

# BEILINSON-FLACH ELEMENTS AND EULER SYSTEMS I: SYNTOMIC REGULATORS AND $p$ -ADIC RANKIN $L$ -SERIES

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## Abstract

This article is the first in a series devoted to the Euler system arising from  $p$ -adic families of *Beilinson-Flach elements* in the first  $K$ -group of the product of two modular curves. It relates the image of these elements under the  $p$ -adic syntomic regulator (as described by Besser (2012)) to the special values at the near-central point of Hida's  $p$ -adic Rankin  $L$ -function attached to two Hida families of cusp forms.

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## 1. Introduction

This article is the first in a series devoted to the Euler system of *Beilinson-Flach elements* in the motivic cohomology of a product of two modular curves. Its main result (see Theorem 4.2 and Corollary 4.4 of §4.2) is a  $p$ -adic analogue of the formula of Beilinson [Bei, Ch. 2, §6] expressing special values of Rankin  $L$ -series in terms of complex regulators. Beilinson's theorem (cf. §4.1 for an explicit version) relates:

- (1) the Rankin  $L$ -series  $L(f \otimes g, s)$  attached to the convolution of weight 2 newforms  $f$  and  $g$  on  $\Gamma_1(N)$ , evaluated at the *near-central point*  $s = 2$ ;
- (2) the image under the complex regulator of certain explicit elements in the motivic cohomology group  $H_{\mathcal{M}}^3(X_1(N)^2, \mathbb{Q}(2))$  or, equivalently, in the higher Chow group  $\mathrm{CH}^2(X_1(N)^2, 1) \otimes \mathbb{Q}$ . These elements, whose definition is recalled in Section 3.1, are constructed from modular units and are referred to in the sequel as *Beilinson-Flach elements*.

In the  $p$ -adic setting, the complex  $L$ -series  $L(f \otimes g, s)$  is replaced by Hida's  $p$ -adic Rankin  $L$ -series attached to two ordinary families of modular forms interpolating  $f$  and  $g$ , whose definition is briefly recalled in Section 2.2. The role of the complex regulator is played by the  $p$ -adic syntomic regulator on  $K_1$  of a surface. Besser's description of it in terms of Coleman integration [Bes3], which is summarised in §3.3, is a key ingredient in the proof of Theorem 4.2.

Our approach also relies crucially on techniques developed in [DR] for relating  $p$ -adic Abel-Jacobi images of diagonal cycles to values of the Garrett-Rankin triple product  $p$ -adic  $L$ -function attached to a triple  $(\mathbf{f}, \mathbf{g}, \mathbf{h})$  of Hida families of cusp forms. Corollary 4.4 deals with the setting where the cuspidal family  $\mathbf{h}$  in the triple  $(\mathbf{f}, \mathbf{g}, \mathbf{h})$  is replaced by a Hida family of Eisenstein series. The reader will also note the close parallel between Theorem 4.2 and the main result of [BD], in which the  $p$ -adic regulators of certain elements in  $K_2(X_1(N))$  are related to the value at  $s = 2$  of the Mazur-Swinnerton-Dyer  $p$ -adic  $L$ -functions attached to weight two cusp forms. The results of the present article are in fact intermediate between those of [DR] and [BD], the latter treating the case where *both*  $\mathbf{g}$  and  $\mathbf{h}$  are replaced by Hida families of Eisenstein series—a setting in which the resulting  $p$ -adic Rankin  $L$ -function factors as a product of two Mazur-Kitagawa  $L$ -functions attached to  $\mathbf{f}$ .

We also remark that a function field analogue of Beilinson's Theorem involving Drinfeld modular curves is described in [Sre2], based on a description of non-Archimedean regulators given in [Sre1]. See also the related work of Ambrus Pál in the setting of the  $K_2$  of Mumford curves [Pa].

Let us conclude this introduction by briefly discussing some eventual arithmetical applications of the main result of this paper.

**I. The Euler system of Beilinson-Flach elements.** The image of Beilinson-Flach elements under the  $p$ -adic étale regulator map gives rise to classes in the global cohomology group  $H^1(\mathbb{Q}, V_f \otimes V_g(2))$ , where  $V_f$  and  $V_g$  are the  $p$ -adic Galois representations attached to  $f$  and  $g$ , respectively. The work in preparation [BDR] explores the theme of the  $p$ -adic variation of the Beilinson-Flach classes attached to Hida families of cusp forms  $\mathbf{f}$  and  $\mathbf{g}$ . In particular, when  $\mathbf{g}$  specialises in weight one to a classical cusp form attached to an odd irreducible Artin representation  $\rho$  and  $\mathbf{f}$  specialises in weight two to the cusp form associated with an elliptic curve  $E$  over  $\mathbb{Q}$ , we expect the associated cohomology class to yield new cases of the Birch and Swinnerton-Dyer conjecture for the complex  $L$ -series  $L(E, \rho, s)$ , proving in particular that  $\rho$  does not occur in the representation  $E(\overline{\mathbb{Q}}) \otimes \mathbb{C}$  when  $L(E, \rho, 1) \neq 0$ .

The idea of using Beilinson elements in Euler system arguments occurs much earlier in the work of Flach [Fl], who used them to construct classes in  $H^1(\mathbb{Q}, \text{Sym}^2(E)(2))$  which are crystalline at  $p$  but ramified at a single prime  $\ell \neq p$ . Applying Kolyagin's method to these classes leads to the finiteness of the Shafarevich-Tate group of  $\text{Sym}^2(E)(2)$  and an explicit annihilator of this group related to the special value  $L(\text{Sym}^2(E), 2)$ , which is critical in the sense of Deligne, unlike the special values  $L(f \otimes g, 2)$  when  $f$  and  $g$  are distinct normalised newforms.

**II. Hida's  $L$ -function for the symmetric square of a modular form.** Theorem 4.2, specialised to the case  $f = g$ , is exploited by S. Dasgupta [Das] to study the Hida  $L$ -function  $L(f \otimes f, s)$  and express it as the product of the Coates-Schmidt  $p$ -adic  $L$ -function attached to  $\text{Sym}^2(f)$  and a Kubota-Leopoldt  $p$ -adic  $L$ -function. This factorisation, which can be viewed as another manifestation of the Artin formalism for  $p$ -adic  $L$ -series, is analogous to a formula of Gross [Gross] expressing the restriction to the cyclotomic line of the Katz two-variable  $p$ -adic  $L$ -function attached to an imaginary quadratic field as a product of two Kubota-Leopoldt  $L$ -functions. The Beilinson-Flach elements play the same role in Dasgupta's proof as elliptic units in the work of Gross.

**III.  $p$ -adic  $L$ -functions and Euler systems over  $\mathbb{Z}_p^2$ -extensions.** The paper in preparation [LLZ] of A. Lei, D. Loeffler and S.L. Zerbes builds on the methods of this paper, in the setting where  $g$  varies over a collection of theta series attached to Hecke characters of an imaginary quadratic field  $K$ , to construct an Euler system for  $V_f$  over the various layers of the *two variable*  $\mathbb{Z}_p$ -extension  $K_\infty$  of  $K$ , thus supplying the global input for their extension

[LZ] of Perrin-Riou's machinery in which the cyclotomic  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$  is replaced by  $K_\infty$ .

## 2. Rankin $L$ -series

Let  $f$  and  $g$  be normalised newforms of weights  $k, \ell$ , levels  $N_f, N_g$ , and nebentypus characters  $\chi_f, \chi_g$  respectively. The  $p$ -adic representations  $V_f$  and  $V_g$  are part of a compatible system of representations which we continue to denote by the same symbol. Let

$$L(V_f \otimes V_g, s) = \prod_p \det((1 - \sigma_p p^{-s})|(V_f \otimes V_g)^{I_p})^{-1}$$

be the motivic  $L$ -function attached to the tensor product  $V_f \otimes V_g$ , where  $I_p$  denotes the inertia subgroup of a decomposition group at  $p$ , and  $\sigma_p$  a corresponding geometric Frobenius element.

The goal of this first chapter is to briefly recall the basic analytic properties of this  $L$ -series, describe Hida's construction of a  $p$ -adic avatar, and—in the special case where  $f$  and  $g$  are both of weight two—present parallel formulae for their special values at the near-central point  $s = 2$ , which is *not* critical in the sense of Deligne.

**2.1. Complex  $L$ -series.** Set  $N := \text{lcm}(N_f, N_g)$  and replace  $\chi_f$  and  $\chi_g$  by their counterparts of modulus  $N$  sending any prime  $r|N$  to 0. It is also convenient to replace  $f$ , as well as  $g$ , by a normalised eigenform of level  $N$  which

- (1) has the same eigenvalues for the Hecke operators  $T_r$  with  $r \nmid N$ ;
- (2) is also an eigenvector for the Hecke operators  $U_r$  attached to the primes  $r$  dividing  $N$ .

This substitution having been made, let

$$(2.1) \quad f(z) = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z}, \quad g(z) = \sum_{n=1}^{\infty} a_n(g) e^{2\pi i n z}$$

be the Fourier expansions of  $f$  and  $g$ , let  $K_f$  and  $K_g \subset \bar{\mathbb{Q}}$  denote the subfields generated by the coefficients  $a_n(f)$  and  $a_n(g)$  respectively, and let  $K_{fg}$  denote the compositum of the two fields. The Hecke polynomials attached to  $f$  can be factored as

$$x^2 - a_p(f)x + \chi_f(p)p^{k-1} = (x - \alpha_p(f))(x - \beta_p(f)),$$

where  $(\alpha_p(f), \beta_p(f)) = (a_p(f), 0)$  when  $p|N$ . Similar notation is adopted for  $g$ . The Rankin  $L$ -function attached to the pair  $(f, g)$  is defined by the formula

$$L(f \otimes g, s) := \prod_p L_{(p)}(f \otimes g, s), \quad \text{where}$$

$$L_{(p)}(f \otimes g, s) := (1 - \alpha_p(f)\alpha_p(g)p^{-s})^{-1}(1 - \alpha_p(f)\beta_p(g)p^{-s})^{-1} \\ \times (1 - \beta_p(f)\alpha_p(g)p^{-s})^{-1}(1 - \beta_p(f)\beta_p(g)p^{-s})^{-1}.$$

The Euler factors at  $p$  defining  $L(V_f \otimes V_g, s)$  and  $L(f \otimes g, s)$  agree for all  $p \nmid N$ , and hence the special values of  $L(V_f \otimes V_g, s)$  and  $L(f \otimes g, s)$  at integer points differ by elementary quantities in  $K_{fg}^\times$ . It will be more convenient, for the sequel, to focus attention on  $L(f \otimes g, s)$ . Assume without loss of generality that the forms  $f$  and  $g$  have been ordered in such a way that  $k \geq \ell$ .

**2.1.1. Rankin’s method.** We begin by recalling the general formula for  $L(f \otimes g, s)$  coming out of Rankin’s method, involving the *non-holomorphic Eisenstein series*

$$(2.2) \quad \tilde{E}_{k-\ell, \chi}(z, s) = \sum'_{(m,n) \in N\mathbb{Z} \times \mathbb{Z}} \frac{\chi^{-1}(n)}{(mz+n)^{k-\ell}} \cdot \frac{y^s}{|mz+n|^{2s}}$$

of weight  $k - \ell$ , level  $N$  and character

$$\chi := \chi_f^{-1} \chi_g^{-1},$$

where the superscript  $'$  in (2.2) indicates that the sum is taken over the non-zero lattice vectors  $(m, n) \in N\mathbb{Z} \times \mathbb{Z}$ . For fixed complex  $s$  with  $\text{Re}(s) \gg 0$ , the product  $\tilde{E}_{k-\ell, \chi}(z, s) \times g(z)$  is a real-analytic  $\mathbb{C}$ -valued function on the Poincaré upper half-plane  $\mathcal{H}$  which transforms like a modular form of weight  $k$ , level  $N$  and character  $\chi_f^{-1}$  and is of rapid decay at infinity. The space of such functions, denoted  $S_k^{\text{ra}}(N, \chi_f^{-1})$ , is equipped with the Petersson scalar product

$$\langle \cdot, \cdot \rangle_{k,N} : S_k^{\text{ra}}(N, \chi_f^{-1}) \times S_k^{\text{ra}}(N, \chi_f^{-1}) \longrightarrow \mathbb{C}$$

given by the formula

$$(2.3) \quad \langle f_1, f_2 \rangle_{k,N} := \int_{\Gamma_0(N) \backslash \mathcal{H}} y^k \overline{f_1(z)} f_2(z) \frac{dx dy}{y^2},$$

which is hermitian linear in the first argument and  $\mathbb{C}$ -linear in the second. Let  $f^* \in S_k(N, \chi_f^{-1})$  denote the modular form obtained from  $f$  by applying complex conjugation to its Fourier coefficients.

**Proposition 2.1** (Shimura). *For all  $s \in \mathbb{C}$  with  $\Re(s) \gg 0$ ,*

$$(2.4) \quad L(f \otimes g, s) = \frac{1}{2} \frac{(4\pi)^s}{\Gamma(s)} \left\langle f^*(z), \tilde{E}_{k-\ell, \chi}(z, s - k + 1) \cdot g(z) \right\rangle_{k,N}.$$

This well-known formula for the Rankin  $L$ -series is taken from equation (14) of [BD].

**2.1.2. Critical values.** Assume here and in §2.1.3 that  $\ell < k$ . The functional equation for  $L(f \otimes g, s)$  arising from Proposition 2.1 reveals that the integer  $j$  is critical for  $L(f \otimes g, s)$  if and only if it lies in the closed interval  $[\ell, k - 1]$ . We now describe a further closed formula for the value at an integer  $j$  belonging to the “right half critical segment”  $[\frac{\ell+k-1}{2}, k - 1]$ , which will be useful in deriving the algebraicity (up to periods) of  $L(f \otimes g, j)$  predicted by the Deligne conjectures and ultimately in constructing Hida’s  $p$ -adic Rankin  $L$ -function by interpolating these quantities  $p$ -adically.

Having fixed an integer  $j \in [\frac{\ell+k-1}{2}, k - 1]$ , let  $t \geq 0$  and  $m \geq 1$  be given by

$$t := k - 1 - j, \quad m := k - \ell - 2t.$$

If  $m \leq 2$ , let us assume also that  $\chi$  is non-trivial. Then the series

$$(2.5) \quad E_{m,\chi}(z) = 2^{-1}L(\chi, 1 - m) + \sum_{n=1}^{\infty} \sigma_{m-1,\chi}(n)q^n,$$

where  $\sigma_{m-1,\chi}(n) = \sum_{d|n} \chi(d)d^{m-1}$  is the  $q$ -expansion of a holomorphic Eisenstein series of weight  $m$  and character  $\chi$ .

The Shimura-Maass derivative operator

$$\delta_m := \frac{1}{2\pi i} \left( \frac{d}{dz} + \frac{im}{2y} \right)$$

transforms modular forms of weight  $m$  into (real analytic) modular forms of weight  $m + 2$  which are *nearly holomorphic* in the sense of [Sh2], and its  $t$ -fold iterate  $\delta_m^t := \delta_{m+2t-2} \cdots \delta_{m+2} \delta_m$  maps the space  $M_m(N, \chi)$  to the space  $M_{m+2t}^{\text{nh}}(N, \chi)$  of nearly holomorphic modular forms of weight  $m + 2t$ . Let

$$C(k, \ell, j) := \frac{(-1)^t 2^{k-1} (2\pi)^{k+m-1} \iota_\chi(iN)^{-m} \tau(\chi^{-1})}{(m+t-1)!(j-1)!}$$

be the elementary constant (in which  $\iota_\chi = 1$  when  $\chi$  is primitive) appearing in equation (18) of [BD]. The following formula for  $L(f \otimes g, j)$  is obtained by setting  $c = j$  in equation (18) of [BD]. (See also Theorem 2 of [Sh1].)

**Proposition 2.2.** *The special value  $L(f \otimes g, j)$  is given by the formula*

$$(2.6) \quad L(f \otimes g, j) = C(k, \ell, j) \langle f^*(z), \delta_m^t E_{m,\chi}(z) \times g(z) \rangle_{k,N}.$$

**2.1.3. Algebraicity and Deligne’s conjecture.** Let  $S_k^{\text{nh}}(N, \chi_f^{-1}; K_{fg}) \subset S_k^{\text{nh}}(N, \chi_f^{-1})$  denote the space of nearly holomorphic cusp forms which are defined over  $K_{fg}$  in the sense of Shimura (cf. Section 2.4 of [DR]). The cusp form

$$(2.7) \quad \Xi(f, g, j) := \delta_m^t E_{m,\chi} \times g \in S_k^{\text{nh}}(N, \chi_f^{-1})$$

appearing in Proposition 2.2 belongs to the  $K_{fg}$ -rational structure  $S_k^{\text{nh}}(N, \chi_f^{-1}; K_{fg})$ . Hence, its image

$$(2.8) \quad \Xi(f, g, j)^{\text{hol}} := \Pi_N^{\text{hol}}(\Xi(f, g, j))$$

under the holomorphic projection  $\Pi_N^{\text{hol}}$  of [DR] belongs to  $S_k(N, \chi_f^{-1}; K_{fg})$ , the space of holomorphic cusp forms with Fourier coefficients in  $K_{fg}$ . In particular, the ratio

$$(2.9) \quad \begin{aligned} L^{\text{alg}}(f \otimes g, j) &:= C(f, g, j)^{-1} \frac{L(f \otimes g, j)}{\langle f^*, f^* \rangle_{k, N}} \\ &= \frac{\langle f^*, \Xi(f, g, j) \rangle_{k, N}}{\langle f^*, f^* \rangle_{k, N}} = \frac{\langle f^*, \Xi(f, g, j)^{\text{hol}} \rangle_{k, N}}{\langle f^*, f^* \rangle_{k, N}} \end{aligned}$$

belongs to  $K_{fg}$ . This algebraicity result is consistent with Deligne’s conjecture which predicts that the period  $C(f, g, j)\langle f^*, f^* \rangle_{k, N}$  is the “transcendental part” of the special value  $L(f \otimes g, j)$ . The associated “algebraic part” appearing in (2.9) will later be interpolated  $p$ -adically to obtain Hida’s  $p$ -adic Rankin  $L$ -function attached to  $f$  and  $g$ .

In order to do this, it will be convenient to give a more geometric description of the quantity  $L^{\text{alg}}(f \otimes g, j)$  appearing in (2.9), in terms of the Poincaré duality on the de Rham cohomology of the modular curve  $X_1(N)$  with values in appropriate sheaves with connection, as described in [DR, §2.3]. To lighten the notation, denote by  $Y$  and by  $X$  the open modular curve  $Y_1(N)$  and the complete modular curve  $X_1(N)$  respectively, classifying (generalised) elliptic curves equipped with an embedding of the finite flat group scheme  $\mu_N$  of  $N$ -th roots of unity.

Let  $K$  be any field containing  $K_{fg}$ . Denote by  $\mathcal{E} \rightarrow Y$  the universal elliptic curve over  $Y$ , and by  $\omega$  the sheaf of relative differentials on  $\mathcal{E}$  over  $Y$ , extended to  $X$  as in [BDP, §1.1]. Recall the Kodaira-Spencer isomorphism  $\omega^2 \simeq \Omega_X^1(\log \text{cusps})$ , where  $\Omega_X^1(\log \text{cusps})$  is the sheaf of regular differentials on  $Y$  with log poles at the cusps.

A modular form  $\phi$  on  $\Gamma_1(N)$  of weight  $k = r + 2$  with Fourier coefficients in  $K$  corresponds to a global section of the sheaf  $\omega^{r+2} = \omega^r \otimes \Omega_X^1(\log \text{cusps})$  over the base-change  $X_K$  of  $X$  to  $K$ . The sheaf  $\omega^r$  can be viewed as a subsheaf of  $\mathcal{L}_r := \text{Sym}^r \mathcal{L}$ , where

$$\mathcal{L} := R^1 \pi_* (\mathcal{E} \rightarrow Y)$$

is the relative de Rham cohomology sheaf on  $Y$ , extended to  $X$  as in [BDP], equipped with the filtration

$$(2.10) \quad 0 \rightarrow \omega \rightarrow \mathcal{L} \rightarrow \omega^{-1} \rightarrow 0$$

arising from the Hodge filtration on the fibers. The sheaf  $\mathcal{L}_r$  is a coherent sheaf over  $X$  of rank  $r+1$ , endowed with the Gauss-Manin connection

$$\nabla : \mathcal{L}_r \longrightarrow \mathcal{L}_r \otimes \Omega_X^1(\log \text{cusps}).$$

Let  $H_{\text{dR}}^1(X_K, \mathcal{L}_r, \nabla)$  be the de Rham cohomology of  $\mathcal{L}_r$ . It is equipped with the perfect Poincaré pairing

$$(2.11) \quad \langle \cdot, \cdot \rangle_{k,X} : H_{\text{dR}}^1(X_K, \mathcal{L}_r, \nabla) \times H_{\text{dR}}^1(X_K, \mathcal{L}_r, \nabla) \longrightarrow K,$$

which is compatible with the exact sequence

$$(2.12) \quad 0 \longrightarrow H^0(X_K, \omega^r \otimes \Omega_X^1) \longrightarrow H_{\text{dR}}^1(X_K, \mathcal{L}_r, \nabla) \longrightarrow H^1(X_K, \omega^{-r}) \longrightarrow 0,$$

in the sense that  $H^0(X_K, \omega^r \otimes \Omega_X^1)$  is an isotropic subspace (cf. Sections 2 and 3 of [Col] for a more detailed account). In particular, Poincaré duality induces a perfect pairing

$$(2.13) \quad \langle \cdot, \cdot \rangle_{k,X} : H^1(X_K, \omega^{-r}) \times H^0(X_K, \omega^r \otimes \Omega_X^1) \longrightarrow K,$$

which is denoted by the same symbol by a slight abuse of notation.

The antiholomorphic differential  $\eta_f^{\text{ah}}$  defined by

$$(2.14) \quad \eta_f^{\text{ah}} := \frac{\bar{f}^*(z)d\bar{z}}{\langle f^*, f^* \rangle_{k,N}}$$

gives rise to a class in  $H_{\text{dR}}^1(X_{\mathbb{C}}, \mathcal{L}_r, \nabla)$ , whose image  $\eta_f$  in  $H^1(X_{\mathbb{C}}, \omega^{-r})$  belongs to  $H^1(X_K, \omega^{-r})$  (cf. Corollary 2.13 of [DR]). The following expression for the algebraic part  $L^{\text{alg}}(f \otimes g, j)$  in terms of the class  $\eta_f$  follows directly from (2.9) in light of the discussion above:

**Proposition 2.3.** *The algebraic part  $L^{\text{alg}}(f \otimes g, j)$  is equal to*

$$(2.15) \quad L^{\text{alg}}(f \otimes g, j) = \langle \eta_f, \Xi(f, g, j)^{\text{hol}} \rangle_{k,X}.$$

**2.1.4. The value at the near central point.** Consider now the case where  $k = \ell = 2$  and  $\chi_f \neq \chi_g^{-1}$ , so that the character  $\chi = \chi_f^{-1} \chi_g^{-1}$  is not the trivial one. The functional equation for  $L(f \otimes g, s)$  relates  $L(f \otimes g, s)$  to  $L(f^* \otimes g^*, 3 - s)$ , and this  $L$ -series has no critical points in the sense of Deligne. Proposition 2.5 below describes its value at the near-central point  $s = 2$  in terms of logarithms of *modular units*.

Enlarge  $K$  so that it contains the field which is cut out by all the Dirichlet characters of modulus  $N$ , and let  $F$  be the field generated over  $K$  by the values of these characters. Let  $\text{Eis}_{\ell}(\Gamma_1(N); F)$  denote the subspace of  $M_{\ell}(\Gamma_1(N); F)$  spanned by the weight  $\ell$  Eisenstein series with coefficients in  $F$ . The logarithmic derivative gives a surjective homomorphism

$$(2.16) \quad \mathcal{O}(Y_K)^{\times} \otimes F \xrightarrow{\text{dlog}} \text{Eis}_2(\Gamma_1(N); F),$$



whose kernel is the subspace  $K^\times \otimes F$  spanned by the non-zero constant functions.

**Definition 2.4.** Let  $u_\chi$  be the modular unit satisfying

$$(2.17) \quad \text{dlog}(u_\chi) = E_{2,\chi}$$

whose value at  $\infty$  is 1 in the sense of [Br, §5].

**Proposition 2.5.** *Given weight two eigenforms  $f$  and  $g$  as above,*

$$(2.18) \quad L(f \otimes g, 2) = 16\pi^3 N^{-2} \tau(\chi^{-1}) \langle f^*(z), \log |u_\chi(z)| \cdot g(z) \rangle_{2,N}.$$

*Proof.* By Proposition 2.1,

$$(2.19) \quad L(f \otimes g, 2) = \frac{1}{2} (4\pi)^2 \left\langle f^*(z), \tilde{E}_{0,\chi}(z, 1) \cdot g(z) \right\rangle_{2,N}.$$

A direct calculation (cf. equation (26) of [BD]) shows that

$$(2.20) \quad \frac{1}{2\pi i} \frac{d}{dz} \tilde{E}_{0,\chi}(z, 1) = -\frac{1}{4\pi} \tilde{E}_{2,\chi}(z) = 2\pi N^{-2} \tau(\chi^{-1}) E_{2,\chi}(z).$$

Having normalized  $u_\chi$  as in Definition 2.4, one obtains the equality

$$(2.21) \quad \tilde{E}_{0,\chi}(z, 1) = 2\pi N^{-2} \tau(\chi^{-1}) \log |u_\chi(z)|,$$

which is compatible with (2.17). Combining (2.19) with (2.21) completes the proof of the proposition.  $\square$

**2.2.  $p$ -adic  $L$ -series.** Let  $p \geq 3$  be a prime, and fix an embedding of  $K$  into  $\mathbb{C}_p$ . This section recalls the definition of the Rankin  $p$ -adic  $L$ -function associated by Hida [Hi] to the convolution of two Hida families of cusp forms. For the sake of brevity, we proceed here, just as in [BD], by specialising the approach and notations of [DR], which constructs the  $p$ -adic  $L$ -function associated to a triple product of three Hida families  $(\mathbf{f}, \mathbf{g}, \mathbf{h})$  of cusp forms. The setting considered here consists, essentially, of letting  $\mathbf{h}$  be a Hida family of Eisenstein series.

**2.2.1. Ordinary projections.** Let  $f, g$  be eigenforms of level  $N$ , weights  $k > \ell$  and nebentypus  $\chi_f, \chi_g$  as in (2.1). Let also  $j \in [\frac{\ell+k-1}{2}, k-1]$  be an integer and set  $t = k-1-j \geq 0$  and  $m = k-\ell-2t \geq 1$  as in §2.1.2. The following ordinariness assumption is important for the constructions described in this section.

**Assumption 2.6.** *The cuspidal eigenforms  $f$  and  $g$  are ordinary at  $p$ , and  $p \nmid N$ .*

Under this assumption, the  $f$ -isotypic part of the exact sequence (2.12) with  $K = \mathbb{C}_p$  admits a canonical *unit root* splitting, arising from the action of Frobenius on de Rham cohomology. Let  $\eta_f^{\text{ur}}$  be the lift of  $\eta_f$  to the unit root subspace  $H_{\text{dR}}^1(X_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)^{f,\text{ur}}$ . The right-hand side of (2.15) is then equal to

$$(2.22) \quad \langle \eta_f, \Xi(f, g, j)^{\text{hol}} \rangle_{k,X} = \langle \eta_f^{\text{ur}}, \Xi(f, g, j)^{\text{hol}} \rangle_{k,X}.$$

Now let  $e_{\text{ord}}$  be Hida’s ordinary projector to  $H_{\text{dR}}^1(Y_K, \mathcal{L}_r, \nabla)^{\text{ord}}$ . By Proposition 2.11 of [DR], the right-hand side of (2.22) can be rewritten after viewing  $\Xi(f, g, j)^{\text{hol}}$  as an overconvergent  $p$ -adic modular form and setting  $\Xi(f, g, j)^{\text{ord}} := e_{\text{ord}}\Xi(f, g, j)^{\text{hol}}$  as

$$(2.23) \quad \langle \eta_f^{\text{ur}}, \Xi(f, g, j)^{\text{hol}} \rangle_{k, X} = \langle \eta_f^{\text{ur}}, \Xi(f, g, j)^{\text{ord}} \rangle_{k, X}.$$

By Proposition 2.8 of [DR],

$$(2.24) \quad \Xi(f, g, j)^{\text{ord}} = e_{\text{ord}}(d^t E_{m, \chi} \cdot g),$$

where  $d = q \frac{d}{dq}$  is Serre’s derivative operator on  $p$ -adic modular forms.

Given a  $p$ -adic modular form  $\phi = \sum c_n q^n$ , let  $\phi^{[p]} := \sum_{p \nmid n} c_n q^n$  denote its “ $p$ -depletion” and set

$$(2.25) \quad \Xi(f, g, j)^{\text{ord}, p} := e_{\text{ord}}(d^t E_{m, \chi}^{[p]} \cdot g).$$

**Proposition 2.7.** *Let  $e_{f^*}$  be the projector to the  $f^*$ -isotypic subspace of  $H_{\text{dR}}^1(Y_K, \mathcal{L}_{k-2}, \nabla)$ . Then*

$$e_{f^*}\Xi(f, g, j)^{\text{ord}, p} = \frac{\mathcal{E}(f, g, j)}{\mathcal{E}(f)} \cdot e_{f^*}\Xi(f, g, j)^{\text{ord}},$$

where

$$\begin{aligned} \mathcal{E}(f, g, j) &= (1 - \beta_p(f)\alpha_p(g)p^{t-k+1})(1 - \beta_p(f)\beta_p(g)p^{t-k+1}) \\ &\quad \times (1 - \beta_p(f)\alpha_p(g)\chi(p)p^{t-k+m})(1 - \beta_p(f)\beta_p(g)\chi(p)p^{t-k+m}), \\ \mathcal{E}(f) &= 1 - \beta_p(f)^2\chi_f^{-1}(p)p^{-k}. \end{aligned}$$

*Proof.* This follows from Corollary 4.17 of [DR], in light of Proposition 2.8 of [DR]. □

**2.2.2. Hida’s  $p$ -adic L-series.** Let  $\mathbf{f}$  and  $\mathbf{g}$  be Hida families of ordinary  $p$ -adic modular forms of tame level  $N$ , indexed by weight variables  $k$  and  $\ell$  in suitable neighborhoods  $U_{\mathbf{f}}$  and  $U_{\mathbf{g}}$  of  $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ , contained in a single residue class modulo  $p-1$ . (These families may be obtained, as shall be the case considered in §4.2, by deforming two given ordinary classical eigenforms  $f$  and  $g$  of possibly equal weights.) Assume likewise that the parameter  $j = k - 1 - t$  belongs to a single residue class modulo  $p-1$  so that the same holds true for the weight  $m = k - \ell - 2t$  of the Eisenstein series  $E_{m, \chi}$ .

For  $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$  and  $\ell \in U_{\mathbf{g}} \cap \mathbb{Z}^{\geq 2}$ , let

$$f_k \in S_k(N, \chi_f), \quad g_\ell \in S_\ell(N, \chi_g)$$

be the classical cusp forms whose  $p$ -stabilisations are the weight  $k$  and  $\ell$  specialisations of  $\mathbf{f}$  and  $\mathbf{g}$  respectively. (We denote by  $\chi_f$ , resp.  $\chi_g$ , the common character of the modular forms  $f_k$ , resp.  $g_k$ .)

The collection of  $p$ -adic modular forms  $\Xi(f_k, g_\ell, j)^{\text{ord}, p}$  (defined as in equation (2.25)) indexed by

$$(2.26) \quad \{(k, \ell, j), \quad k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}, \quad \ell \in U_{\mathbf{g}} \cap \mathbb{Z}^{\geq 2}, \quad \frac{\ell + k - 1}{2} \leq j \leq k - 1\}$$

has Fourier coefficients which extend analytically to  $U_{\mathbf{f}} \times U_{\mathbf{g}} \times \mathbb{Z}_p$  as functions in  $k, \ell$  and  $j$ . Hence, they can be viewed as a (three-variable)  $\Lambda$ -adic family of modular forms of level  $N$  in the sense of [DR, §2.7].

Set

$$\mathcal{E}^*(f_k) := 1 - \beta_p(f_k)^2 \chi_f^{-1}(p) p^{1-k}.$$

Proposition 4.10 of [DR] shows that the expression

$$(2.27) \quad L_p(\mathbf{f}, \mathbf{g})(k, \ell, j) := \frac{1}{\mathcal{E}^*(f_k)} \langle \eta_{f_k}^{\text{ur}}, \Xi(f_k, g_\ell, j)^{\text{ord}, p} \rangle_{k, X},$$

defined on the triples  $(k, \ell, j)$  in the set in (2.26), extends to an analytic function  $L_p(\mathbf{f}, \mathbf{g})$  on  $U_{\mathbf{f}} \times U_{\mathbf{g}} \times \mathbb{Z}_p$ , which we refer to as the Hida  $p$ -adic Rankin  $L$ -function attached to  $\mathbf{f}$  and  $\mathbf{g}$ . This appellation is justified by noting that for all triples  $(k, \ell, j)$  in the range of ‘‘classical interpolation’’, i.e. belonging to (2.26), the function  $L_p(\mathbf{f}, \mathbf{g})(k, \ell, j)$  satisfies the interpolation property

$$L_p(\mathbf{f}, \mathbf{g})(k, \ell, j) = \frac{\mathcal{E}(f_k, g_\ell, j)}{\mathcal{E}^*(f_k)\mathcal{E}(f_k)} L^{\text{alg}}(f_k \otimes g_\ell, j).$$

This follows from a direct calculation combining (2.27), Proposition 2.7, (2.23), (2.22) and (2.15).

Note that the point  $(2, 2, 2)$  lies outside the region of classical interpolation for this function. (In fact, there are no critical values for the pair of weights  $(2, 2)$ .) Corollary 4.4 of Section 4.2 relates the value of  $L_p(\mathbf{f}, \mathbf{g})$  at  $(2, 2, 2)$  to the  $p$ -adic regulator attached in Section 3.3 to the triple of modular forms  $(f = f_2, g = g_2, E_{2, X})$ .

Generalising our setting somewhat, we do not assume now that the modular form  $g \in S_2(N, \chi_g)$  is ordinary, so that  $g$  may not necessarily be viewed as the weight 2 specialisation of a Hida family. In this case, the above construction still allows us to define a two-variable  $p$ -adic  $L$ -function  $L_p(\mathbf{f}, g)(k, j)$  on  $U_{\mathbf{f}} \times \mathbb{Z}_p$  by the equation

$$(2.28) \quad L_p(\mathbf{f}, g)(k, j) := \frac{1}{\mathcal{E}^*(f_k)} \langle \eta_{f_k}^{\text{ur}}, \Xi(f_k, g, j)^{\text{ord}, p} \rangle_{k, X},$$

for  $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$  and  $(k + 1)/2 \leq j \leq k - 1$ . Theorem 4.2 relates  $L_p(\mathbf{f}, g)(2, 2)$  to the  $p$ -adic regulator attached to  $(f, g, E_{2, X})$ .

### 3. Beilinson-Flach elements

**3.1. Definition and basic properties.** Let  $S$  be a quasi-projective variety over a field  $K$ , and  $K_j(S)$  denote Quillen’s algebraic  $K$ -groups of  $S$ . The motivic cohomology groups  $H_{\mathcal{M}}^i(S, \mathbb{Q}(n)) = K_{2n-i}^{(n)}(S)$  of  $S$  were defined by Beilinson [Bei, §2] as the  $n$ -th graded piece of the Adams filtration on  $K_{2n-i}(S) \otimes \mathbb{Q}$ . In parallel with Beilinson’s motivic cohomology groups, Bloch [Bl] introduced the higher Chow groups  $\text{CH}^i(S, n)$  of  $S$ .

In this note we shall focus on the smooth projective surface  $S := X \times X$ , where  $X$  is the modular curve over the field  $K$  of §2.1.4. For  $i = 3$  and  $n = 2$ ,  $H_{\mathcal{M}}^3(S, \mathbb{Q}(2)) = K_1^{(2)}(S)$  is identified with  $\text{CH}^2(S, 1) \otimes \mathbb{Q}$ . The higher Chow group  $\text{CH}^2(S, 1)$  may be explicitly described (cf. also [Sc]) as the first homology of the *Gersten complex*

$$(3.1) \quad K_2(K(S)) \xrightarrow{\partial} \bigoplus_{Z \subset S} K(Z)^\times \xrightarrow{\text{div}} \bigoplus_{P \in S} \mathbb{Z},$$

where

- (1)  $K_2(K(S))$  denotes the second Milnor  $K$ -group of the rational function field  $K(S)$ , and  $\partial$  is the map whose “component at  $Z$ ” is the tame symbol attached to the valuation  $\text{ord}_Z$ ;
- (2) the group

$$\Theta := \bigoplus_{Z \subset S} K(Z)^\times$$

is the set of finite formal linear combinations  $\sum_i (Z_i, u_i)$ , where the  $Z_i$  are irreducible curves in  $S$  and  $u_i$  is a rational function on  $Z_i$ ;

- (3) the map  $\text{div}$  is the divisor map, and the direct sum defining its target is taken over all closed points  $P \in S_K$ .

Given a closed point  $P \in X$  and a rational function  $u$  on  $X$ , an element of  $\Theta$  of the form  $(\{P\} \times X, u)$  (resp. of the form  $(X \times \{P\}, u)$ ) is said to be *vertical* (resp. *horizontal*). A linear combination of vertical and horizontal terms is said to be *negligible*. Similar definitions apply to the tensor product  $\Theta \otimes F$  over any field  $F$ .

Let  $\Delta \subset S$  be a copy of the curve  $X$  diagonally embedded in  $S$ . Let  $F$  denote the field introduced in §2.1.4, let  $u \in \mathcal{O}(Y_K)^\times \otimes F$  be a modular unit with coefficients in  $F$ , and consider the element  $(\Delta, u) \in \Theta \otimes F$ .

**Lemma 3.1.** *There exists a negligible element  $\theta_u \in \Theta \otimes F$  satisfying*

$$\text{div}(\theta_u) = \text{div}(\Delta, u).$$

*Proof.* Let  $D_u = \text{div}(\Delta, u) \in \coprod_{P \in S} F$  be the image of the element  $(\Delta, u) \in \Theta$  under the divisor map. Since  $D_u$  is an  $F$ -linear combination of elements

of the form  $(c_1, c_1) - (c_2, c_2)$  where  $c_1$  and  $c_2$  are cusps of the modular curve  $X_K$ , it is enough to construct a negligible element  $\theta \in \Theta \otimes \mathbb{Q}$  satisfying

$$(3.2) \quad \text{div}(\theta) = (c_1, c_1) - (c_2, c_2).$$

By the Manin-Drinfeld theorem, there is an element  $\alpha \in \mathcal{O}(Y_K)^\times \otimes \mathbb{Q}$  whose divisor is  $c_1 - c_2$ , and the negligible element given by

$$\theta = (\{c_1\} \times X, \alpha) + (X \times \{c_2\}, \alpha)$$

satisfies (3.2). The lemma follows. □

Thanks to Lemma 3.1, we can associate to any element of the form  $(\Delta, u) \in \Theta \otimes F$  the element

$$(3.3) \quad \Delta_u := \text{class of } (\Delta, u) - \theta_u \quad \text{in } H^3_{\mathcal{M}}(S, F(2)).$$

These elements were introduced by Beilinson in [Bei, Ch. 2, §6]. A variant ([Fl, Prop. 2.1]) of the above construction was later exploited by Flach in [Fl] to prove the finiteness of the Tate-Shafarevic group of the symmetric square of an elliptic curve, using the method of Kolyvagin. We call  $\Delta_u$  the *Beilinson-Flach element* attached to the modular unit  $u \in \mathcal{O}(Y_K)^\times \otimes F$ . Strictly speaking,  $\Delta_u$  is not a well-defined element in  $H^3_{\mathcal{M}}(S, F(2))$ , as it is only well-defined modulo the  $F$ -vector space generated by the classes of negligible elements. However, this inherent ambiguity will not lead to problems because the image of  $\Delta_u$  under the relevant piece of the regulator maps will turn out to depend only on  $u$  and not on the choice of  $\theta_u$  made in defining  $\Delta_u$ . See Proposition 3.3 below for more details.

**3.2. Complex regulators.** Fix an embedding of  $K$  into the field of complex numbers. Following the definitions in [Bei, §2], [DS, §2], the *complex regulator* on  $H^3_{\mathcal{M}}(S_{\mathbb{C}}, \mathbb{Q}(2))$  may be regarded as a map

$$(3.4) \quad \mathbf{reg}_{\mathbb{C}} : H^3_{\mathcal{M}}(S_{\mathbb{C}}, \mathbb{Q}(2)) \longrightarrow (\text{Fil}^1 H^2_{\text{dR}}(S/\mathbb{C}))^\vee,$$

where here the superscript  $\vee$  denotes the complex linear dual. It sends the class of  $\theta = \sum_i (Z_i, u_i)$  to the element  $\mathbf{reg}_{\mathbb{C}}(\theta)$  defined by

$$\mathbf{reg}_{\mathbb{C}}(\theta)(\omega) = \frac{1}{2\pi i} \sum_i \int_{Z_i - Z_i^{\text{sing}}} \omega \log |u_i|.$$

Recall the modular unit  $u_\chi$  associated to the Dirichlet character  $\chi$ , and the class  $\eta_f^{\text{ah}} \in H^1_{\text{dR}}(X/\mathbb{C})$  attached to the cusp form  $f$ . Moreover, write as customary  $\omega_g \in \text{Fil}^1 H^1_{\text{dR}}(X/\mathbb{C})$  for the class associated to the regular differential  $2\pi i g(z) dz$ .

The tensor product  $\omega_g \otimes \eta_f^{\text{ah}}$  of these classes gives rise, via the Künneth decomposition of  $H^2_{\text{dR}}(S/\mathbb{C})$ , to an element of  $\text{Fil}^1 H^2_{\text{dR}}(S/\mathbb{C})$ .

**Proposition 3.2.** *With notation as above, we have*

$$\begin{aligned} \mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ah}}) &= (-2i)[\Gamma_0(N) : \Gamma_1(N)(\pm 1)] \langle f^*, f^* \rangle_{2,N}^{-1} \\ &\quad \times \langle f^*(z), \log |u_\chi(z)| \cdot g(z) \rangle_{2,N}. \end{aligned}$$

*Proof.* Since the differential  $\omega_g \otimes \eta_f^{\text{ah}}$  vanishes identically on the horizontal and vertical curves on  $S$ , the negligible element  $\theta_{u_\chi}$  arising in the definition of  $\Delta_{u_\chi}$  does not contribute to the value of the regulator at that class. Hence

$$\begin{aligned} \mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ah}}) &= \int_{X(\mathbb{C})} \frac{\bar{f}^*(z)}{\langle f^*, f^* \rangle_{2,N}} g(z) \log |u_\chi(z)| dz d\bar{z} \\ &= (-2i)[\Gamma_0(N) : \Gamma_1(N)(\pm 1)] \langle f^*, f^* \rangle_{2,N}^{-1} \langle f^*(z), \log |u_\chi(z)| \cdot g(z) \rangle_{2,N}, \end{aligned}$$

where the last equality follows from the explicit formula for the Petersson scalar product on  $S_k^{\text{ra}}(N, \chi_f^{-1})$ .  $\square$

**3.3.  $p$ -adic regulators.** Let  $K_p$  be a finite extension of  $\mathbb{Q}_p$  containing  $K$  and fix an embedding of  $K_p$  in  $\mathbb{C}_p$ . Write  $\mathcal{O}_p$ , resp.  $k_p$ , for the ring of integers, resp. the residue field of  $K_p$ . Let  $\mathcal{X}$  denote the (Deligne-Rapoport) smooth model of  $X$  over  $\mathcal{O}_p$ , and  $\tilde{\mathcal{X}}/k_p$  its special fiber. Define  $\mathcal{S} = \mathcal{X} \times \mathcal{X}$ , which is a smooth projective model of  $S_{K_p}$  over  $\mathcal{O}_p$ .

In analogy with the complex regulator (3.4), there is a  $p$ -adic syntomic regulator map

$$(3.5) \quad \mathbf{reg}_p : H_{\mathcal{M}}^3(S_{K_p}, \mathbb{Q}(2)) \longrightarrow (\text{Fil}^1 H_{\text{dR}}^2(S/K_p))^\vee := \text{Hom}(\text{Fil}^1 H_{\text{dR}}^2(S/K_p), K_p)$$

arising from the syntomic Chern character in  $K$ -theory (cf. [Gros], [Ni], [Bes3]).

After possibly enlarging the field  $K_p$ , let  $\{P_1, \dots, P_t\} \subset \mathcal{X}(\mathcal{O}_p)$  be a set of points consisting of the cusps and of a choice of a lift of every supersingular point in  $\tilde{\mathcal{X}}(\bar{\mathbb{F}}_p)$ . Set

$$\mathcal{X}' = \mathcal{X} \setminus \{P_1, \dots, P_t\}, \quad X' = \mathcal{X}' \times_{\text{spec } \mathcal{O}_p} \text{spec } K_p.$$

Let  $\text{red} : \mathcal{X}(\mathcal{O}_p) \longrightarrow \tilde{\mathcal{X}}(k_p)$  denote the reduction map and let  $\mathcal{A} \subset X(K_p)$  be the affinoid subspace of the rigid analytic variety underlying  $X$  defined by

$$\mathcal{A} := X(K_p) - \text{red}^{-1}(\{\tilde{P}_1, \dots, \tilde{P}_t\}), \quad \tilde{P}_j := \text{red}(P_j).$$

Fix a system  $\{\mathcal{W}_\epsilon\}_{\epsilon > 0}$  of wide open neighborhoods of  $\mathcal{A}$  as in [DR, §2.1] and denote by  $\Phi$  the canonical lift of Frobenius on  $X$  as in [DR, §2.2]. As explained there, restriction from  $X'$  to  $\mathcal{W}_\epsilon$  gives rise to an isomorphism

$$(3.6) \quad H_{\text{dR}}^1(X') \xrightarrow{\text{comp}_\epsilon} H_{\text{rig}}^1(\mathcal{W}_\epsilon)$$

between the de Rham cohomology of the open curve  $X'$  and the rigid cohomology  $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$  of  $\mathcal{W}_\epsilon$ . The inclusion  $X' \subset X$  yields by restriction a

monomorphism

$$H_{\text{dR}}^1(X) \hookrightarrow H_{\text{dR}}^1(X'),$$

and the image of  $H_{\text{dR}}^1(X)$  under  $\text{comp}_\epsilon$  consists of those classes in  $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$  whose annular residues about all the points  $\{P_i\}$  vanish. The lift  $\Phi$  of Frobenius induces a linear endomorphism of  $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$  which preserves the subspace  $H_{\text{dR}}^1(X)$ .

Label now two copies of  $X$  as  $X_1$  and  $X_2$ , denote by  $\Phi_1$  and  $\Phi_2$  the corresponding canonical lifts of Frobenius on the system of wide open neighborhoods  $\mathcal{W}_\epsilon$ , and write  $\Phi_{12} := (\Phi_1, \Phi_2)$  for the associated lift of Frobenius on the product  $X_1 \times X_2$ .

Choose a polynomial  $P(x) \in \mathbb{C}_p[x]$  such that

- (i)  $P(\Phi_{12})$  annihilates the class of  $\omega_g \otimes \frac{du_\chi}{u_\chi}$  in  $H_{\text{rig}}^2(\mathcal{W}_\epsilon^2)$ ;
- (ii)  $P(\Phi)$  is an invertible endomorphism on  $H_{\text{dR}}^1(X')$ .

Such a polynomial exists, since the eigenvalues of  $\Phi_{12}$  acting on the space spanned by the Frobenius translates of  $\omega_g \otimes \frac{du_\chi}{u_\chi}$  have complex absolute value  $p^{3/2}$ , while  $\Phi$  acts on  $H_{\text{dR}}^1(X')$  with eigenvalues of complex absolute value  $p^{1/2}$  and  $p$ .

Thanks to (i), there exists a rigid analytic one-form

$$(3.7) \quad \varrho_P \in \Omega^1(\mathcal{W}_\epsilon^2) \text{ such that } d(\varrho_P) = P(\Phi_{12}) \left( \omega_g \otimes \frac{du_\chi}{u_\chi} \right).$$

This form, which depends on the choice of  $P$ , is determined only up to *closed* forms in  $\Omega^1(\mathcal{W}_\epsilon^2)$  by (3.7).

In order to adapt our calculations to Besser’s in [Bes2] and [Bes3], it will be convenient to fix a particular choice of polynomial  $P$  and form  $\varrho_P$ . (In the next section we shall exploit the fact that the computations performed there hold independently of the choice of  $P$  and will work with a different polynomial so that we can take advantage of the results obtained in [DR].)

Let  $P_g(t) \in \mathbb{C}_p[t]$  be a polynomial such that  $P_g(\Phi)$  annihilates the class of  $\omega_g$  in  $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$ . Specifically, we may set  $P_g(t) := t^2 - a_p(g)t + \chi_g(p)p$ , and let  $F_g \in \mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon)$  be a Coleman integral of  $\omega_g$ , that is to say, a rigid analytic function such that

$$(3.8) \quad pdF_g = p\omega_{g^{[p]}} = P_g(\Phi)\omega_g$$

(cf. for example equation (127) of [DR]). Likewise, let  $P_{E_\chi}(t) \in \mathbb{C}_p[t]$  be a polynomial such that  $P_{E_\chi}(\Phi)$  annihilates the class of the Eisenstein series  $E_\chi = \frac{du_\chi}{u_\chi}$  in  $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$ . Here we make the specific choice  $P_{E_\chi}(t) := t^h - p^h$ , where  $h$  is the order of the root of unity  $\chi(p)$  (in other words,  $\Phi^h/p^h$  fixes the class of  $E_\chi$ ). Although a more optimal choice for  $P_{E_\chi}(t)$  would have been the linear polynomial  $t - \chi(p)p$ , we made here a choice corresponding to the

one made in the definition of the modified syntomic regulator  $\text{reg}(u_\chi)$  of the function  $u_\chi$  (cf. [Bes2, Prop. 10.3]). The rigid analytic function

$$(3.9) \quad F_{E_\chi} := p^{-h} P_{E_\chi}(\Phi) \log(u_\chi) \in \mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon)$$

is a Coleman integral of  $E_\chi$ , satisfying

$$p^h dF_{E_\chi} = P_{E_\chi}(\Phi) E_\chi.$$

Given two choices as above of polynomials  $P_g(t) = \prod_i (t - \alpha_i)$  and  $P_{E_\chi}(t) = \prod_j (t - \beta_j)$ , it is clear that the polynomial

$$(3.10) \quad P(t) := P_g(t) \star P_{E_\chi}(t) := \prod_{i,j} (t - \alpha_i \beta_j)$$

satisfies (i) above. Moreover, as explained in [Bes1, Lemma 4.2, (4)], there exist polynomials  $a(t_1, t_2), b(t_1, t_2)$  such that  $P(t_1 \cdot t_2) = p^{-1} a(t_1, t_2) P_g(t_1) + p^{-h} b(t_1, t_2) P_{E_\chi}(t_2) \in \mathbb{C}_p[t_1, t_2]$ , and one checks that

$$(3.11) \quad \varrho_P = a(\Phi_1, \Phi_2) \left( F_g \otimes \frac{du_\chi}{u_\chi} \right) + b(\Phi_1, \Phi_2) (\omega_g \otimes F_{E_\chi}) \in \Omega^1(\mathcal{W}_\epsilon^2)$$

then satisfies (3.7).

There is a certain degree of ambiguity in (3.11): neither the Coleman primitives  $F_g, F_{E_\chi}$  nor the polynomials  $a(t_1, t_2), b(t_1, t_2)$  are unique. But all solutions of the differential equation (3.7) are of the form (3.11); moreover, given one such  $\varrho_P$ , all of them can be written as  $\varrho_P + \varrho_0$  with  $\varrho_0$  a closed 1-form on  $\mathcal{W}_\epsilon^2$ .

We can single out a canonical choice of  $\varrho_P$  (up to exact 1-forms on  $\mathcal{W}_\epsilon^2$ ) by setting  $F_g(\infty) = F_{E_\chi}(\infty) = 0$  in (3.11); more precisely, in doing this, two different choices of pairs  $(a(t_1, t_2), b(t_1, t_2)), (a'(t_1, t_2), b'(t_1, t_2))$  allowed by [Bes1, Lemma 4.2, (4)] give rise to forms  $\varrho_{P,a,b}, \varrho_{P,a',b'}$  such that  $\varrho_0 = \varrho_{P,a,b} - \varrho_{P,a',b'}$  is exact on  $\mathcal{W}_\epsilon^2$  and therefore the class of  $\varrho_0$  in  $H_{\text{rig}}^1(\mathcal{W}_\epsilon^2)$  vanishes.

Imposing  $F_g(\infty) = 0$  amounts to normalizing the  $q$ -expansion of  $F_g$  to be

$$(3.12) \quad F_g(q) = \sum_{p \nmid n} \frac{a_n(g)}{n} q^n,$$

and the condition  $F_{E_\chi}(\infty) = 0$  is equivalent to normalizing the modular unit  $u_\chi$  as was done in Definition 2.4. This way  $F_{E_\chi}$  also equals the modified syntomic regulator  $\text{reg}(u_\chi)$  of  $u_\chi$  defined in [Bes2, Prop. 10.3].

Let  $\Delta \subset \mathcal{W}_\epsilon^2$  denote the diagonal and define

$$(3.13) \quad \xi'_P := [\varrho_P|_\Delta] \in H_{\text{rig}}^1(\mathcal{W}_\epsilon) \simeq H_{\text{dR}}^1(X').$$



The above discussion shows that the class  $\xi'_P$  in  $H^1_{\text{rig}}(\mathcal{W}_\epsilon) = \frac{\Omega^1(\mathcal{W}_\epsilon)}{d\mathcal{O}(\mathcal{W}_\epsilon)}$  is well-defined. Moreover, in view of condition (ii), we can now set

$$(3.14) \quad \xi' := P(\Phi)^{-1} \cdot \xi'_P \in H^1_{\text{dR}}(X'),$$

which is directly seen to be independent of the choice of  $P$ .

Finally, let  $\text{spl}_X : H^1_{\text{dR}}(X') \rightarrow H^1_{\text{dR}}(X)$  denote the Frobenius equivariant splitting of the short exact sequence

$$(3.15) \quad 0 \rightarrow H^1_{\text{dR}}(X) \rightarrow H^1_{\text{dR}}(X') \rightarrow K_p(-1)^{t-1} \rightarrow 0$$

and set  $\xi := \text{spl}_X(\xi') \in H^1_{\text{dR}}(X)$ .

**Proposition 3.3.** *With notation as above, we have*

$$\text{reg}_p(\Delta_{u_x})(\omega_g \otimes \eta_f^{\text{ur}}) = \langle \eta_f^{\text{ur}}, \xi \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the pairing on  $H^1_{\text{dR}}(X)$  induced by Poincaré duality.

*Proof.* Thanks to the work of Besser [Bes3], the  $p$ -adic syntomic regulator (3.5) admits the following description in terms of Coleman integration. Let  $\theta = \sum_i (Z_i, u_i)$  be an element in  $K_1^{(2)}(S)$  and write  $\iota_i : Z_i \hookrightarrow S$  for the embedding of  $Z_i$  in  $S$  given by inclusion. Assume for simplicity that the curves  $Z_i$  are all non-singular and that  $\theta$  is *integral*, by which we mean that for each  $i$ :

- the curve  $Z_i$  admits a smooth integral model  $\mathcal{Z}_i$  over  $\mathcal{O}_p$  and
- the divisor of  $u_i$ , when regarded as a function on  $\mathcal{Z}_i$ , does not contain the special fiber.

Note that these conditions are satisfied in our setting.

Under this assumption,  $\theta$  lies in the image of the natural restriction map  $K_1(\mathcal{S}) \rightarrow K_1(S)$ .

Let  $\Omega^{\text{II}}(X_{K_p})$  denote the space of differential forms of the second kind on  $X_{K_p}$ , that is to say, the space of meromorphic 1-forms whose residue at any point of the curve is zero. There is an exact sequence

$$0 \rightarrow K_p(X)^\times \xrightarrow{d} \Omega^{\text{II}}(X_{K_p}) \rightarrow H^1_{\text{dR}}(X/K_p) \rightarrow 0$$

and for any  $\eta \in \Omega^{\text{II}}(X_{K_p})$  we write  $[\eta]$  for its class in  $H^1_{\text{dR}}(X/K_p)$ .

Instead of invoking the description of the  $p$ -adic syntomic regulator in terms of Besser-de Jeu’s global triple index as stated in the main theorem of [Bes3], it will be more convenient for us to exploit [Bes3, Prop. 6.3], which provides a formula for (3.5) in the language of Besser’s finite polynomial cohomology [Bes1]. In order to state this formula, let  $H^*_{\text{ms}}$  and  $H^*_{\text{fp}}$  denote, respectively, Besser’s modified version of syntomic cohomology and finite polynomial cohomology; cf. e.g. [Bes3, §2] for a quick review of both and their interactions.

Let  $\omega \in \Omega^1(X_{K_p})$  be a regular form on  $X$  and  $\eta \in \Omega^{\text{II}}(X_{K_p})$  be a differential of the second kind, regular on some affine curve  $X^0 \subset X$ . Write

$$\omega_1 = \pi_1^*(\omega) \in \Omega^1(S), \quad \eta_2 = \pi_2^*(\eta) \in \Omega^{\text{II}}(S)$$

for the pull-back of  $\omega$  and  $\eta$  under the projection of  $S$  into the first and second component, respectively.

Then the class  $\omega_1 \wedge [\eta_2]$  is an element of  $\text{Fil}^1 H_{\text{dR}}^2(S)$  and, according to [Bes3, Theorem 1.1, Prop. 6.3]:

$$(3.16) \quad \mathbf{reg}_p(\theta)(\omega_1 \otimes [\eta_2]) = \sum_i \langle \iota_i^* \tilde{\eta}_2, \iota_i^* \tilde{\omega}_1 \cup \mathbf{reg}(u_i) \rangle_{\mathcal{Z}_i^0, \text{fp}},$$

where

- $\mathcal{Z}_i^0 = Z_i \cap (X \times X^0)$ ,  $\mathcal{Z}_i^0$  is the model for  $Z_i^0$  deduced from  $\mathcal{Z}_i$ ,
- $\mathbf{reg}(u_i) \in H_{\text{ms}}^1(\mathcal{Z}_i^0, 1) \subseteq H_{\text{fp}}^1(\mathcal{Z}_i^0, 1, 2)$  is the regulator of the function  $u_i$  as defined in [Bes2, Prop. 10.3],
- $\iota_i^* \tilde{\omega}_1 \in H_{\text{fp}}^1(\mathcal{Z}_i, 1, 1)$  is a Coleman primitive of  $\iota_i^* \omega \in \Omega^1(Z_i)$ ,
- $\iota_i^* \tilde{\eta}_2 \in H_{\text{fp},c}^1(\mathcal{Z}_i^0, 0, 1)$  is the single lift of  $\iota_i^*([\eta_2])$  under the isomorphism

$$(3.17) \quad \mathfrak{p} : H_{\text{fp},c}^1(\mathcal{Z}_i^0, 0, 1) \xrightarrow{\sim} H_{\text{dR}}^1(Z_i)$$

of [Bes3, Lemma 6.2], and

$$(3.18) \quad \langle \cdot, \cdot \rangle_{\mathcal{Z}_i^0, \text{fp}} : H_{\text{fp},c}^1(\mathcal{Z}_i^0, 0, 1) \times H_{\text{fp}}^2(\mathcal{Z}_i^0, 2, 3) \longrightarrow H_{\text{fp},c}^3(\mathcal{Z}_i^0, 2, 4) \simeq H_{\text{dR},c}^2(\mathcal{Z}_i^0) \stackrel{\text{tr}}{\simeq} K_p$$

is the pairing induced by Poincaré duality in finite polynomial cohomology. Here  $H_{\text{fp},c}^*$  stands for finite polynomial cohomology with compact support, as introduced in [Bes3, §4]. The cup-product (3.18) is constructed in [Bes3], where it is also shown that it satisfies the projection formula.

At the time [Bes3] was written, the results were subject to the compatibility of pushforward maps in syntomic and motivic cohomology, as specified in [Bes3, Conjecture 4.2]. At present this compatibility has been checked by Déglise and Mazzari [DM], and thus (3.16) holds unconditionally.

Let us now apply (3.16) to the Beilinson-Flach element  $\Delta_{u_x}$  that was introduced in (3.3). Recall that the curves in  $X \times X$  on which  $\Delta_{u_x}$  is supported are the images of  $X$  under the diagonal embedding  $\iota_{12}(x) = (x, x)$  and the various horizontal and vertical embeddings  $\iota_{1,c}(x) = (x, c)$  and  $\iota_{2,c}(x) = (c, x)$ , where  $c$  is a cusp on the modular curve  $X$ .

We firstly claim that the terms on the right-hand side of (3.16) corresponding to  $\iota_{1,c}$  and  $\iota_{2,c}$  vanish and the one corresponding to  $\iota_{12}$  is independent of the choices of lifts to finite polynomial cohomology.

To see that, put  $\omega = \omega_g$  and  $\eta = \eta_f^{\text{ur}}$  and recall  $X' = \mathcal{X}' \times K_p$  is the curve obtained from  $X$  by removing a finite set of points including all the cusps. Note first that  $\iota_{1,c}^*([\eta_2]) = 0 \in H_{\text{dR}}^1(X)$ , because the composition  $\pi_2 \circ \iota_{1,c}$  is the constant function  $c$  on  $X$ . Hence, since the map  $p$  in (3.17) is an isomorphism, the class of the lift  $\iota_{1,c}^*(\tilde{\eta}_2)$  is also trivial and

$$\langle \iota_{1,c}^* \tilde{\eta}_2, \iota_{1,c}^* \tilde{\omega}_1 \cup \text{reg}(u) \rangle_{\mathcal{X}', \text{fp}} = 0,$$

for any rational function  $u$ .

We similarly have  $\iota_{2,c}^*(\omega_1) = 0 \in \Omega^1(X)$  because  $\pi_1 \circ \iota_{2,c} = c$ . Notice however that a lift of 0 to  $H_{\text{fp}}^1(\mathcal{X}, 1, 1)$  is not necessarily trivial, but represented by a pair in  $\mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^1(X)$  of the form  $[(\lambda, 0)]$ , where  $\lambda$  is a constant. Then, if  $u$  is a modular unit on  $X$ , the cup-product

$$\iota_{2,c}^* \tilde{\omega}_1 \cup \text{reg}(u) \in H_{\text{fp}}^2(\mathcal{X}', 2, 3) \simeq H_{\text{dR}}^1(X') \simeq H_{\text{rig}}^1(\mathcal{W}_\epsilon)$$

may be represented by the pair  $(\lambda \frac{du}{u}|_{\mathcal{W}_\epsilon}, 0)$ . But then

$$(3.19) \quad \langle \iota_{2,c}^* \tilde{\eta}_2, \iota_{2,c}^* \tilde{\omega}_1 \cup \text{reg}(u) \rangle_{\mathcal{X}', \text{fp}} = \lambda \langle \eta_f^{\text{ur}}, \frac{du}{u} \rangle_{\text{dR}} = 0$$

because the cusp form  $f$  is orthogonal to the Eisenstein series  $\frac{du}{u}$ . This accounts for the vanishing of the horizontal and vertical terms and explains why we call them negligible.

As for the diagonal term, let us show that  $\langle \iota_{12}^* \tilde{\eta}_2, \iota_{12}^* \tilde{\omega}_1 \cup \text{reg}(u_\chi) \rangle_{\mathcal{X}', \text{fp}}$  is independent of the choices of lifts to finite polynomial cohomology. Since  $\pi_1 \circ \iota_{12}$  and  $\pi_2 \circ \iota_{12}$  are both the identity map on  $X$ , this is just  $\langle \tilde{\eta}_f^{\text{ur}}, \tilde{\omega}_g \cup \text{reg}(u_\chi) \rangle_{\mathcal{X}', \text{fp}}$ . Again there is a single choice for  $\tilde{\eta}_f^{\text{ur}}$ , but the Coleman integral  $F_g$  of  $\omega_g$  is only well-defined up to a constant. The difference between any two choices is then equal to

$$\langle \tilde{\eta}_f^{\text{ur}}, [(\lambda, 0)] \cup \text{reg}(u_\chi) \rangle_{\mathcal{X}', \text{fp}} = \lambda \left\langle \eta_f^{\text{ur}}, \frac{du_\chi}{u_\chi} \right\rangle_{\text{dR}}$$

for some  $\lambda \in K_p$ , and the same orthogonality argument between cusp and Eisenstein forms again shows that this is 0. The claim follows.

Summing up, we obtain from (3.16) that

$$(3.20) \quad \text{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}) = \langle \tilde{\eta}_f^{\text{ur}}, \tilde{\omega}_g \cup \text{reg}(u_\chi) \rangle_{\mathcal{X}', \text{fp}}.$$

Recall that  $\tilde{\omega}_g$  may be represented by the pair  $(F_g, \omega_g)$  where  $F_g \in \mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon)$  is a Coleman integral of  $\omega_g$ , which in light of the above claim we are entitled to normalize as was done in (3.12). Besides, by [Bes2, Prop. 10.3] the class  $\text{reg}(u_\chi)$  is represented by the pair  $(F_{E_\chi}, \frac{du_\chi}{u_\chi}) \in \mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^1(X')$  where  $F_{E_\chi}$  is the Coleman integral of  $\frac{du_\chi}{u_\chi}$  introduced in (3.9) and normalized as we explained right after (3.12).

By definition,  $\tilde{\omega}_g \cup \text{reg}(u_\chi)$  is the restriction to the diagonal of  $\pi_1^* \tilde{\omega}_g \wedge \pi_2^* \text{reg}(u_\chi)$ . Note that the polynomial  $P$  defined in equation (3.10) satisfies the properties (i) and (ii) above. The class  $\pi_1^* \tilde{\omega}_g \wedge \pi_2^* \text{reg}(u_\chi)$  in  $H_{\text{fp}}^2(\mathcal{X}'^2, 2, 3)$  may then be represented by the pair

$$(3.21) \quad (\varrho_P, \pi_1^* \omega_g \wedge \pi_2^* \frac{du_\chi}{u_\chi}) \in \Omega_{\text{rig}}^1(\mathcal{W}_\epsilon^2) \oplus \Omega^2(X'^2),$$

where  $\varrho_P$  is the form introduced in (3.11), which satisfies

$$(3.22) \quad d\varrho_P = P(\Phi_{12})(\pi_1^* \omega_g \wedge \pi_2^* \frac{du_\chi}{u_\chi}).$$

Let us again remark that this differential equation does not determine  $\varrho_P$  uniquely, but that the above normalizations of  $F_g$  and  $F_{E_\chi}$  completely determine it up to exact 1-forms on  $\mathcal{W}_\epsilon^2$ . Obviously, when we restrict (3.21) to the diagonal, this ambiguity does not affect the class we obtain in  $H_{\text{fp}}^2(\mathcal{X}', 2, 3)$ , because exact 1-forms on  $\mathcal{W}_\epsilon$  vanish in  $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$ .

In conclusion, the class  $\tilde{\omega}_g \cup \text{reg}(u_\chi)$  in  $H_{\text{fp}}^2(\mathcal{X}', 2, 3)$  may be represented by the pair

$$(\iota_{12}^*(\varrho_P), 0) \in \Omega_{\text{rig}}^1(\mathcal{W}_\epsilon) \oplus \Omega^2(X'),$$

where  $\varrho_P$  is as above and  $\iota_{12}^*(\varrho_P)$  is the form denoted by  $\xi'_P$  in (3.13).

As in [Bes1, (14)] there is a commutative diagram

$$(3.23) \quad \begin{array}{ccc} H_{\text{fp},c}^1(\mathcal{X}', 0, 1) \times H_{\text{dR}}^1(X') & \xrightarrow{\text{Id} \times \text{i}} & H_{\text{fp},c}^1(\mathcal{X}', 0, 1) \times H_{\text{fp}}^2(\mathcal{X}', 2, 3) \\ \downarrow \text{p} \times \text{Id} & & \downarrow \langle \cdot, \cdot \rangle_{\text{fp}} \\ H_{\text{dR},c}^1(X')^{w=1} \times H_{\text{dR}}^1(X') & \xrightarrow{\langle \cdot, \cdot \rangle_{\text{dR}}} & H_{\text{dR},c}^2(X') \simeq H_{\text{fp},c}^3(\mathcal{X}', 2, 4), \end{array}$$

where  $H_{\text{dR},c}^1(X')^{w=1}$  stands for the pure submodule of weight 1 of  $H_{\text{dR},c}^1(X')$ . In fact both maps

$$H_{\text{dR}}^1(X') \xrightarrow{\text{i}} H_{\text{fp}}^1(\mathcal{X}', 2, 3) \quad \text{and} \quad H_{\text{fp},c}^1(\mathcal{X}', 0, 1) \xrightarrow{\text{p}} H_{\text{dR},c}^1(X')^{w=1}$$

are isomorphisms, as follows from [Bes3, (2.7) and the first assertion of Lemma 2.8].

By definition of  $\text{i}$ , the preimage of  $\tilde{\omega}_g \cup \text{reg}(u_\chi) = [(\xi'_P, 0)]$  under  $\text{i}$  is the class in  $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$  of the 1-form  $P(\Phi)^{-1}(\xi'_Q) = \xi'$ . To conclude, we now deduce from the commutativity of (3.23) that

$$\langle \eta_f^{\text{ur}}, \xi' \rangle_{\text{dR}} = \langle \tilde{\eta}_f^{\text{ur}}, \text{i}(\xi') \rangle_{\text{fp}} = \langle \tilde{\eta}_f^{\text{ur}}, \tilde{\omega}_g \cup \text{reg}(u_\chi) \rangle_{\text{fp}}.$$

Since the class  $\eta_f^{\text{ur}}$  is orthogonal to the complement of  $H_{\text{dR}}^1(X)$  in  $H_{\text{dR}}^1(X')$  under the Frobenius equivariant splitting of (3.15), we have  $\langle \eta_f^{\text{ur}}, \xi' \rangle_{\text{dR}} = \langle \eta_f^{\text{ur}}, \xi \rangle_{\text{dR}}$ , and the proposition follows.  $\square$

**4. The Beilinson formula**

Let  $f \in S_2(N, \chi_f)$ ,  $g \in S_2(N, \chi_g)$  be eigenforms of weight 2 as in Section 2.1.4. Recall that  $f$  and  $g$  are not assumed to be newforms. Moreover, we insist on the condition  $\chi_f \neq \chi_g^{-1}$ , which implies that  $\chi = \chi_f^{-1}\chi_g^{-1}$  is non-trivial.

**4.1. The complex setting.** In [Bei, Ch. 2, §6], Beilinson relates the image of  $\Delta_{u_\chi}$  under the complex regulator map to the value at  $s = 2$  of the Rankin  $L$ -series attached to  $f \otimes g$ . The following explicit version of Beilinson’s theorem is a slight generalisation of the results of [BaSr].

**Proposition 4.1.** *For cusp forms  $f$  and  $g$  of weight two as in Section 2.1.4, we have*

$$\frac{L(f \otimes g, 2)}{\langle f^*, f^* \rangle_{2,N}} = (8i)\pi^3[\Gamma_0(N) : \Gamma_1(N)(\pm 1)]^{-1}N^{-2}\tau(\chi^{-1})\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ah}}).$$

*Proof.* This follows by combining the explicit formula for  $L(f \otimes g, 2)$  obtained in Proposition 2.5 with the explicit expression for  $\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})$  given in Proposition 3.2. □

**4.2. The  $p$ -adic setting.** Let  $p \geq 3$  be a prime which does not divide  $N$ . Assume that  $f$  is ordinary at  $p$  (with respect to a fixed embedding of the field  $K_f$  in  $\mathbb{C}_p$ ). Let  $\mathbf{f}$  be the Hida family whose specialisation in weight 2 is the  $p$ -stabilisations of  $f$ , and let  $L_p(\mathbf{f}, g)(k, j)$  be the  $p$ -adic  $L$ -function defined in Section 2.2.2.

Let  $\mathcal{E}(f)$ ,  $\mathcal{E}^*(f)$  and  $\mathcal{E}(f, g, 2)$  be the  $p$ -adic multipliers defined in Sections 2.2.1 and 2.2.2. Recall that

$$\begin{aligned} \mathcal{E}(f, g, 2) &= (1 - \beta_p(f)\alpha_p(g)p^{-2})(1 - \beta_p(f)\beta_p(g)p^{-2}) \\ &\quad \times (1 - \beta_p(f)\alpha_p(g)\chi(p)p^{-1})(1 - \beta_p(f)\beta_p(g)\chi(p)p^{-1}). \end{aligned}$$

The following  $p$ -adic Beilinson formula is the main result of this paper.

**Theorem 4.2.** *For cusp forms  $f$  and  $g$  of weight two as in Section 2.1.4, we have*

$$L_p(\mathbf{f}, g)(2, 2) = \frac{\mathcal{E}(f, g, 2)}{\mathcal{E}(f) \cdot \mathcal{E}^*(f)} \times \mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}).$$

*Proof.* By the description of the  $p$ -adic  $L$ -function given in equation (2.28),

$$L_p(\mathbf{f}, g)(k, j) = \frac{1}{\mathcal{E}^*(f_k)} \langle \eta_{f_k}^{\text{ur}}, \Xi(f_k, g, j)^{\text{ord}, p} \rangle_{k, X}$$

for all triples  $(k, \ell, j)$  belonging to the set (2.26). Since the terms in the above expression vary analytically, taking the limit to  $k = \ell = j = 2$  yields, in light

of equation (2.25),

$$(4.1) \quad L_p(\mathbf{f}, g)(2, 2) = \frac{1}{\mathcal{E}^*(f)} \left\langle \eta_f^{\text{ur}}, e_{\text{ord}}(d^{-1}E_{2,\chi}^{[p]} \cdot g) \right\rangle_{2,X}.$$

On the other hand, by Proposition 3.3,

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}) = \langle \eta_f^{\text{ur}}, \xi \rangle.$$

Since

$$\Phi(\eta_f^{\text{ur}}) = \alpha_p(f)\eta_f^{\text{ur}}, \quad \langle \Phi(\eta_f^{\text{ur}}), \Phi(\xi) \rangle = p\langle \eta_f^{\text{ur}}, \xi \rangle, \quad \alpha_p(f)\beta_p(f) = \chi_f(p)p,$$

we deduce by multi-linearity that

$$\langle \eta_f^{\text{ur}}, \xi \rangle = P(\chi_f^{-1}(p)\beta_p(f))^{-1} \langle \eta_f^{\text{ur}}, \xi'_P \rangle.$$

Since  $f$  is an ordinary eigenform, the quantity  $\langle \eta_f^{\text{ur}}, \xi'_P \rangle$  depends only on the  $f^*$ -isotypical ordinary projection of  $\xi'_P$ , that is,  $\langle \eta_f^{\text{ur}}, \xi'_P \rangle = \langle \eta_f^{\text{ur}}, e_{f^*}e_{\text{ord}}\xi'_P \rangle$ .

Choose the polynomial  $P(x)$  satisfying conditions (i) and (ii) to be

$$P(x) := (x - \alpha_p(g)) \cdot (x - \alpha_p(g)\chi(p)p) \cdot (x - \beta_p(g)) \cdot (x - \beta_p(g)\chi(p)p).$$

This choice of  $P$  has the advantage of allowing us to directly invoke the calculations already performed in [DR, Prop. 5.4]. They give

$$e_{f^*}e_{\text{ord}}\xi'_P = \chi_f(p)^{-2}p^4\mathcal{E}(f) \cdot e_{f^*}e_{\text{ord}}(d^{-1}E_{2,\chi}^{[p]} \cdot g).$$

A direct calculation shows that

$$\mathcal{E}(f, g, 2) = p^{-4}\chi_f(p)^{-2}P(\chi_f^{-1}(p)\beta_p(f)).$$

By combining the above remarks, we find the following expression for the  $p$ -adic regulator:

$$(4.2) \quad \mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}) = \frac{\mathcal{E}(f)}{\mathcal{E}(f, g, 2)} \times \left\langle \eta_f^{\text{ur}}, e_{f^*}e_{\text{ord}}(d^{-1}E_{2,\chi}^{[p]} \cdot g) \right\rangle_{2,X}.$$

The theorem follows by comparing equations (4.1) and (4.2). □

**Remark 4.3.** Note that the modular form  $g$  that arises in Theorem 4.2 is fixed throughout the argument and is thus not required to be ordinary at  $p$ . Assume now that both  $f$  and  $g$  are ordinary at  $p$  (with respect to a fixed embedding of the field  $K_{fg}$  in  $\mathbb{C}_p$ ). Let  $\mathbf{f}$  and  $\mathbf{g}$  be the Hida families whose specialisations in weight 2 are the  $p$ -stabilisations of  $f$  and  $g$ , respectively, and let  $L_p(\mathbf{f}, \mathbf{g})(k, \ell, j)$  be the  $p$ -adic  $L$ -function defined in Section 2.2.2. The following corollary is an immediate consequence of Theorem 4.2.

**Corollary 4.4.** *For cusp forms  $f$  and  $g$  of weight two as in Section 2.1.4, we have*

$$L_p(\mathbf{f}, \mathbf{g})(2, 2, 2) = \frac{\mathcal{E}(f, g, 2)}{\mathcal{E}(f) \cdot \mathcal{E}^*(f)} \times \mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}).$$

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### References

- [BaSr] Srinath Baba and Ramesh Sreekantan, *An analogue of circular units for products of elliptic curves*, Proc. Edinb. Math. Soc. (2) **47** (2004), no. 1, 35–51, DOI 10.1017/S001309150200113X. MR2064735 (2005c:11073)
- [BD] Massimo Bertolini and Henri Darmon, *Kato’s Euler system and rational points on elliptic curves I: a  $p$ -adic Beilinson formula*, Israel J. Math. **199** (2014), no. 1, 163–188, DOI 10.1007/s11856-013-0047-2. MR3219532
- [BDP] Massimo Bertolini, Henri Darmon, and Kartik Prasanna, *Generalized Heegner cycles and  $p$ -adic Rankin  $L$ -series*, with an appendix by Brian Conrad, Duke Math. J. **162** (2013), no. 6, 1033–1148, DOI 10.1215/00127094-2142056. MR3053566
- [BDR] M. Bertolini, H. Darmon, and V. Rotger, *Beilinson-Flach elements and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin  $L$ -series*, submitted.
- [Bei] A. A. Beilinson, *Higher regulators and values of  $L$ -functions* (Russian), Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238. MR760999 (86h:11103)
- [Bes1] Amnon Besser, *A generalization of Coleman’s  $p$ -adic integration theory*, Invent. Math. **142** (2000), no. 2, 397–434, DOI 10.1007/s002220000093. MR1794067 (2001i:14032)
- [Bes2] Amnon Besser, *Syntomic regulators and  $p$ -adic integration. I. Rigid syntomic regulators*, Proceedings of the Conference on  $p$ -adic Aspects of the Theory of Automorphic Representations (Jerusalem, 1998), Israel J. Math. **120** (2000), part B, 291–334, DOI 10.1007/BF02834843. MR1809626 (2002c:14035)
- [Bes3] Amnon Besser, *On the syntomic regulator for  $K_1$  of a surface*, Israel J. Math. **190** (2012), 29–66, DOI 10.1007/s11856-011-0188-0. MR2956231
- [Bl] Spencer Bloch, *Algebraic cycles and higher  $K$ -theory*, Adv. in Math. **61** (1986), no. 3, 267–304, DOI 10.1016/0001-8708(86)90081-2. MR852815 (88f:18010)
- [Br] François Brunault, *Valeur en 2 de fonctions  $L$  de formes modulaires de poids 2: théorème de Beilinson explicite* (French, with English and French summaries), Bull. Soc. Math. France **135** (2007), no. 2, 215–246. MR2430191 (2009e:11096)
- [Col] Robert F. Coleman, *A  $p$ -adic Shimura isomorphism and  $p$ -adic periods of modular forms*,  $p$ -adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), Contemp. Math., vol. 165, Amer. Math. Soc., Providence, RI, 1994, pp. 21–51, DOI 10.1090/conm/165/01602. MR1279600 (96a:11050)
- [DR] Henri Darmon and Victor Rotger, *Diagonal cycles and Euler systems I: a  $p$ -adic Gross-Zagier formula* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 779–832. MR3250064
- [Das] S. Dasgupta, *Greenberg’s conjecture on symmetric square  $p$ -adic  $L$ -functions*, preprint, 2014.
- [DM] F. Déglise and N. Mazzari, *The rigid syntomic spectrum*, preprint, 2012.

- [DS] Christopher Deninger and Anthony J. Scholl, *The Beilinson conjectures, L-functions and arithmetic* (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, pp. 173–209, DOI 10.1017/CBO9780511526053.007. MR1110393 (92h:11056)
- [Fl] Matthias Flach, *A finiteness theorem for the symmetric square of an elliptic curve*, Invent. Math. **109** (1992), no. 2, 307–327, DOI 10.1007/BF01232029. MR1172693 (93g:11066)
- [Gros] Michel Gros, *Régulateurs syntomiques et valeurs de fonctions L p-adiques. II* (French), Invent. Math. **115** (1994), no. 1, 61–79, DOI 10.1007/BF01231754. MR1248079 (95f:11044)
- [Gross] Benedict H. Gross, *On the factorization of p-adic L-series*, Invent. Math. **57** (1980), no. 1, 83–95, DOI 10.1007/BF01389819. MR564185 (82a:12010)
- [Hi] Haruzo Hida, *Elementary theory of L-functions and Eisenstein series*, London Mathematical Society Student Texts, vol. 26, Cambridge University Press, Cambridge, 1993. MR1216135 (94j:11044)
- [LLZ] Antonio Lei, David Loeffler, and Sarah Livia Zerbes, *Euler systems for Rankin-Selberg convolutions of modular forms*, Ann. of Math. (2) **180** (2014), no. 2, 653–771, DOI 10.4007/annals.2014.180.2.6. MR3224721
- [LZ] David Loeffler and Sarah Livia Zerbes, *Iwasawa theory and p-adic L-functions over  $\mathbb{Z}_p^2$ -extensions*, Int. J. Number Theory **10** (2014), no. 8, 2045–2095, DOI 10.1142/S1793042114500699. MR3273476
- [Ni] Wiesława Nizioł, *On the image of p-adic regulators*, Invent. Math. **127** (1997), no. 2, 375–400, DOI 10.1007/s002220050125. MR1427624 (98a:14031)
- [Pa] Ambrus Pál, *A rigid analytical regulator for the  $K_2$  of Mumford curves*, Publ. Res. Inst. Math. Sci. **46** (2010), no. 2, 219–253, DOI 10.2977/PRIMS/8. MR2722778 (2011m:19002)
- [Sc] Anthony J. Scholl, *Integral elements in K-theory and products of modular curves*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 467–489. MR1744957 (2001i:11077)
- [Sh1] Goro Shimura, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), no. 6, 783–804. MR0434962 (55 #7925)
- [Sh2] Goro Shimura, *On a class of nearly holomorphic automorphic forms*, Ann. of Math. (2) **123** (1986), no. 2, 347–406, DOI 10.2307/1971276. MR835767 (88b:11025a)
- [Sre1] Ramesh Sreekantan, *Non-Archimedean regulator maps and special values of L-functions*, Cycles, motives and Shimura varieties, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2010, pp. 469–492. MR2906033
- [Sre2] Ramesh Sreekantan,  *$K_1$  of products of Drinfeld modular curves and special values of L-functions*, Compos. Math. **146** (2010), no. 4, 886–918, DOI 10.1112/S0010437X10004720. MR2660677 (2011g:11095)

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