mathrm{et}}(\bar{Y}_1, F(2)(\chi_1)) = \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_p}}(\mathbb{Q}_p, H^1_{\mathrm{et}}(\bar{Y}_1, F(2)(\chi_1))),$$

where the group of extensions is taken in the category of continuous p-adic representations of $G_{\mathbb{Q}_p}$ which are cristalline. On the other hand, $\mathbf{reg}_p\{u_1, u_2\}$ belongs to

$$H^1_{\mathrm{dR}}(Y_1) = H^1_{\mathrm{dR}}(Y_1)/\mathrm{Fil}^2 H^1_{\mathrm{dR}}(Y_1) = \mathrm{Ext}^1_{\mathrm{ffm}}(\mathbb{Q}_p, H^1_{\mathrm{dR}}(Y_1)(2)),$$

where the group of extensions is taken in the category of admissible filtered Frobenius modules. Fontaine's comparison functor sets up an isomorphism

$$\mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_p}}(\mathbb{Q}_p, H^1_{\mathrm{et}}(\bar{Y}_1, F(2)(\chi_1))) \xrightarrow{\mathrm{comp}} \mathrm{Ext}^1_{\mathrm{ffm}}(\mathbb{Q}_p, H^1_{\mathrm{dR}}(Y_1)(2)). \tag{62}$$

More generally, the Eisenstein series E_1 and E_2 of weight $\ell_0 = r+2$ introduced in Section 4.2 give rise to classes

$$\delta_1 \in H^1_{\mathrm{et}}(Y_1, \mathcal{L}_r^{\mathrm{et}}(1)), \qquad \delta_2 \in H^1_{\mathrm{et}}(Y_1, \mathcal{L}_r^{\mathrm{et}}(1)(\chi_1)),$$

where $\mathcal{L}_r^{\mathrm{et}}$ is the étale p-adic sheaf associated to the local system $(\mathcal{L}_r, \nabla)$ of Section 2.5 (tensored with the field F containing the values of the characters

χ_1 and χ_2). Imitating the complex treatment of Beilinson [Bei], we define the p-adic étale regulator of (E_1, E_2) to be

$$\mathbf{reg}_{\mathrm{et}}\{E_1, E_2\} := \delta_1 \cup \delta_2 \in H^2_{\mathrm{et}}(Y_1, F(r+2)(\chi_1)) = H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\bar{Y}_1, F(r+2)(\chi_1))),$$

where we have used the pairing $\mathcal{L}_r^{\mathrm{et}} \times \mathcal{L}_r^{\mathrm{et}} \longrightarrow F(r)$, and as before the last identification follows from the Hochschild–Serre spectral sequence. Note that when $r = 0$, $\mathbf{reg}_{\mathrm{et}}\{E_1, E_2\}$ is equal to $\mathbf{reg}_{\mathrm{et}}\{u_1, u_2\}$, with $E_j = \mathrm{dlog}(u_j)$. As in the case $r = 0$, there is an isomorphism

(63)
$$\mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_p}}(\mathbb{Q}_p, H^1_{\mathrm{et}}(\bar{Y}_1, F(r+2)(\chi_1))) \xrightarrow{\mathrm{comp}} \mathrm{Ext}^1_{\mathrm{ffm}}(\mathbb{Q}_p, H^1_{\mathrm{dR}}(Y_1)(r+2)),$$

and we have

Proposition 5.3: *For all* $u_1, u_2 \in \mathcal{O}(Y_1)^\times$,

$$\mathbf{reg}_p\{u_1, u_2\} = \mathrm{comp}(\mathrm{res}_p(\mathbf{reg}_{\mathrm{et}}\{u_1, u_2\})).$$

More generally, for all $E_1, E_2 \in \mathrm{Eis}_{\ell_0}(\Gamma_1(N), \mathbb{Q})$,

$$\mathbf{reg}_p\{E_1, E_2\} = \mathrm{comp}(\mathrm{res}_p(\mathbf{reg}_{\mathrm{et}}\{E_1, E_2\})).$$

Proof. See Proposition 9.11 and Corollary 9.10 of [Bes1], and the references therein. ∎

Proposition 5.3 leads to an alternate definition of the p-adic regulator, which is the one that enters in the p-adic Beilinson formulae of [Br2] and [Ge]. Theorem 5.1 and Corollary 5.2 thus give a different proof of the main results of [Ge] and [Br2] respectively. The strategy followed in loc. cit. builds on the work of Kato, in which a collection of norm-compatible elements in the K_2 of a tower of modular curves is used to construct a Λ-adic cohomology class $\kappa \in H^1(\mathbb{Q}, \mathbb{V}_p(E))$ with values in the Λ-adic representation $\mathbb{V}_p(E)$ of $G_{\mathbb{Q}}$ interpolating the Tate twists $V_p(E)(j)$ for all $j \in \mathbb{Z}$. Kato's reciprocity law relates the image of κ in $H^1(\mathbb{Q}_p, \mathbb{V}_p(E))$ to the p-adic L-function attached to E. Both [Br2] and [Ge] exploit deep local results of Perrin-Riou ([PR2], [Colz1]) to parlay this relation into a precise connection between the p-adic étale regulator of the Beilinson elements and the special values at $\ell \geq 2$ of the Mazur–Swinnerton–Dyer p-adic L-function. The proof proposed in the present work can be viewed as somewhat more direct, insofar as it does not rely on Kato's Λ-adic classes or on any facts about the behaviour of the Bloch–Kato logarithm and dual exponential maps in p-adic families.

References

[Bei] A. A. Beilinson, *Higher regulators of modular curves,* in *Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Part I, II (Boulder, Colo., 1983)*, Contemporary Mathematics, Vol. 55, American Mathematical Society, Providence, RI, 1986, pp. 1–34.

[BDP] M. Bertolini, H. Darmon and K. Prasanna, *Generalised Heegner cycles and p-adic Rankin L-series*, Duke Mathematical Journal **162** (2013), 1033–1148.

[BD1] M. Bertolini and H. Darmon, *Hida families and rational points on elliptic curves,* Inventiones Mathematicae **168** (2007), 371–431.

[BD2] M. Bertolini and H. Darmon, *Kato's Euler system and rational points on elliptic curves II: The explicit reciprocity law*, in preparation.

[BD3] M. Bertolini and H. Darmon, *Kato's Euler system and rational points on elliptic curves III: The conjecture of Perrin-Riou*, in preparation.

[BDR] M. Bertolini, H. Darmon and V. Rotger, *Beilinson–Flach elements and Euler systems I: syntomic regulators and p-adic Rankin L-series*, submitted.

[Bes1] A. Besser, *Syntomic regulators and p-adic integration. I. Rigid syntomic regulators,* in *Proceedings of the Conference on p-adic Aspects of the Theory of Automorphic Representations (Jerusalem, 1998)*, Israel Journal of Mathematics **120** (2000), 291–334.

[Bes2] A. Besser, *Syntomic regulators and p-adic integration. II. K_2 of curves,* in *Proceedings of the Conference on p-adic Aspects of the Theory of Automorphic Representations (Jerusalem, 1998),* Israel Journal of Mathematics **120** (2000), 335–359.

[BK] K. Bannai and G. Kings, *p-adic elliptic polylogarithm, p-adic Eisenstein series and Katz measure*, American Journal of Mathematics **132** (2010), 1609–1654.

[Bl] S. J. Bloch, *Higher Regulators, Algebraic K-theory, and Zeta Functions of Elliptic Curves*, CRM Monograph Series, Vol. 11, American Mathematical Society, Providence, RI, 2000.

[Br1] F. Brunault, *Valeur en 2 de fonctions L de formes modulaires de poids 2: théorème de Beilinson explicite,* Bulletin de la Société Mathématique de France **135** (2007), 215–246.

[Br2] F. Brunault, *Régulateurs p-adiques explicites pour le K_2 des courbes elliptiques,* Actes de la Conférence "Fonctions L et Arithmétique", Publ. Math. Besançon Algèbre Théorie Nr., Lab. Math. Besançon, Besançon, 2010, pp. 29–57.

[CodS] R. F. Coleman and E. de Shalit, *p-adic regulators on curves and special values of p-adic L-functions,* Inventiones Mathematicae **93** (1988), 239–266.

[Col94] R. F. Coleman, *A p-adic Shimura isomorphism and p-adic periods of modular forms*, in *p-adic Monodromy and the Birch and Swinnerton–Dyer Conjecture (Boston, MA, 1991),* Contemporary Mathematics, Vol. 165, American Mathematical Society, Providence, RI, 1994, pp. 21–51.

[Colz1] P. Colmez, *Fonctions L p-adiques,* Séminaire Bourbaki, Vol. 1998/99, Astérisque No. **266** (2000), Exp. No. 851, 3, 21–58.

[Colz2] P. Colmez, *La conjecture de Birch et Swinnerton–Dyer p-adique,* (French) Astérisque No. **294** (2004), ix, 251–319.

[DR] H. Darmon and V. Rotger, *Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula*, Annales Scientifiques de l'École Normale Supérieure, to appear.

[Ge] M. Gealy, *On the Tamagawa number conjecture for motives attached to modular forms,* PhD Thesis, California Institute of Technology, 2006.

[Hi93] H. Hida, *Elementary Theory of L-functions and Eisenstein Series,* London Mathematical Society Student Texts, Vol.26, Cambridge University Press, Cambridge, 1993.

[Kato] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms,* in *Cohomologies p-adiques et applications arithmétiques. III*, Astérisque No. **295** (2004), ix, 117–290.

[Ki] K. Kitagawa, *On standard p-adic L-functions of families of elliptic cusp forms,* in *p-adic Monodromy and the Birch and Swinnerton–Dyer Conjecture* (Boston, MA, 1991), Contemporary Mathematics, Vol. 165, American Mathematical Society, Providence, RI, 1994, pp. 81–110.

[Nik] M. Niklas, *Rigid syntomic regulators and the p-adic L-function of a modular form,* Regensburg PhD Thesis, 2010, available at http://epub.uni-regensburg.de/19847/

[MTT] B. Mazur, J. Tate and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton–Dyer,* Inventiones Mathematicae **84** (1986), 1–48.

[PR1] B. Perrin-Riou, *Fonctions L p-adiques d'une courbe elliptique et points rationnels,* Annales de l'Institut Fourier (Grenoble) **43** (1993), 945–995.

[PR2] B. Perrin-Riou, *Théorie d'Iwasawa des représentations p-adiques sur un corps local,* with an appendix by Jean-Marc Fontaine, Inventiones Mathematicae **115** (1994), 81–161.

[Sh1] G. Shimura, *The special values of the zeta functions associated with cusp forms,* Communications on Pure and Applied Mathematics **29** (1976), 783–804.

[Sh2] G. Shimura, *On a class of nearly holomorphic automorphic forms,* Annals of Mathematics **123** (1986), 347–406.