p-ADIC RANKIN L-SERIES AND RATIONAL POINTS ON CM ELLIPTIC CURVES

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1. Introduction

The aim of this article is to present a new proof of a theorem of Karl Rubin (see [Ru] and Thm. 1 below) relating values of the Katz p-adic L-function of an imaginary quadratic field at certain points outside its range of classical interpolation to the formal group logarithms of rational points on CM elliptic curves. This theorem has been seminal in providing a motivation for Perrin-Riou's formulation ([PR2], [PR3]) of the p-adic Beilinson conjectures. The new proof described in this work is based on the p-adic Gross-Zagier type formula of [BDP-gz], and only makes use of Heegner points (as opposed to the original proof which relied on on a comparison between Heegner points and elliptic units). Hence it should be adaptable to more general situations, for example to the setting of general CM fields.

Let A be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the ring of integers of a quadratic imaginary field K. A classical result of Deuring identifies the Hasse-Weil L-series $L(A, s)$ of A with the L-series $L(\nu_A, s)$ attached to a Hecke character ν_A of K of infinity type $(1, 0)$. When p is a prime which splits in K and does not divide the conductor of A, the Hecke L-function $L(\nu_A, s)$ has a p-adic analog, namely the Katz two-variable p-adic L-function attached to K. It is a p-adic analytic function, denoted $\nu \mapsto \mathscr{L}_p(\nu)$, on the space of Hecke characters equipped with its natural p-adic analytic structure. Section 3.1 recalls the definition of this L-function: the values $\mathscr{L}_p(\nu)$ at Hecke characters of infinity type $(1 + j_1, -j_2)$ with $j_1, j_2 \geq 0$ are defined by interpolation of the classical L-values $L(\nu^{-1}, 0)$. Letting $\nu^* := \nu \circ c$, where c denotes complex conjugation on the ideals of K, it is readily seen by comparing Euler factors that $L(\nu, s) = L(\nu^*, s)$. A similar equality need not hold in the p-adic setting, because the involution $\nu \mapsto \nu^*$ corresponds to the map $(j_1, j_2) \mapsto (j_2, j_1)$ on weight space and therefore does not preserve the lower right quadrant of weights of Hecke characters that lie in the range of classical interpolation. Since ν_A lies in the domain of classical interpolation, the p-adic L-value $\mathscr{L}_p(\nu_A)$ is a simple multiple of $L(\nu_A^{-1}, 0) = L(A, 1)$. Suppose that it vanishes. (This implies, by the Birch and Swinnerton-Dyer conjecture, that $A(\mathbb{Q})$ is infinite.) The value $\mathscr{L}_p(\nu_A^*)$ is a second, a priori more mysterious p-adic avatar of the leading term of $L(A, s)$ at $s = 1$. Rubin's theorem gives a formula for this quantity:

Theorem 1 (Rubin). *Let ν_A be a Hecke character of type $(1, 0)$ attached to an elliptic curve $A/\mathbb{Q}$ with complex multiplication. Then there exists a global point $P \in A(\mathbb{Q})$ such that*

$$\mathscr{L}_p(\nu_A^*) = \Omega_p(A)^{-1} \log_{\omega_A}(P)^2 \pmod{K^\times}, \tag{1.1}$$

where

- *$\Omega_p(A)$ is the p-adic period attached to A as in Section 2.3;*
- *$\omega_A \in \Omega^1(A/\mathbb{Q})$ is a regular differential on A over $\mathbb{Q}$, and $\log_{\omega_A} : A(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ denotes the p-adic formal group logarithm with respect to ω_A.*

The point P is of infinite order if and only if $L(A, s)$ has a simple zero at $s = 1$.

(For a more precise statement without the $K^\times$ ambiguity, see [Ru].) Formula (1.1) is peculiar to the p-adic world and suggests that p-adic L-functions encode arithmetic information that is not readily apparent in their complex counterparts.

The proof of Theorem 1 given in [Ru] breaks up naturally into two parts:

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(1) Rubin exploits the Euler system of elliptic units to construct a global cohomology class κ_A belonging to a pro-p Selmer group $\mathrm{Sel}_p(A/\mathbb{Q})$ attached to A. The close connection between elliptic units and the Katz L-function is then parlayed into the explicit evaluation of two natural p-adic invariants attached to κ_A: the p-adic formal group logarithm $\log_{A,p}(\kappa_A)$ and the cyclotomic p-adic height $\langle \kappa_A, \kappa_A \rangle$:

$$\log_{A,p}(\kappa_A) = (1-\beta_p^{-1})^{-1}\mathscr{L}_p(\nu_A^*)\Omega_p(A), \tag{1.2}$$

$$\langle \kappa_A, \kappa_A \rangle = (1-\alpha_p^{-1})^{-2}\mathscr{L}_p'(\nu_A)\mathscr{L}_p(\nu_A^*), \tag{1.3}$$

where

- α_p and β_p denote the roots of the Hasse polynomial $x^2 - a_p(A)x + p$, ordered in such a way that $\mathrm{ord}_p(\alpha_p) = 0$ and $\mathrm{ord}_p(\beta_p) = 1$;
- the quantity $\mathscr{L}_p'(\nu_A)$ denotes the derivative of $\mathscr{L}_p$ at ν_A in the direction of the cyclotomic character.

If $\mathscr{L}_p'(\nu_A)$ is non-zero, then an argument based on Perrin-Riou's p-adic analogue of the Gross-Zagier formula and the work of Kolyvagin implies that $\mathrm{Sel}_p(A/\mathbb{Q}) \otimes \mathbb{Q}$ is a one-dimensional $\mathbb{Q}_p$-vector space with κ_A as a generator. (Cf. Thm. 8.1 and Cor. 8.3 of [Ru].) Equations (1.2) and (1.3) then make it possible to evaluate the ratio

$$\frac{\log^2_{A,p}(\kappa)}{\langle \kappa, \kappa \rangle} = \frac{(1-\beta_p^{-1})^{-2}\mathscr{L}_p(\nu_A^*)\Omega_p(A)^2}{(1-\alpha_p^{-1})^{-2}\mathscr{L}_p'(\nu_A)}, \tag{1.4}$$

a quantity which does not depend on the choice of generator κ of the $\mathbb{Q}_p$-vector space $\mathrm{Sel}_p(A/\mathbb{Q}) \otimes \mathbb{Q}$.

(2) Independently of the construction of κ_A, the theory of Heegner points can be used to construct a canonical point $P \in A(\mathbb{Q})$, which is of infinite order when $\mathscr{L}_p'(\nu_A) \neq 0$. Its image $\kappa_P \in \mathrm{Sel}_p(A/\mathbb{Q})$ under the connecting homomorphism of Kummer theory supplies us with a second generator for $\mathrm{Sel}_p(A/\mathbb{Q}) \otimes \mathbb{Q}$. Furthermore, the p-adic analogue of the Gross-Zagier formula proved by Perrin-Riou in [PR1] shows that

$$\langle \kappa_P, \kappa_P \rangle = \mathscr{L}_p'(\nu_A)\Omega_p(A)^{-1} \pmod{K^\times}. \tag{1.5}$$

Rubin obtains Theorem 1 by setting $\kappa = \kappa_P$ in (1.4) and using (1.5) to eliminate the quantities involving $\langle \kappa_P, \kappa_P \rangle$ and $\mathscr{L}_p'(\nu_A)$.

The reader will note the key role that is played in Rubin's proof by both the Euler systems of elliptic units and of Heegner points. The new approach to Theorem 1 described in this paper relies solely on Heegner points, and requires neither elliptic units nor Perrin-Riou's p-adic height calculations. Instead, the key ingredient in this approach is the p-adic variant of the Gross-Zagier formula arising from the results of [BDP-gz] which is stated in Theorem 3.12. This formula expresses p-adic logarithms of Heegner points in terms of the special values of a p-adic Rankin L-function attached to a cusp form f and an imaginary quadratic field K, and may be of some independent interest insofar as it exhibits a strong analogy with Rubin's formula but applies to arbitrary—not necessarily CM—elliptic curves over $\mathbb{Q}$. When f is the theta series attached to a Hecke character of K, Theorem 1 follows from the factorisation of the associated p-adic Rankin L-function into a product of two Katz L-functions, a factorisation which is a simple manifestation of the Artin formalism for these p-adic L-series.

One might expect that the statement of Theorem 1 should generalise to the setting where ν_A is replaced by an algebraic Hecke character ν of infinity type $(1,0)$ of a quadratic imaginary field K (of arbitrary class number) satisfying

$$\nu|_{\mathbb{A}_\mathbb{Q}} = \varepsilon_K \cdot \mathbf{N}, \tag{1.6}$$

where ε_K denotes the quadratic Dirichlet character associated to $K/\mathbb{Q}$ and $\mathbf{N} : \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{R}^\times$ is the adèlic norm character. Chapter 3 treats this more general setting, which (although probably amenable as well to the original methods of [Ru]) is not yet covered in the literature. Assumption (1.6) implies that the classical functional equation relates $L(\nu, s)$ to $L(\nu, 2-s)$. Assume further that the sign w_ν in this functional equation satisfies

$$w_\nu = -1, \tag{1.7}$$

so that $L(\nu, s)$ vanishes to odd order at $s = 1$. For technical reasons, it will also be convenient to make two further assumptions. Firstly, we assume that

(1.8) The discriminant $-D$ of K is odd.

Secondly, we note that assumption (1.6) implies that $\mathfrak{d}_K := \sqrt{-D}$ necessarily divides the conductor of ν, and we further restrict the setting by imposing the assumption that

(1.9) The conductor of ν is exactly divisible by $\mathfrak{d}_K$.

The statement of Theorem 2 below requires some further notions which we now introduce. Let E_ν be the subfield of $\mathbb{C}$ generated by the values of the Hecke character ν, and let T_ν be its ring of integers. A general construction which is recalled in Sections 2.2 and 3.6 attaches to ν an abelian variety B_ν over K of dimension $[E_\nu : K]$, equipped with inclusions

$$T_\nu \subset \mathrm{End}_K(B_\nu), \qquad E_\nu \subset \mathrm{End}_K(B_\nu) \otimes \mathbb{Q}.$$

Given $\lambda \in T_\nu$, denote by $[\lambda]$ the corresponding endomorphism of B_ν, and set

$$\Omega^1(B_\nu/E_\nu)^{T_\nu} := \left\{\omega \in \Omega^1(B_\nu/E_\nu) \quad \text{such that} \quad [\lambda]^*\omega = \lambda\omega, \quad \forall \lambda \in T_\nu\right\}, \tag{1.10}$$

$$(B_\nu(K) \otimes E_\nu)^{T_\nu} := \{P \in B_\nu(K) \otimes_{\mathbb{Z}} E_\nu \quad \text{such that} \quad [\lambda]P = \lambda P, \quad \forall \lambda \in T_\nu\}. \tag{1.11}$$

The vector space $\Omega^1(B_\nu/E_\nu)^{T_\nu}$ is one-dimensional over E_ν. The results of Gross-Zagier and Kolyvagin, which continue to hold in the setting of abelian variety quotients of modular curves, also imply that $(B_\nu(K) \otimes E_\nu)^{T_\nu}$ is one-dimensional over E_ν when $L(\nu, s)$ has a simple zero at $s = 1$.

After fixing a p-adic embedding $K \subset \mathbb{Q}_p$, the formal group logarithm on B_ν gives rise to a bilinear pairing

$$\begin{array}{rcl} \langle\ ,\ \rangle : \Omega^1(B_\nu/K) \times B_\nu(K) & \longrightarrow & \mathbb{Q}_p \\ (\omega, P) & \mapsto & \log_\omega P, \end{array}$$

satisfying $\langle [\lambda]^*\omega, P\rangle = \langle \omega, [\lambda]P\rangle$ for all $\lambda \in T_\nu$. This pairing can be extended by E_ν-linearity to an $E_\nu \otimes \mathbb{Q}_p$-valued pairing between $\Omega^1(B_\nu/E_\nu)$ and $B_\nu(K) \otimes E_\nu$. When ω and P belong to these E_ν-vector spaces, we will continue to write $\log_\omega(P)$ for $\langle \omega, P\rangle$.

Theorem 2. *Let ν be an algebraic Hecke character of infinity type $(1, 0)$ satisfying (1.6), (1.7), (1.8) and (1.9) above. Then there exists $P_\nu \in B_\nu(K)$ such that*

$$\mathscr{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1} \log_{\omega_\nu}(P_\nu)^2 \pmod{E_\nu^\times},$$

where $\Omega_p(\nu^*) \in \mathbb{C}_p$ is the p-adic period attached to ν in Definition 2.13, and ω_ν is a non-zero element of $\Omega^1(B_\nu/E_\nu)^{T_\nu}$. The point P_ν is non-zero if and only if $L'(\nu, 1) \neq 0$.

Remark 3. Assumptions (1.8) and (1.9) could certainly be relaxed with more work. For instance, (1.8) is needed since the main theorem of [BDP-gz] is only proved for imaginary quadratic fields of odd discriminant. Likewise, removing (1.9) would require generalizing the main result of loc. cit. to the case of Shimura curves over $\mathbb{Q}$.

Remark 4. In [BDP-ch], we give a conjectural construction of rational points on CM elliptic curves (called Chow-Heegner points) using cycles on higher dimensional varieties. While this construction of points is contingent on a certain case of the Tate conjecture, the corresponding contruction at the level of cohomology classes can be made unconditionally. The results of this paper, combined with those of [BDP-gz], are used in [BDP-ch] to establish that these cohomology classes indeed correspond to global points via the Kummer map.

Remark 5. The methods used in the proof of Theorem 2 also give information about the special values $\mathscr{L}_p(\nu^*)$ for Hecke characters ν of type $(1 + j, -j)$ satisfying (1.6) with $j \geq 0$. A discussion of this point will be taken up in future work. (See [BDP-co].)

2. Hecke characters and periods

Throughout this article, all number fields that arise are viewed as embedded in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. A complex embedding $\bar{\mathbb{Q}} \longrightarrow \mathbb{C}$ and p-adic embeddings $\bar{\mathbb{Q}} \longrightarrow \mathbb{C}_p$ for each rational prime p are also fixed from the outset, so that any finite extension of $\mathbb{Q}$ is simultaneously realised as a subfield of $\mathbb{C}$ and of $\mathbb{C}_p$.

2.1. **Algebraic Hecke characters.** We will recall briefly some key definitions regarding algebraic Hecke characters, mainly to fix notation. The reader is referred to [Scha] Ch. 0 for more details. Let K and E be number fields. Given a $\mathbb{Z}$-linear combination

$$\phi = \sum_\sigma n_\sigma \sigma \in \mathbb{Z}[\mathrm{Hom}(K, \bar{\mathbb{Q}})]$$

of embeddings of K into $\bar{\mathbb{Q}}$, we define

$$\alpha^\phi := \prod_\sigma (\sigma\alpha)^{n_\sigma},$$

for all $\alpha \in K^\times$. Let $I_\mathfrak{f}$ denote the group of fractional ideals of K which are prime to a given integral ideal $\mathfrak{f}$ of K, and let

$$J_\mathfrak{f} := \{(\alpha) \text{ such that } \alpha \gg 0 \text{ and } \alpha - 1 \in \mathfrak{f}\} \quad \subseteq \quad I_\mathfrak{f}.$$

Definition 2.1. An E-valued *algebraic Hecke character* (or simply Hecke character) of K of infinity type ϕ and conductor *dividing* $\mathfrak{f}$ is a homomorphism

$$\chi : I_\mathfrak{f} \to E^\times$$

such that

$$\chi((\alpha)) = \alpha^\phi, \quad \text{for all } (\alpha) \in J_\mathfrak{f}. \tag{2.1}$$

The smallest integral ideal $\mathfrak{g}$ such that χ can be extended to a Hecke character of conductor dividing $\mathfrak{g}$ is called the *conductor* of χ, and is denoted $\mathfrak{f}_\chi$.

The most basic examples of algebraic Hecke characters are the norm characters on $\mathbb{Q}$ and on K respectively, which are given by

$$\mathbf{N}((a)) = |a|, \qquad \mathbf{N}_K := \mathbf{N} \circ \mathrm{N}^K_\mathbb{Q}.$$

Note that the infinity type ϕ of a Hecke character χ must be trivial on all totally positive units congruent to $1 \mod \mathfrak{f}$. Hence the existence of such a χ implies there is an integer $w(\chi)$ (called the *weight* of χ or of ϕ) such that for any choice of embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$,

$$n_\sigma + n_{\bar\sigma} = w(\chi), \quad \text{for all } \sigma \in \mathrm{Hom}(K, \bar{\mathbb{Q}}).$$

Let $U_\mathfrak{f} \subset U'_\mathfrak{f} \subset \mathbb{A}_K^\times$ be the subgroups defined by

$$U'_\mathfrak{f} := \left\{ (x_v) \in \mathbb{A}_K^\times \text{ such that } \begin{array}{ll} x_v \equiv 1 \pmod{\mathfrak{f}}, & \text{for all } v|\mathfrak{f}, \\ x_v > 0, & \text{for all real } v \end{array} \right\},$$

and

$$U_\mathfrak{f} := \{(x_v) \in U'_\mathfrak{f} \text{ such that } x_v \in \mathcal{O}_{K_v}^\times, \text{ for all non-archimedean } v\}.$$

A Hecke character χ of conductor dividing $\mathfrak{f}$ may also be viewed as a character on $\mathbb{A}_K^\times/U_\mathfrak{f}$ (denoted by the same symbol by a common abuse of notation),

$$\chi : \mathbb{A}_K^\times/U_\mathfrak{f} \to E^\times, \qquad \text{satisfying} \qquad \chi|_{K^\times} = \phi. \tag{2.2}$$

To wit, given $x \in \mathbb{A}_K^\times$, we define $\chi(x)$ by choosing $\alpha \in K^\times$ such that αx belongs to $U'_\mathfrak{f}$, and setting

$$\chi(x) = \chi(i(\alpha x))\phi(\alpha)^{-1}, \tag{2.3}$$

where the symbol $i(x)$ denotes the fractional ideal of K associated to x. This definition is independent of the choice of α by (2.1). In the opposite direction, given a character χ as in (2.2), we can set

$$\chi(\mathfrak{a}) = \chi(x), \quad \text{for any } x \in U'_\mathfrak{f} \text{ such that } i(x) = \mathfrak{a}.$$

The subfield of E generated by the values of χ on $I_\mathfrak{f}$ is easily seen to be independent of the choice of $\mathfrak{f}$ and will be denoted E_χ.

Definition 2.2. The *central character* ε_η of a Hecke character η of K is the finite order character of $\mathbb{Q}$ given by

$$\eta|_{\mathbb{A}_\mathbb{Q}^\times} = \varepsilon_\eta \cdot \mathbf{N}^{w(\eta)}.$$

The infinity type ϕ defines a homomorphism $\mathrm{Res}_{K/\mathbb{Q}}(\mathbf{G}_m) \to \mathrm{Res}_{E/\mathbb{Q}}(\mathbf{G}_m)$ of algebraic groups and therefore induces a homomorphism

$$\phi_{\mathbb{A}} : \mathbb{A}_K^\times \to \mathbb{A}_E^\times$$

on adelic points. Given a Hecke character χ with values in E and a place λ of E (either finite or infinite), we may use $\phi_{\mathbb{A}}$ to define an idèle class character

$$\chi_\lambda : \mathbb{A}_K^\times / K^\times \to E_\lambda^\times,$$

by setting

$$\chi_\lambda(x) = \chi(x)/\phi_{\mathbb{A}}(x)_\lambda.$$

If λ is an infinite place, the character χ_λ is a Grossencharacter of K of type A_0. If λ is a finite place, then χ_λ factors through G_K^{ab} and gives a Galois character (denoted $\rho_{\chi,\lambda}$) valued in $E_\lambda^\times$, satisfying

$$\rho_{\chi,\lambda}(\mathrm{Frob}_{\mathfrak{p}}) = \chi(\mathfrak{p})$$

for any prime ideal $\mathfrak{p}$ of K not dividing $\mathfrak{f}\lambda$.

Let $\mathfrak{g}$ be any integral ideal of K. The L-function (and L-function with modulus $\mathfrak{g}$) attached to χ are defined by

$$L(\chi, s) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\right)^{-1}, \qquad L_{\mathfrak{g}}(\chi, s) = \prod_{\mathfrak{p} \nmid \mathfrak{g}} \left(1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\right)^{-1}.$$

Note that $L(\chi, s) = L_{\mathfrak{f}_\chi}(\chi, s)$.

The following definition will only be used in Sec. 3.6.

Definition 2.3. Let $E = \prod_i E_i$ be a product of number fields. An *E-valued algebraic Hecke character* of conductor dividing $\mathfrak{f}$ is a character

$$\chi : I_{\mathfrak{f}} \to E^\times$$

whose projection to each component E_i is an algebraic Hecke character in the sense defined above.

2.2. **Abelian varieties associated to characters of type $(1, 0)$.** In this section, we limit the discussion to the case where K is an imaginary quadratic field. Let $\tau : K \mapsto \mathbb{C}$ be the given complex embedding of K. A Hecke character of infinity type $\phi = n_\tau \tau + n_{\bar\tau}\bar\tau$ will also be said to be of infinity type $(n_\tau, n_{\bar\tau})$.

Let ν be a Hecke character of K of infinity type $(1, 0)$ and conductor $\mathfrak{f}_\nu$, let $E_\nu \supset K$ denote the subfield of $\bar{\mathbb{Q}}$ generated by its values, and let T_ν be the ring of integers of E_ν. The Hecke character ν gives rise to a compatible system of one-dimensional ℓ-adic representations of G_K with values in $(E_\nu \otimes \mathbb{Q}_\ell)^\times$, denoted $\rho_{\nu,\ell}$, satisfying

$$\rho_{\nu,\ell}(\sigma_{\mathfrak{a}}) = \nu(\mathfrak{a}), \quad \text{for all } \mathfrak{a} \in I_{\mathfrak{f}_\nu \ell},$$

where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(\bar K/K)$ denotes Frobenius conjugacy class attached to $\mathfrak{a}$. The theory of complex multiplication realises the representations $\rho_{\nu,\ell}$ on the division points of CM abelian varieties:

Definition 2.4. A *CM abelian variety* over K is a pair (B, E) where

(1) B is an abelian variety over K;
(2) E is a product of CM fields equipped the structure of a K-algebra and an inclusion

$$i : E \longrightarrow \mathrm{End}_K(B) \otimes \mathbb{Q},$$

satisfying $\dim_K(E) = \dim B$;
(3) for all $\lambda \in K \subset E$, the endomorphism $i(\lambda)$ acts on the cotangent space $\Omega^1(B/K)$ as multiplication by λ.

The abelian varieties (B, E) over K with complex multiplication by a fixed E form a category denoted $\mathcal{CM}_{K,E}$ in which a morphism from B_1 to B_2 is a morphism $j : B_1 \longrightarrow B_2$ of abelian varieties over K for which the diagrams

$$\begin{array}{ccc} B_1 & \xrightarrow{j} & B_2 \\ \downarrow e & & \downarrow e \\ B_1 & \xrightarrow{j} & B_2 \end{array}$$

commute, for all $e \in E$ which belong to both $\mathrm{End}_K(B_1)$ and $\mathrm{End}_K(B_2)$. An isogeny in $\mathcal{CM}_{K,E}$ is simply a morphism in this category arising from an isogeny on the underlying abelian varieties.

If (B,E) is a CM abelian variety, its endomorphism ring over K contains a finite index subring T^0 of the integral closure T of $\mathbb{Z}$ in E. After replacing B by the K-isogenous abelian variety $\mathrm{Hom}_{T_0}(T,B)$, we can assume that $\mathrm{End}_K(B)$ contains T. This assumption, which is occasionally convenient, will consistently be made from now on.

Let (B,E) be a CM-abelian variety with E a field, and let $E' \supset E$ be a finite extension of E with ring of integers T'. The abelian variety $B \otimes_T T'$ is defined to be the variety whose L-rational points, for any $L \supset K$, are given by

$$(B \otimes_T T')(L) = (B(\bar{\mathbb{Q}}) \otimes_T T')^{\mathrm{Gal}(\bar{\mathbb{Q}}/L)}.$$

This abelian variety is equipped with an action of T' by K-rational endomorphisms, described by multiplication on the right, and therefore $(B \otimes_T T', E')$ is an object of $\mathcal{CM}_{K,E'}$. Note that $B \otimes_T T'$ is isogenous to $t := \dim_E(E')$ copies of B, and that the action of T on $B \otimes_T T'$ agrees with the "diagonal" action of T on B^t.

Let ℓ be a rational prime. For each CM abelian variety (B,E), let

$$T_\ell(B) := \lim_{\leftarrow,n} B[\ell^n](\bar{K}), \qquad V_\ell(B) := T_\ell(B) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

be the ℓ-adic Tate module and ℓ-adic representation of G_K attached to B. The $\mathbb{Q}_\ell$-vector space $V_\ell(B)$ is a free $E \otimes \mathbb{Q}_\ell$-module of rank one via the action of E by endomorphisms. The natural action of $G_K := \mathrm{Gal}(\bar{K}/K)$ on $V_\ell(B)$ commutes with this $E \otimes \mathbb{Q}_\ell$-action, and the collection $\{V_\ell(B)\}$ thus gives rise to a compatible system of one-dimensional ℓ-adic representations of G_K with values in $(E \otimes \mathbb{Q}_\ell)^\times$, denoted $\rho_{B,\ell}$. We note in passing that for any extension $E' \supset E$ where T' is the integral closure of T in E', we have

$$T_\ell(B \otimes_T T') = T_\ell(B) \otimes_T T', \qquad V_\ell(B \otimes_T T') = V_\ell(B) \otimes_E E'.$$

The following result is due to Casselman (cf. Theorem 6 of [Shi]).

Theorem 2.5. *Let ν be a Hecke character of K of type $(1,0)$ as above, and let $\rho_{\nu,\ell}$ be the associated one-dimensional ℓ-adic representation with values in $(E_\nu \otimes \mathbb{Q}_\ell)^\times$. Then*

(1) *There exists a CM abelian variety (B_ν, E_ν) satisfying*

$$\rho_{B_\nu,\ell} \simeq \rho_{\nu,\ell}.$$

(2) *The CM abelian variety B_ν is unique up to isogeny over K. More generally, if (B,E) is any CM abelian variety with $E \supset E_\nu$ satisfying $\rho_{B,\ell} \simeq \rho_{\nu,\ell} \otimes_{E_\nu} E$ as $(E \otimes \mathbb{Q}_\ell)[G_K]$-modules, then there is an isogeny in $\mathcal{CM}_{K,E}$ from B to $B_\nu \otimes_{T_\nu} T$.*

Let ψ be a Hecke character of infinity type $(1,0)$, and let χ be a finite order Hecke character of K, so that $\psi\chi^{-1}$ also has infinity type $(1,0)$. In comparing the abelian varieties B_ψ and $B_{\psi\chi^{-1}}$, it is useful to introduce a CM abelian variety $B_{\psi,\chi}$ over K, which we now describe.

Let E_χ denote the field generated by K and the values of χ. We denote by $E_{\psi,\chi}$ the compositum of E_ψ and E_χ, and by $T_{\psi,\chi} \subset E_{\psi,\chi}$ its ring of integers. We also write H_χ for the abelian extension of K which is cut out by χ viewed as a Galois character of G_K. Consider first the abelian variety over K with endomorphisms by $T_{\psi,\chi}$:

$$B^0_{\psi,\chi} := B_\psi \otimes_{T_\psi} T_{\psi,\chi}.$$

The natural inclusion $i_\psi : T_\psi \longrightarrow T_{\psi,\chi}$ induces a morphism

$$i : B_\psi \longrightarrow B^0_{\psi,\chi} \tag{2.4}$$

with finite kernel, which is compatible with the T_ψ-actions on both sides and is given by

$$i(P) = P \otimes 1.$$

Lemma 2.6. *Let F be any number field containing $E_{\psi,\chi}$. With notations as in equation* (1.10) *of the Introduction, the restriction map i^* induces an isomorphism*

$$i^* : \Omega^1(B^0_{\psi,\chi}/F)^{T_{\psi,\chi}} \longrightarrow \Omega^1(B_\psi/F)^{T_\psi} \tag{2.5}$$

of one-dimensional F-vector spaces.

Proof. The fact that B_ψ and $B^0_{\psi,\chi}$ are CM abelian varieties over F implies that the spaces $\Omega^1(B_\psi/F)$ and $\Omega^1(B^0_{\psi,\chi}/F)$ of regular differentials over F are free of rank one over $T_\psi \otimes_{\mathcal{O}_K} F$ and $T_{\psi,\chi} \otimes_{\mathcal{O}_K} F$ respectively. In particular, the source and target in (2.5) are both one-dimensional over F. The space $\Omega^1(B^0_{\psi,\chi}/F)$ is canonically identified with $\mathrm{Hom}_{T_\psi}(T_{\psi,\chi}, \Omega^1(B_\psi/F))$, and under this identification, the pullback

$$i^* : \Omega^1(B^0_{\psi,\chi}/F) \longrightarrow \Omega^1(B_\psi/F)$$

corresponds to the natural restriction

$$\mathrm{Hom}_{T_\psi}(T_{\psi,\chi}, \Omega^1(B_\psi/F)) \longrightarrow \Omega^1(B_\psi/F)$$

sending the function φ to $\varphi(1)$. (To see this, consider the map i_* on tangent spaces and dualize.) It follows directly from this description that $\ker(i^*) \cap \Omega^1(B^0_{\psi,\chi}/F)^{T_{\psi,\chi}} = 0$, and hence that the restriction of i^* to the one-dimensional F-vector space $\Omega^1(B^0_{\psi,\chi}/F)^{T_{\psi,\chi}}$ is injective. □

We define $\omega^0_{\psi,\chi} \in \Omega^1(B^0_{\psi,\chi}/\bar{\mathbb{Q}})^{T_{\psi,\chi}}$ by

$$i^*(\omega^0_{\psi,\chi}) = \omega_\psi, \text{ where } \omega_\psi \in \Omega^1(B_\psi/E_\psi)^{T_\psi}. \tag{2.6}$$

It follows from Lemma 2.6 that $\omega^0_{\psi,\chi}$ exists and is unique (once ω_ψ has been chosen), and that $\omega^0_{\psi,\chi}$ belongs to $\Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})$.

The character $\chi^{-1} : \mathrm{Gal}(H_\chi/K) \longrightarrow T_\chi^\times$ can be viewed as a one-cocycle in

$$H^1(\mathrm{Gal}(H_\chi/K), T^\times_{\psi,\chi}) \subset H^1(\mathrm{Gal}(H_\chi/K), \mathrm{Aut}_K(B^0_{\psi,\chi})).$$

Let

$$B_{\psi,\chi} := (B^0_{\psi,\chi})^{\chi^{-1}} \tag{2.7}$$

denote the twist of $B^0_{\psi,\chi}$ by this cocycle. There is a natural identification $B^0_{\psi,\chi}(\bar{K}) = B_{\psi,\chi}(\bar{K})$ of sets, arising from an isomorphism of varieties over H_χ, where H_χ is the extension of K cut out by χ. The actions of G_K on $B^0_{\psi,\chi}(\bar{K})$ and $B_{\psi,\chi}(\bar{K})$, denoted $*_0$ and $*$ respectively, are related by

$$\sigma * P = (\sigma *_0 P) \otimes \chi^{-1}(\sigma), \qquad \text{for all } \sigma \in G_K. \tag{2.8}$$

In particular, for any $L \supset K$, we have:

$$B_{\psi,\chi}(L) \quad = \quad \left\{ P \in B_\psi(\bar{\mathbb{Q}}) \otimes_{T_\psi} T_{\psi,\chi} \text{ such that } \sigma P = P \otimes \chi(\sigma), \quad \forall \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L) \right\}. \tag{2.9}$$

Likewise, the natural actions of G_K on $\Omega^1(B^0_{\psi,\chi}/\bar{K})$ and on $\Omega^1(B_{\psi,\chi}/\bar{K})$ are related by

$$\sigma * \omega = [\chi^{-1}(\sigma)]^*(\sigma *_0 \omega) \qquad \text{for all } \sigma \in G_K. \tag{2.10}$$

The isomorphism of $B^0_{\psi,\chi}$ and $B_{\psi,\chi}$ as CM abelian varieties over H_χ gives natural identifications

$$\Omega^1(B^0_{\psi,\chi}/H_\chi) = \Omega^1(B_{\psi,\chi}/H_\chi), \qquad \Omega^1(B^0_{\psi,\chi}/E'_{\psi,\chi})^{T_{\psi,\chi}} = \Omega^1(B_{\psi,\chi}/E'_{\psi,\chi})^{T_{\psi,\chi}},$$

where $E'_{\psi,\chi}$ denotes the subfield of $\bar{\mathbb{Q}}$ generated by H_χ and $E_{\psi,\chi}$. Let $\omega^0_{\psi,\chi}$ and $\omega_{\psi,\chi}$ be $E_{\psi,\chi}$ vector space generators of $\Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$ and $\Omega^1(B_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$ respectively, the former being chosen to satisfy (2.6) above. Since they both generate $\Omega^1(B_{\psi,\chi}/E'_{\psi,\chi})^{T_{\psi,\chi}}$ as an $E'_{\psi,\chi}$-vector space, they necessarily differ by a non-zero scalar in $E'_{\psi,\chi}$.

To spell out the relation between $\omega^0_{\psi,\chi}$ and $\omega_{\psi,\chi}$ more precisely, it will be useful to introduce the notion of a *generalised Gauss sum* attached to any finite order character χ of G_K. Given such a character, let

$$E\{\chi\} := \{\lambda \in E_\chi H_\chi \text{ such that } \lambda^\sigma = \chi(\sigma)\lambda, \quad \forall \sigma \in \mathrm{Gal}(E_\chi H_\chi/E_\chi)\}.$$

This set is a one-dimensional E_χ-vector space in a natural way. It is not closed under multiplication, but

$$E\{\chi_1\} \cdot E\{\chi_2\} = E\{\chi_1\chi_2\} \pmod{(E_{\chi_1}E_{\chi_2})^\times}. \tag{2.11}$$

Definition 2.7. An E_χ-vector space generator of $E\{\chi\}$ is called a *Gauss sum* attached to the character χ, and is denoted $\mathfrak{g}(\chi)$.

By definition, the Gauss sum $\mathfrak{g}(\chi)$ belongs to $E\{\chi\} \cap (E_\chi H_\chi)^\times$, but is only well-defined up to multiplication by $E_\chi^\times$. It follows from (2.11) that

$$\mathfrak{g}(\chi_1\chi_2) = \mathfrak{g}(\chi_1)\mathfrak{g}(\chi_2) \pmod{(E_{\chi_1}E_{\chi_2})^\times}, \qquad \mathfrak{g}(\chi^{-1}) = \mathfrak{g}(\chi)^{-1} \pmod{E_\chi^\times}. \tag{2.12}$$

The following lemma pins down the relationship between the differentials $\omega^0_{\psi,\chi}$ and $\omega_{\psi,\chi}$.

Lemma 2.8. *For all Hecke characters ψ and χ as above,*

$$\omega_{\psi,\chi} = \mathfrak{g}(\chi)\omega^0_{\psi,\chi} \pmod{E^\times_{\psi,\chi}}.$$

Proof. Let $\lambda \in (H_\chi E_{\psi,\chi})^\times$ be the scalar satisfying

$$\omega_{\psi,\chi} = \lambda\omega^0_{\psi,\chi}. \tag{2.13}$$

Since $\omega_{\psi,\chi}$ is an $E_{\psi,\chi}$-rational differential on $B_{\psi,\chi}$, for all $\sigma \in \mathrm{Gal}(\bar{K}/E_{\psi,\chi})$ we have

$$\omega_{\psi,\chi} = \sigma * \omega_{\psi,\chi} = [\chi^{-1}(\sigma)]^*\sigma *_0 \omega_{\psi,\chi} = \chi^{-1}(\sigma)\lambda^\sigma\omega^0_{\psi,\chi}, \tag{2.14}$$

where the second equality follows from equation (2.10) and the last from the fact that the differential $\omega^0_{\psi,\chi}$ belongs to $\Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$. Comparing (2.13) and (2.14) gives $\lambda^\sigma = \chi(\sigma)\lambda$, and hence $\lambda = \mathfrak{g}(\chi)$ $\pmod{E^\times_{\psi,\chi}}$. □

The following lemma relates the abelian varieties $B_{\psi,\chi}$ and B_ν, where $\nu = \psi\chi^{-1}$.

Lemma 2.9. *There is an isogeny defined over K:*

$$i_\nu : B_{\psi,\chi} \longrightarrow B_\nu \otimes_{T_\nu} T_{\psi,\chi}$$

which is compatible with the action of $T_{\psi,\chi}$ by endomorphisms on both sides.

Proof. The pair $(B^0_{\psi,\chi}, E_{\psi,\chi})$ is a CM abelian variety having ψ (viewed as taking values in $E_{\psi,\chi}$) as its associated Hecke character. The Hecke character attached to the Galois twist $B_{\psi,\chi}$ is therefore $\psi\chi^{-1} = \nu$. The second part of Theorem 2.5 implies that $B_{\psi,\chi}$ and $B_\nu \otimes_{T_\nu} T_{\psi,\chi}$ are isogenous over K as CM abelian varieties. □

2.3. **Complex periods and special values of L-functions.** This section recalls certain periods attached to the quadratic imaginary field K and to Hecke characters of this field. We begin by fixing:

(1) An elliptic curve A with complex multiplication by $\mathcal{O}_K$, defined over a finite extension F of K. (Note that F necessarily contains the Hilbert class field of K.)
(2) A regular differential $\omega_A \in \Omega^1(A/F)$.
(3) A non-zero element γ of $H_1(A(\mathbb{C}), \mathbb{Q})$.

The complex period attached to this data is defined by

$$\Omega(A) := \frac{1}{2\pi i}\int_\gamma \omega_A \qquad \pmod{F^\times}. \tag{2.15}$$

Note that $\Omega(A)$ depends on the pair (ω, γ). A different choice of ω or γ has the effect of multiplying $\Omega(A)$ by a scalar in $F^\times$, and therefore $\Omega(A)$ can be viewed as a well-defined element of $\mathbb{C}^\times/F^\times$.

For any Hecke character ψ of K, recall that ψ^* is the Hecke character defined as in the Introduction by $\psi^*(x) = \psi(\bar{x})$. Suppose that ψ is of infinity type $(1,0)$, and as before let $E_\psi \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ denote the field generated by the values of ψ (or, equivalently, ψ^*). Choose (arbitrary) non-zero elements

$$\omega_\psi \in \Omega^1(B_\psi/E_\psi)^{T_\psi}, \qquad \gamma \in H_1(B_\psi(\mathbb{C}), \mathbb{Q}),$$

where B_ψ is the CM abelian variety attached to ψ by Theorem 2.5, and $\Omega^1(B_\psi/E_\psi)^{T_\psi}$ is defined in equation (1.10) of the Introduction. The period $\Omega(\psi^*)$ attached to ψ^* is defined by setting

$$\Omega(\psi^*) = \frac{1}{2\pi i}\int_\gamma \omega_\psi \qquad \pmod{E_\psi^\times}.$$

Note that the complex number $\Omega(\psi^*)$ does not depend, up to multiplication by $E_\psi^\times$, on the choices of ω_ψ and γ that were made to define it.

Lemma 2.10. *If ψ is a Hecke character of infinity type $(1,0)$, and χ is a finite order character, then*

$$\Omega(\psi^*\chi) = \Omega(\psi^*)\mathfrak{g}(\chi^*)^{-1} \pmod{E_{\psi,\chi}^\times}. \tag{2.16}$$

Proof. Choose a non-zero generator γ of $H_1(B^0_{\psi,\chi}(\mathbb{C}), \mathbb{Q}) = H_1(B_{\psi,\chi}(\mathbb{C}), \mathbb{Q})$ (viewed as a one-dimensional $E_{\psi,\chi}$ vector space via the endomorphism action). By definition,

$$\Omega((\psi\chi^{-1})^*) = \int_\gamma \omega_{\psi,\chi} = \mathfrak{g}(\chi)\int_\gamma \omega^0_{\psi,\chi} = \mathfrak{g}(\chi)\Omega(\psi^*) \pmod{E_{\psi,\chi}^\times},$$

where the second equality follows from Lemma 2.8. The result now follows after substituting χ^{*-1} for χ. □

As in [Scha] §1.8, one can also attach a period $\Omega(\psi)$ to an arbitrary Hecke character ψ of K; these satisfy the following relations:

Proposition 2.11. *Let ψ be a Hecke character of infinity type (k,j). Then*

(1) *The ratio*

$$\frac{\Omega(\psi^*)}{(2\pi i)^j\Omega(A)^{k-j}}$$

is algebraic.

(2) *For all ψ and ψ',*

$$\Omega(\psi\psi') = \Omega(\psi)\Omega(\psi') \pmod{E_{\psi,\psi'}^\times},$$

where $E_{\psi,\psi'}$ is the subfield of $\bar{\mathbb{Q}}$ generated by E_ψ and $E_{\psi'}$.

The following theorem is due to Goldstein and Schappacher [GS] in certain cases and Blasius [Bl] in the general case (even CM fields).

Theorem 2.12. *Suppose that ψ has infinity type (k,j) with $k > j$, and that m is a critical integer for $L(\psi^{-1}, s)$. Then*

$$\frac{L(\psi^{-1}, m)}{(2\pi i)^m\Omega(\psi^*)} \text{ belongs to } E_\psi,$$

and for all $\tau \in \mathrm{Gal}(E_\psi/K)$,

$$\left(\frac{L(\psi^{-1}, m)}{(2\pi i)^m\Omega(\psi^*)}\right)^\tau = \frac{L((\psi^{-1})^\tau, m)}{(2\pi i)^m\Omega((\psi^*)^\tau)}.$$

2.4. **p-adic periods.** Fix a prime p that splits in K. We will need p-adic analogs of the periods $\Omega(A)$ and $\Omega(\nu^*)$. The p-adic analogue $\Omega_p(A)$ of $\Omega(A)$ is obtained by considering the base change $A_{\mathbb{C}_p}$ of A to $\mathbb{C}_p$ (via our fixed embedding of F into $\mathbb{C}_p$). Assume that A has good reduction at the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$, i.e., that $A_{\mathbb{C}_p}$ extends to a smooth proper model $A_{\mathcal{O}_{\mathbb{C}_p}}$ over $\mathcal{O}_{\mathbb{C}_p}$. The p-adic completion $\hat{A}_{\mathcal{O}_{\mathbb{C}_p}}$ of A along its special fiber is isomorphic to $\hat{\mathbf{G}}_m$. Following [deS] II, §4.4, choose an isomorphism $\iota_p : \hat{A} \longrightarrow \hat{\mathbf{G}}_m$ over $\mathcal{O}_{\mathbb{C}_p}$, and define $\Omega_p(A) \in \mathbb{C}_p^\times$ by the rule

$$\omega_A = \Omega_p(A) \cdot \iota_p^*(du/u), \tag{2.17}$$

where u is the standard coordinate on $\hat{\mathbf{G}}_m$. The invariant $\Omega_p(A) \in \mathbb{C}_p^\times$ thus defined depends on the choices of ω_A and ι_p, but only up to multiplication by a scalar in $F^\times$. Observe also that $\Omega(A)$ and $\Omega_p(A)$ each depend linearly in the same way on the choice of the global differential ω_A.

The p-adic period $\Omega_p(A)$ can be used to define p-adic analogs of the complex periods that appear in the statement of Theorem 2.12.

Definition 2.13. Let ν be a Hecke character of K of type $(1,0)$. The p-adic period $\Omega_p(\nu^*)$ is defined by

$$\Omega_p(\nu^*) := \Omega_p(A) \cdot \frac{\Omega(\nu^*)}{\Omega(A)}.$$

More generally, for any character ν of infinity type (k,j), we define

$$\Omega_p(\nu^*) := \Omega_p(A)^{k-j} \cdot \frac{\Omega(\nu^*)}{(2\pi i)^j\Omega(A)^{k-j}}.$$

It can be seen from this definition that the period $\Omega_p(\nu^*)$, like its complex counterpart $\Omega(\nu^*)$, is well-defined up to multiplication by a scalar in $E_\nu^\times$. The following p-adic analog of Lemma 2.10 is a direct consequence of this lemma combined with the definition of $\Omega_p(\psi)$:

Lemma 2.14. *If ψ is a Hecke character of infinity type $(1,0)$, and χ is a finite order character, then*

$$\Omega_p(\psi^*\chi) = \Omega_p(\psi^*)\mathfrak{g}(\chi^*)^{-1} \pmod{E_{\psi,\chi}^\times}. \tag{2.18}$$

Likewise, Proposition 2.11 implies:

Proposition 2.15. *Let ψ be a Hecke character of infinity type (k,j). Then*

1. *The ratio*
$$\frac{\Omega_p(\psi^*)}{(2\pi i)^j \Omega_p(A)^{k-j}}$$
is algebraic.
2. *For all ψ and ψ',*

$$\Omega_p(\psi\psi') = \Omega_p(\psi)\Omega_p(\psi') \pmod{E_{\psi,\psi'}^\times}. \tag{2.19}$$

3. p-adic L-functions and Rubin's formula

3.1. **The Katz p-adic L-function.** Throughout this chapter, we will fix a prime p that is split in K. Let $\mathfrak{c}$ be an integral ideal of K which is prime to p, and let $\Sigma(\mathfrak{c})$ denote the set of all Hecke characters of K of conductor dividing $\mathfrak{c}$. Denote by $\mathfrak{p}$ the prime above p corresponding to the chosen embedding $K \hookrightarrow \bar{\mathbb{Q}}_p$.

A character $\nu \in \Sigma(\mathfrak{c})$ is called a *critical character* if $L(\nu^{-1}, 0)$ is a critical value in the sense of Deligne, i.e., if the Γ-factors that arise in the functional equation for $L(\nu^{-1}, s)$ are non-vanishing and have no poles at $s = 0$. The set $\Sigma_{\rm crit}(\mathfrak{c})$ of critical characters can be expressed as the disjoint union

$$\Sigma_{\rm crit}(\mathfrak{c}) = \Sigma^{(1)}_{\rm crit}(\mathfrak{c}) \cup \Sigma^{(2)}_{\rm crit}(\mathfrak{c}),$$

where

$$\Sigma^{(1)}_{\rm crit}(\mathfrak{c}) = \{\nu \in \Sigma(\mathfrak{c}) \quad \text{of type } (\ell_1, \ell_2) \text{ with } \ell_1 \le 0, \quad \ell_2 \ge 1\},$$
$$\Sigma^{(2)}_{\rm crit}(\mathfrak{c}) = \{\nu \in \Sigma(\mathfrak{c}) \quad \text{of type } (\ell_1, \ell_2) \text{ with } \ell_1 \ge 1, \quad \ell_2 \le 0\}.$$

The possible infinity types of Hecke characters in these two critical regions are sketched in Figure 1. Note in particular that when $\mathfrak{c} = \bar{\mathfrak{c}}$, the regions $\Sigma^{(1)}_{\rm crit}(\mathfrak{c})$ and $\Sigma^{(2)}_{\rm crit}(\mathfrak{c})$ are interchanged by the involution $\nu \mapsto \nu^*$. The set $\Sigma_{\rm crit}(\mathfrak{c})$ is endowed with a natural p-adic topology as described in Section 5.2 of [BDP-gz]. The subsets $\Sigma^{(1)}_{\rm crit}(\mathfrak{c})$ and $\Sigma^{(2)}_{\rm crit}(\mathfrak{c})$ are dense in the completion $\hat{\Sigma}_{\rm crit}(\mathfrak{c})$ relative to this topology.

Recall that $\mathfrak{p}$ is the prime above p induced by our chosen embedding of K into $\mathbb{C}_p$. The following theorem on the existence of the p-adic L-function is due to Katz. The statement below is a restatement of [deS] (II, Thm. 4.14) with a minor correction, and restricted to characters unramified at p. We remark that since our characters are unramified at p, the Gauss sum in the interpolation formula in loc. cit. is equal to 1.

Theorem 3.1. *There exists a p-adic analytic function $\nu \mapsto \mathscr{L}_{p,\mathfrak{c}}(\nu)$ (valued in $\mathbb{C}_p$) on $\hat{\Sigma}_{\rm crit}(\mathfrak{c})$ which is determined by the interpolation property:*

$$\frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(A)^{\ell_1-\ell_2}} = \left(\frac{\sqrt{D}}{2\pi}\right)^{\ell_2} (\ell_1 - 1)!(1 - \nu(\mathfrak{p})/p)(1 - \nu^{-1}(\bar{\mathfrak{p}}))\frac{L_\mathfrak{c}(\nu^{-1}, 0)}{\Omega(A)^{\ell_1-\ell_2}}, \tag{3.1}$$

for all critical characters $\nu \in \Sigma^{(2)}_{\rm crit}(\mathfrak{c})$ of infinity type (ℓ_1, ℓ_2).

The right hand side of (3.1) belongs to $\bar{\mathbb{Q}}$, by Part 1 of Proposition 2.11 and Theorem 2.12 with $m = 0$. Equation (3.1) should be interpreted to mean that the left hand side also belongs to $\bar{\mathbb{Q}}$, viewed as a subfield of $\mathbb{C}_p$ under the chosen embeddings, and agrees with the right hand side. Note that although both sides of (3.1) depend on the choice of the differential ω_A that was made in the definition of the periods $\Omega(A)$ and $\Omega_p(A)$, the quantity $\mathscr{L}_{p,\mathfrak{c}}(\nu)$, just like its complex counterpart $L_\mathfrak{c}(\nu^{-1}, 0)$, does not depend on this choice.

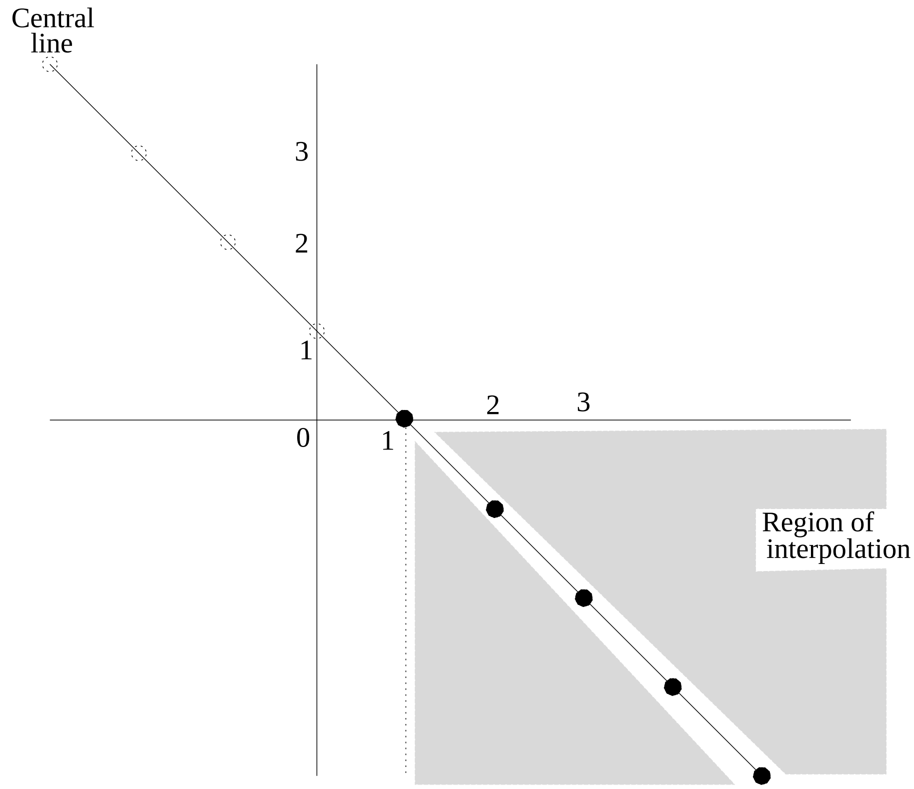

FIGURE 1. Critical infinity types for the Katz p-adic L-function

Remark 3.2. Once a choice of $\mathfrak{c}$ is fixed, we shall often drop the subscript $\mathfrak{c}$ and simply write $\mathscr{L}_p$ for the p-adic L-function.

The following corollary is the p-adic counterpart of Thm. 2.12.

Corollary 3.3. *Suppose that* $\nu \in \Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c})$. *Then*

$$\frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(\nu^*)} \text{ belongs to } E_\nu.$$

Proof. Suppose that ν has infinity type (ℓ_1, ℓ_2). By the definition of $\Omega_p(\nu^*)$ and the the interpolation property of the Katz p-adic L-function in Thm. 3.1, we have

$$\begin{aligned}\frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(\nu^*)} &= \frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(A)^{\ell_1-\ell_2}} \times \frac{(2\pi i)^{\ell_2}\Omega(A)^{\ell_1-\ell_2}}{\Omega(\nu^*)} \\ &= \sqrt{-D}^{\ell_2} \cdot (\ell_1-1)!(1-\nu(\mathfrak{p}))(1-\nu^{-1}(\bar{\mathfrak{p}}))\frac{L_{\mathfrak{c}}(\nu^{-1},0)}{\Omega(\nu^*)}.\end{aligned}$$

The result is now a direct consequence of Theorem 2.12 with $m = 0$. □

Cor. 3.3 expresses $\mathscr{L}_{p,\mathfrak{c}}(\nu)$ as an E_ν-multiple of a p-adic period $\Omega_p(\nu^*)$, when ν lies in the range $\Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c})$ of classical interpolation for the Katz p-adic L-function. On the other hand, the characters in $\Sigma_{\mathrm{crit}}^{(1)}(\mathfrak{c})$ are outside the range of interpolation, and so Cor. 3.3 does not directly say anything about these values, and indeed the main goal of this paper is to obtain analogous results for certain characters in $\Sigma_{\mathrm{crit}}^{(1)}(\mathfrak{c})$. It turns out that the methods of this paper only allow us to study $\mathscr{L}_{p,\mathfrak{c}}(\nu)$ for characters ν in $\Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c})$ satisfying the following auxiliary (but not unnatural) condition:

(3.2) ν is a *self-dual* Hecke character with $\varepsilon_\nu = \varepsilon_K$.

For the benefit of the reader, we now recall this key definition.

Definition 3.4. A Hecke character $\nu \in \Sigma_{\rm crit}(\mathfrak{c})$ is said to be *self-dual* or *anticyclotomic* if

$$\nu\nu^* = \mathbf{N}_K.$$

The reason for the terminology in Definition 3.4 is that the functional equation for the L-series $L(\nu^{-1}, s)$ relates $L(\nu^{-1}, s)$ to $L(\nu^{-1}, -s)$, and therefore $s = 0$ is the *central critical point* for this complex L-series. Note that a self-dual character is necessarily of infinity type $(1+j, -j)$ for some $j \in \mathbb{Z}$. Also the conductor of a self-dual character is clearly invariant under complex conjugation. If $\mathfrak{c}$ is an integral ideal such that $\mathfrak{c} = \bar{\mathfrak{c}}$, we denote by $\Sigma_{\rm sd}(\mathfrak{c})$ the set of self-dual Hecke characters of conductor *exactly* $\mathfrak{c}$, and write

$$\Sigma^{(1)}_{\rm sd}(\mathfrak{c}) = \Sigma^{(1)}_{\rm crit}(\mathfrak{c}) \cap \Sigma_{\rm sd}(\mathfrak{c}), \qquad \Sigma^{(2)}_{\rm sd}(\mathfrak{c}) = \Sigma^{(2)}_{\rm crit}(\mathfrak{c}) \cap \Sigma_{\rm sd}(\mathfrak{c}).$$

In particular, the possible infinity types of characters in $\Sigma^{(2)}_{\rm sd}(\mathfrak{c})$ correspond to the black dots in Figure 1.

For convenience, we restate Thm. 3.1 for self-dual characters.

Proposition 3.5. *For all characters $\nu \in \Sigma^{(2)}_{\rm sd}(\mathfrak{c})$ of infinity type $(1+j, -j)$ with $j \geq 0$,*

$$\frac{\mathscr{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p(A)^{1+2j}} = (1 - \nu^{-1}(\bar{\mathfrak{p}}))^2 \times \frac{j!(2\pi)^j L_{\mathfrak{c}}(\nu^{-1}, 0)}{\sqrt{D}^j\,\Omega(A)^{1+2j}}. \tag{3.3}$$

Remark 3.6. In the proposition above, we could equally write $L(\nu^{-1}, 0)$ instead of $L_{\mathfrak{c}}(\nu^{-1}, 0)$ since ν has conductor exactly equal to c.

Remark 3.7. The *central character* of such a ν is very restricted. Indeed, for any Hecke character ν it is clear that $\varepsilon_{\bar{\nu}} = \overline{\varepsilon_\nu}$, while $\varepsilon_{\nu^*} = \varepsilon_\nu$. If further ν is a self-dual character, it follows that for any $x \in \mathbb{A}_K^\times$,

$$\nu(\mathrm{N}^K_{\mathbb{Q}}(x)) = \nu(x\bar{x}) = (\nu\nu^*)(x) = \mathbf{N}_K(x) = \mathbf{N}(\mathrm{N}^K_{\mathbb{Q}}(x)).$$

Hence

$$\nu|_{\mathrm{N}^K_{\mathbb{Q}}\mathbb{A}_K^\times} = \mathbf{N} \qquad \text{and} \qquad \varepsilon_\nu|_{\mathrm{N}^K_{\mathbb{Q}}\mathbb{A}_K^\times} = 1.$$

This implies that the central character ε_ν of a self-dual character ν is either 1 or ε_K, where ε_K denotes the quadratic Dirichlet character corresponding to the extension $K/\mathbb{Q}$. Conversely, it is easy to see that if ν is a Hecke character with $w(\nu) = 1$ and $\varepsilon_\nu = 1$ or ε_K, then ν is a self-dual character.

We define:

$$\Sigma_{\rm sd}(\mathfrak{c})^+ := \{\nu \in \Sigma_{\rm sd}(\mathfrak{c});\ \ \varepsilon_\nu = 1\}, \quad \Sigma_{\rm sd}(\mathfrak{c})^- := \{\nu \in \Sigma_{\rm sd}(\mathfrak{c});\ \ \varepsilon_\nu = \varepsilon_K\}. \tag{3.4}$$

The sets $\Sigma^{(1)}_{\rm sd}(\mathfrak{c})^\pm$ and $\Sigma^{(2)}_{\rm sd}(\mathfrak{c})^\pm$ are defined similarly.

Our approach to studying $\mathscr{L}_{p,\mathfrak{c}}(\nu)$ for characters ν satisfying (3.2), i.e., those ν lying in $\Sigma^{(1)}_{\rm sd}(\mathfrak{c})^-$ for some $\mathfrak{c}$, relies on a different kind of p-adic L-function. This latter p-adic L-function is attached to Rankin-Selberg L-series and is recalled in the following section.

3.2. **p-adic Rankin L-series.** In this section, we consider p-adic L-functions obtained by interpolating special values of Rankin-Selberg L-series associated to modular forms and Hecke characters of a quadratic imaginary field K of odd discriminant. We briefly recall the definition of this p-adic L-function that is given in Sec. 5 of [BDP-gz], referring the reader to loc.cit. for a more detailed description.

Let $S_k(\Gamma_0(N), \varepsilon)$ denote the space of cusp forms of weight $k \geq 2$ and character ε on $\Gamma_0(N)$. Let $f \in S_k(\Gamma_0(N), \varepsilon)$ be a normalized newform and let E_f denote the subfield of $\mathbb{C}$ generated by its Fourier coefficients.

Definition 3.8. The pair (f, K) is said to satisfy the *Heegner hypothesis* if $\mathcal{O}_K$ contains a cyclic ideal of norm N, i.e., an integral ideal $\mathfrak{N}$ of $\mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} = \mathbb{Z}/N\mathbb{Z}$.

Assume from now on that (f, K) satisfies the Heegner hypothesis, and let $\mathfrak{N}$ be a cyclic $\mathcal{O}_K$-ideal of norm N. We write $\mathfrak{N}_\varepsilon$ for the unique ideal dividing $\mathfrak{N}$ of norm N_ε.

Definition 3.9. A Hecke character χ of K of infinity type (ℓ_1, ℓ_2) is said to be *central critical* for f if

$$\ell_1 + \ell_2 = k \qquad \text{and} \qquad \varepsilon_\chi = \varepsilon.$$

The reason for the terminology of Definition 3.9 is that when χ satisfies these hypotheses, the complex Rankin L-series $L(f, \chi^{-1}, s)$ is self-dual and $s = 0$ is its central (critical) point.

Definition 3.10. Let c be a rational integer prime to pN. Then $\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ is defined to be the set of Hecke characters χ of K such that

(1) χ is central critical for f.
(2) $\mathfrak{f}_\chi = c \cdot \mathfrak{N}_\varepsilon$.
(3) The local sign $\varepsilon_q(f, \chi^{-1}) = +1$ for all finite primes q.

It is easily checked that this agrees with the definition of $\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ given in [BDP-gz] §4.1 where this is just denoted $\Sigma_{\rm cc}(\mathfrak{N})$. Further, as in loc. cit., given conditions (1) and (2) above, condition (3) is automatic except possibly for primes q lying in the set S_f defined by:

$$S_f := \{q : \quad q \mid (N, D), q \nmid N_\varepsilon\}.$$

The set $\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ can be expressed as a disjoint union

$$\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon) = \Sigma^{(1)}_{\rm cc}(c, \mathfrak{N}, \varepsilon) \cup \Sigma^{(2)}_{\rm cc}(c, \mathfrak{N}, \varepsilon),$$

where $\Sigma^{(1)}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ and $\Sigma^{(2)}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ denote the subsets consisting of characters of infinity type $(k + j, -j)$ with $1 - k \leq j \leq -1$ and $j \geq 0$ respectively. We shall also denote by $\hat{\Sigma}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ the completion of $\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ relative to the p-adic compact open topology on $\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ which is defined in Section 5.2 of [BDP-gz]. The infinity types of Hecke characters in $\Sigma^{(1)}_{cc}(c, \mathfrak{N}, \varepsilon)$ and $\Sigma^{(2)}_{cc}(c, \mathfrak{N}, \varepsilon)$ correspond respectively to the white and black dots in the shaded regions in Figure 2. We note that the set $\Sigma^{(2)}_{cc}(c, \mathfrak{N}, \varepsilon)$ of classical central critical characters "of type 2" is dense in $\hat{\Sigma}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$.

For all $\chi \in \Sigma^{(2)}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ of infinity type $(k+j, -j)$ with $j \geq 0$, let $E_{f,\chi}$ denote the subfield of $\mathbb{C}$ generated by E_f and the values of χ, and let $E_{f,\chi,\varepsilon}$ be the field generated by $E_{f,\chi}$ and by the abelian extension of $\mathbb{Q}$ cut out by ε. The *algebraic part* of $L(f, \chi^{-1}, 0)$ is defined by the rule

$$L_{\rm alg}(f, \chi^{-1}, 0) := w(f, \chi)^{-1} C(f, \chi, c) \cdot \frac{L(f, \chi^{-1}, 0)}{\Omega(A)^{2(k+2j)}}, \tag{3.5}$$

where $w(f, \chi)^{-1} \in E_{f,\chi,\varepsilon}$ and $C(f, \chi, c)$ are respectively the scalar (of complex norm 1) and the explicit real constant defined in equation (5.1.11) and Theorem 4.6 of [BDP-gz]; we have

$$C(f, \chi, c) = \frac{2^{k+2j-2} \pi^{k+2j-1} j!(k + j - 1)! w_K}{\sqrt{D}^{k+2j-1} c^{k+2j-1}} \prod_{q|c} \frac{q - \varepsilon_K(q)}{q - 1}, \tag{3.6}$$

where $w_K = \#\mathcal{O}_K^\times$ is the number of roots of unity in K. Thm. 5.5 and Thm. 5.10 of loc. cit. show respectively that the values $L_{\rm alg}(f, \chi^{-1}, 0)$ belongs to $\overline{\mathbb{Q}}$, and that they interpolate p-adically:

Proposition 3.11. *Let $\chi \mapsto L_p(f, \chi)$ be the function on $\Sigma^{(2)}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ defined by*

$$L_p(f, \chi) := \Omega_p(A)^{2(k+2j)} (1 - \chi^{-1}(\bar{\mathfrak{p}}) a_p(f) + \chi^{-2}(\bar{\mathfrak{p}}) \varepsilon(p) p^{k-1})^2 L_{\rm alg}(f, \chi^{-1}, 0), \tag{3.7}$$

for χ of infinity type $(k + j, -j)$ with $j \geq 0$. This function extends (uniquely) to a p-adically continuous function on $\hat{\Sigma}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$.

The function $\chi \mapsto L_p(f, \chi)$ on $\hat{\Sigma}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ will be referred to as the *p-adic Rankin L-function* attached to the cusp form f.

3.3. **A p-adic Gross-Zagier formula.** In this section, we specialise to the case where the newform f is of weight $k = 2$, and assume that χ is a finite order Hecke character of K satisfying

$$\chi \mathbf{N}_K \quad \text{belongs to} \quad \Sigma^{(1)}_{\rm cc}(c, \mathfrak{N}, \varepsilon).$$

In particular, the character $\chi \mathbf{N}_K$ lies outside the domain $\Sigma^{(2)}_{\rm cc}(c, \mathfrak{N}, \varepsilon)$ of classical interpolation defining $L_p(f, -)$. The p-adic Gross-Zagier formula alluded to in the title of this section relates the special value $L_p(f, \chi \mathbf{N}_K)$ to the formal group logarithm of a Heegner point on the modular abelian variety attached to f.

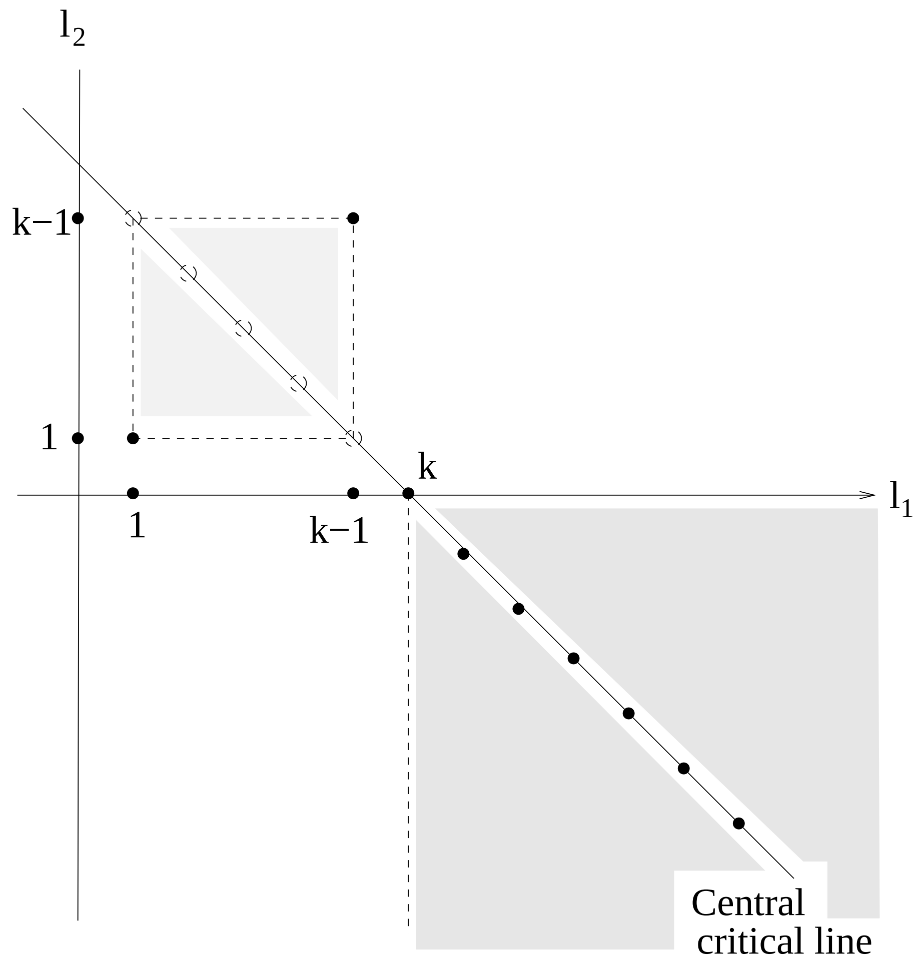

FIGURE 2. Critical infinity types for the p-adic Rankin L-function

The Eichler-Shimura construction associates to f an abelian variety B_f with endomorphism by an order in the ring of integers $T_f \subset E_f$, and a surjective morphism

$$\Phi_f : J_1(N) \longrightarrow B_f$$

of abelian varieties over $\mathbb{Q}$, called the *modular parametrisation*, which is well-defined up to a rational isogeny. Let

$$\omega_f = 2\pi i f(\tau) d\tau \in \Omega^1(X_1(N)/E_f)$$

be the differential form on $X_1(N)$ attached to f; we use the same symbol ω_f to denote the associated one-form on $J_1(N)$. Let $\omega_{B_f} \in \Omega^1(B_f/E_f)^{T_f}$ be the unique one-form satisfying

$$\Phi_f^*(\omega_{B_f}) = \omega_f. \tag{3.8}$$

Let A' be an elliptic curve with endomorphism ring isomorphic to the order $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$ of conductor c, defined over the ring class field H_c of conductor c. The pair $(A', A'[\mathfrak{N}])$ corresponds to a point on $X_0(N)(H_c)$. Let t be any generator of $A'[\mathfrak{N}]$. Then the triple $(A', A'[\mathfrak{N}], t)$ corresponds to a point in $X_1(N)$, whose field of definition $H_{c,\mathfrak{N}}$ is an abelian extension of K, independent of the choice of t, and the finite order Hecke character χ can be viewed as a character

$$\chi : \mathrm{Gal}(H_{c,\mathfrak{N}}/K) \longrightarrow E_\chi.$$

Fix a cusp ∞ of $X_1(N)$ which is defined over $\mathbb{Q}$, and let

$$\Delta = [A', A'[\mathfrak{N}], t] - (\infty) \quad \in J_1(N)(H_{c,\mathfrak{N}}). \tag{3.9}$$

To the pair (f, χ) we associate a Heegner point by letting $G = \mathrm{Gal}(H_{c,\mathfrak{N}}/K)$ and setting

$$P_f(\chi) := \sum_{\sigma \in G} \chi^{-1}(\sigma)\Phi_f(\Delta^\sigma) \quad \in B_f(H_{c,\mathfrak{N}}) \otimes_{T_f} E_{f,\chi}. \tag{3.10}$$

Note that, since $P_f(\chi)^\sigma = P_f(\chi)$ for any $\sigma \in \mathrm{Gal}(H_{c,\mathfrak{N}}/H_\chi)$, the point $P_f(\chi)$ lies in the subspace $B_f(H_\chi) \otimes_{T_f} E_{f,\chi}$. The embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$ that was fixed from the outset allows us to consider the formal group logarithm

$$\log_{\omega_{B_f}} : B_f(H_{c,\mathfrak{N}}) \longrightarrow \mathbb{C}_p.$$

We extend this function to $B_f(H_{c,\mathfrak{N}}) \otimes_{T_f} E_{f,\chi}$ by $E_{f,\chi}$-linearity.

Theorem 3.12. *With notations and assumptions as above,*

$$L_p(f, \chi \mathbf{N}_K) = (1 - \chi^{-1}(\bar{\mathfrak{p}})p^{-1}a_p(f) + \chi^{-2}(\bar{\mathfrak{p}})\varepsilon(p)p^{-1})^2 \log^2_{\omega_{B_f}}(P_f(\chi)).$$

Proof. Let

$$\mathscr{E}(f, \chi) := (1 - \chi^{-1}(\bar{\mathfrak{p}})p^{-1}a_p(f) + \chi^{-2}(\bar{\mathfrak{p}})\varepsilon(p)p^{-1})^2 \quad \in \quad E_{f,\chi}^\times$$

be the Euler factor appearing in the statement of Theorem 3.12. Let F' denote the p-adic completion of $H_{c,\mathfrak{N}}$. Theorem 5.13 of [BDP-gz] in the case $k = 2$ and $r = j = 0$, with χ replaced by $\chi \mathrm{N}_K$, gives

$$L_p(f, \chi \mathbf{N}_K) = \mathscr{E}(f, \chi) \times \left(\sum_{\sigma \in G} \chi^{-1}(\sigma) \cdot \mathrm{AJ}_{F'}(\Delta^\sigma)(\omega_f) \right)^2. \tag{3.11}$$

Note that in this context, the p-adic Abel-Jacobi map $\mathrm{AJ}_{F'}$ that appears in (3.11) is related to the formal group logarithm by

$$\mathrm{AJ}_{F'}(\Delta)(\omega_f) = \log_{\omega_f}(\Delta).$$

Therefore,

$$L_p(f, \chi \mathbf{N}_K) = \mathscr{E}(f, \chi) \left(\sum_{\sigma \in G} \chi^{-1}(\sigma) \log_{\omega_f}(\Delta^\sigma) \right)^2. \tag{3.12}$$

Theorem 3.12 follows from this formula and the fact that, by (3.8),

$$\log_{\omega_f}(\Delta) = \log_{\Phi_f^*(\omega_{B_f})}(\Delta) = \log_{\omega_{B_f}}(\Phi_f(\Delta)).$$

□

In the special case where f has rational Fourier coefficients and $\chi = 1$ is the trivial character, the abelian variety B_f is an elliptic curve quotient of $J_0(N)$ and the Heegner point $P_f := P_f(1)$ belongs to $B_f(K)$. Theorem 3.12 implies in this case that

$$L_p(f, \mathbf{N}_K) = \left(\frac{p + 1 - a_p(f)}{p} \right)^2 \log^2(P_f), \tag{3.13}$$

where $\log : B_f(K_\mathfrak{p}) \longrightarrow K_\mathfrak{p}$ is the formal group logarithm attached to a rational differential on $B_f/\mathbb{Q}$. Equation (3.13) exhibits a strong analogy with Theorem 1 of the Introduction, although it applies to arbitrary (modular) elliptic curves and not just elliptic curves with complex multiplication.

The remainder of Chapter 3 explains how Theorem 3.12 can in fact be used to prove Theorems 1 and 2 of the Introduction. The key to this proof is a relation between the Katz p-adic L-function of Section 3.1 and the p-adic Rankin L-function $L_p(f, \chi)$ of Section 3.2 in the special case where f is a theta series attached to a Hecke character of the imaginary quadratic field K. This explicit relation is described in the following section.

3.4. **A factorisation of the p-adic Rankin L-series.** This section focuses on the Rankin L-function $L_p(f,\chi)$ of f and K in the special case where f is a theta series associated to a Hecke character of the *same* imaginary quadratic field K.

More precisely, let ψ be a fixed Hecke character of K of infinity type $(k-1,0)$ with $k=r+2\geq 2$. Consider the associated theta series:

$$\theta_\psi := \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}} = \sum_{n=1}^{\infty} a_n(\theta_\psi) q^n,$$

where the first sum is taken over integral ideals of K. The Fourier coefficients of θ_ψ generate a number field E_{θ_ψ} which is clearly contained in E_ψ.

The following classical proposition is due to Hecke and Schoenberg. (Cf. [Ogg] or Sec. 3.2 of [Za]).

Proposition 3.13. *The theta series θ_ψ belongs to $S_k(\Gamma_0(N),\varepsilon)$, where*

(1) *The level N is equal to DM, with $M = N^K_{\mathbb{Q}}\mathfrak{f}_\psi$,*
(2) *The Nebentypus character ε is equal to $\varepsilon_K\varepsilon_\psi$.*

Lemma 3.14. *If the conductor $\mathfrak{f}_\psi$ of ψ is a cyclic ideal $\mathfrak{m}$ of norm M prime to D, then $f := \theta_\psi$ satisfies the Heegner hypothesis relative to K.*

Proof. In this case, the modular form θ_ψ is of level $N = DM$, by Proposition 3.13. But then the ideal

$$\mathfrak{N} := \mathfrak{d}_K\mathfrak{m}, \tag{3.14}$$

with $\mathfrak{d}_K := (\sqrt{-D_K})$ is a cyclic ideal of K of norm N. □

We will assume from now on that the condition in Lemma 3.14 is satisfied. Furthermore, we will always take $\mathfrak{N}$ to be the ideal in (3.14).

The goal of this section is to factor the p-adic Rankin L-function $L_p(\theta_\psi,\cdot)$ as a product of two Katz p-adic L-functions. As a preparation to stating the main result we record the following lemma:

Lemma 3.15. *Let c be an integer prime to pN and let χ be any character in $\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$.*

(1) *If χ belongs to $\Sigma^{(2)}_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$, then $\psi^{-1}\chi$ belongs to $\Sigma^{(2)}_{\mathrm{sd}}(c\mathfrak{d}_K)^-$ and $\psi^{*-1}\chi$ belongs to $\Sigma^{(2)}_{\mathrm{sd}}(c\mathfrak{d}_K M)^-$.*
(2) *If χ belongs to $\Sigma^{(1)}_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$, then $\psi^{-1}\chi$ belongs to $\Sigma^{(1)}_{\mathrm{sd}}(c\mathfrak{d}_K)^-$ and $\psi^{*-1}\chi$ belongs to $\Sigma^{(2)}_{\mathrm{sd}}(c\mathfrak{d}_K M)^-$.*

Proof. We first note that when χ is of type $(k+j,-j)$ then $\psi^{-1}\chi$ is of infinity type $(1+j,-j)$ and $\psi^{*-1}\chi$ is of infinity type $(k+j,1-(k+j))$. Since $\chi\in\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$, we have

$$\varepsilon_\chi = \varepsilon = \varepsilon_\psi\cdot\varepsilon_K.$$

Thus $\varepsilon_{\psi^{-1}\chi}$ equals ε_K and the same holds for $\varepsilon_{\psi^{*-1}\chi}$ since $\varepsilon_{\psi^*}=\varepsilon_\psi$. It follows then from Remark 3.7 that $\psi^{-1}\chi$ and $\psi^{*-1}\chi$ are self-dual characters.

Let q be a rational prime dividing M. Since $\mathfrak{m}$ is a cyclic $\mathcal{O}_K$-ideal, it follows that $q=\mathfrak{q}\bar{\mathfrak{q}}$ must be split in K, and exactly one of $\mathfrak{q}$,$\bar{\mathfrak{q}}$ divides $\mathfrak{m}$. From this it is easy to see that ε_ψ has conductor exactly M, hence ε has conductor exactly N and $\mathfrak{N}_\varepsilon=\mathfrak{N}$. Thus $\mathfrak{f}_\chi = c\mathfrak{N} = c\mathfrak{d}_K\mathfrak{m}$ and $\mathfrak{f}_{\psi^{*-1}\chi} = c\mathfrak{d}_K\mathfrak{m}\bar{\mathfrak{m}} = c\mathfrak{d}_K M$. On the other hand, since $\varepsilon_\chi=\varepsilon_\psi\varepsilon_K$, it follows that $\mathfrak{f}_{\psi^{-1}\chi} = c\mathfrak{d}_K$.

The preceding remarks imply that if χ is in $\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$, then $\psi^{-1}\chi$ lies in $\Sigma_{\mathrm{sd}}(c\mathfrak{d}_K)^-$ and $\psi^{*-1}\chi$ lies in $\Sigma_{\mathrm{sd}}(c\mathfrak{d}_K M)^-$. To finish we note that if $j\geq 0$, then both $\psi^{-1}\chi$ and $\psi^{*-1}\chi$ lie in $\Sigma^{(2)}_{\mathrm{sd}}$, while if $-(k-1)\leq j\leq -1$, then $\psi^{*-1}\chi$ is in $\Sigma^{(2)}_{\mathrm{sd}}$ while $\psi^{-1}\chi$ lies in $\Sigma^{(1)}_{\mathrm{sd}}$. □

Theorem 3.16. *For all $\chi\in\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$,*

$$L_p(\theta_\psi,\chi) = \frac{w(\theta_\psi,\chi)^{-1}w_K}{2c^{k+2j-1}} \prod_{q|c} \frac{q-\varepsilon_K(q)}{q-1} \times \mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{-1}\chi) \times \mathscr{L}_{p,c\mathfrak{d}_K M}(\psi^{*-1}\chi). \tag{3.15}$$

Proof. Since $\Sigma_{cc}^{(2)}(c,\mathfrak{N},\varepsilon)$ is dense in $\hat{\Sigma}_{cc}^{(2)}(c,\mathfrak{N},\varepsilon)$, it suffices to prove the formula for the characters χ in this range, where it follows directly from the interpolation properties defining the respective p-adic L-functions. More precisely, by (3.7),

$$\frac{L_p(\theta_\psi,\chi)}{\Omega_p(A)^{2(k+2j)}} = \left((1-\psi\chi^{-1}(\bar{\mathfrak{p}}))(1-\psi^*\chi^{-1}(\bar{\mathfrak{p}})\right)^2 L_{\mathrm{alg}}(\theta_\psi,\chi^{-1},0). \tag{3.16}$$

Let $\delta_c := \prod_{q|c}\frac{q-\varepsilon_K(q)}{q-1}$. By the definition of $L_{\mathrm{alg}}(\theta_\psi,\chi^{-1},0)$ given in (3.5) and (3.6),

$$\begin{aligned} L_{\mathrm{alg}}(\theta_\psi,\chi^{-1},0) &= w(\theta_\psi,\chi)^{-1}C(\theta_\psi,\chi,c)\frac{L(\theta_\psi,\chi^{-1},0)}{\Omega(A)^{2(k+2j)}} \\ &= w(\theta_\psi,\chi)^{-1}w_K\delta_c\frac{2^{r+2j}\pi^{k+2j-1}j!(k+j-1)!}{\sqrt{D}^{k+2j-1}c^{k+2j-1}}\times\frac{L(\psi\chi^{-1},0)L(\psi^*\chi^{-1},0)}{\Omega(A)^{2(k+2j)}} \qquad (3.17)\\ &= \frac{w(\theta_\psi,\chi)^{-1}w_K\delta_c}{2c^{k+2j-1}}\left(\frac{j!(2\pi)^jL(\psi\chi^{-1},0)}{\sqrt{D}^j\Omega(A)^{1+2j}}\right)\times\left(\frac{(k+j-1)!(2\pi)^{k+j-1}L(\psi^*\chi^{-1},0)}{\sqrt{D}^{k+j-1}\Omega(A)^{1+2(k+j-1)}}\right). \end{aligned}$$

Combining (3.16) and (3.17) with the interpolation property of the Katz p-adic L-function given in Proposition 3.5, we obtain

$$\frac{L_p(\theta_\psi,\chi)}{\Omega_p(A)^{2(k+2j)}} = \frac{w(\theta_\psi,\chi)^{-1}w_K\delta_c}{2c^{k+2j-1}}\times\frac{\mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{-1}\chi)}{\Omega_p(A)^{1+2j}}\times\frac{\mathscr{L}_{p,c\mathfrak{d}_KM}(\psi^{*-1}\chi)}{\Omega_p(A)^{1+2(k+j-1)}}. \tag{3.18}$$

Clearing the powers of $\Omega_p(A)$ on both sides gives the desired result. □

The Nebentypus character ε can be viewed as a finite order Galois character of $G_{\mathbb{Q}}$. Recall that $E_{\psi,\chi,\varepsilon}$ denotes the smallest extension of $E_{\psi,\chi}$ containing the field through which this character factors.

Corollary 3.17. *For all $\chi\in\Sigma_{cc}(c,\mathfrak{N},\varepsilon)$,*

$$L_p(\theta_\psi,\chi) = \mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{-1}\chi)\times\mathscr{L}_{p,c\mathfrak{d}_KM}(\psi^{*-1}\chi) \pmod{E_{\psi,\chi,\varepsilon}^\times}.$$

Proof. This follows from Theorem 3.16 in light of the fact that the constant that appears on the right hand side of (3.15) belongs to $E_{\psi,\chi,\varepsilon}^\times$. □

3.5. **Proof of Rubin's Theorem.** The goal of this section is to prove Theorem 2 of the Introduction. Let $\mathfrak{c}=\bar{\mathfrak{c}}$ be an integral ideal in $\mathcal{O}_K$ invariant under complex conjugation and let $\nu\in\Sigma_{sd}(\mathfrak{c})^-$ be a Hecke character of K of infinity type $(1,0)$. Since $\varepsilon_\nu=\varepsilon_K$, it follows that $\mathfrak{d}_K|\mathfrak{c}$. We will also assume that ν satisfies the following additional conditions:

(i) The sign w_ν of the functional equation of the L-function $L(\nu,s)$ is -1.
(ii) $\mathfrak{d}_K||\mathfrak{c}$. Thus $\mathfrak{c}=(c)\mathfrak{d}_K$ for a unique positive rational integer c that is prime to D.

Let p be a rational prime split in K that is prime to c.

Definition 3.18. A pair (ψ,χ) of Hecke characters is said to be *good* for ν if it satisfies the following conditions.

(1) The character ψ is of type $(1,0)$ and has conductor $\mathfrak{m}$, where $\mathfrak{m}$ is a cyclic $\mathcal{O}_K$-ideal prime to pD. Thus θ_ψ is a newform in $S_2(\Gamma_0(N),\varepsilon)$ where $N=MD$ and $\varepsilon=\varepsilon_\psi\varepsilon_K$ is a Dirichlet character of conductor exactly N. Let $\mathfrak{N}:=\mathfrak{m}\mathfrak{d}_K$.
(2) The character χ is of finite order, and $\chi\mathbf{N}_K$ belongs to $\Sigma_{cc}^{(1)}(c,\mathfrak{N},\varepsilon)$. This implies (on account of Lemma 3.15 applied to $\chi\mathbf{N}_K$) that $\psi^{-1}\chi\mathbf{N}_K$ lies in $\Sigma_{sd}^{(1)}(\mathfrak{c})$ and $\psi^{*-1}\chi\mathbf{N}_K$ lies in $\Sigma_{sd}^{(2)}(\mathfrak{c}M)$.
(3) The character $\psi\chi^{-1}$ is equal to ν, i.e., $\psi^{-1}\chi\mathbf{N}_K=\nu^*$.
(4) The classical L-value $L(\psi^*\chi^{-1}\mathbf{N}_K^{-1},0)$ is non-zero, i.e., $\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi\mathbf{N}_K)\neq 0$.

Remark 3.19. Suppose that a pair (ψ,χ) satisfies (1) and (3) above with $\mathfrak{m}$ prime to $\mathfrak{c}$. Then such a pair automatically satisfies (2) also. Indeed, the character $\chi\mathbf{N}_K=\psi\nu^*$ is of type $(1,1)$ and its central character is equal to

$$\varepsilon_\chi=\varepsilon_\psi\varepsilon_{\nu^*}=\varepsilon_\psi\varepsilon_K=\varepsilon,$$

where ε is the nebentype character attached to θ_ψ. Further, $\mathfrak{f}_\chi=\mathfrak{f}_\psi\mathfrak{f}_{\nu^*}=\mathfrak{m}\cdot c\mathfrak{d}_K$. It follows that the character $\chi\mathbf{N}_K$ belongs to $\Sigma_{cc}(c,\mathfrak{N},\varepsilon)$ with $\mathfrak{N}=\mathfrak{d}_K\mathfrak{m}$. (The set S_{θ_ψ} in the discussion below Defn. 3.10 is empty since $D\mid N_\varepsilon$.)

Remark 3.20. Suppose that a pair (ψ,χ) satisfies conditions (1), (2) and (3) above. Since $\chi\mathbf{N}_K$ lies in $\Sigma^{(1)}_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$, the sign in the functional equation of $L(\theta_\psi,\chi^{-1},s)$ is -1. As seen previously, this L-function factors as

$$L(\theta_\psi,(\chi\mathbf{N}_K)^{-1},s)=L(\psi\chi^{-1}\mathbf{N}_K^{-1},s)L(\psi^*\chi^{-1}\mathbf{N}_K^{-1},s)=L(\nu\mathbf{N}_K^{-1},s)L(\psi^*\chi^{-1}\mathbf{N}_K^{-1},s).$$

The normalization here is such that the central point is $s=0$. Since the sign of $L(\nu,s)$ is -1, it follows that the sign of $L(\psi^*\chi^{-1}\mathbf{N}_K^{-1},s)$ is $+1$. Hence condition (4) would be expected to hold generically.

The modular abelian variety B_{θ_ψ} attached to ψ is a CM abelian variety in the sense of Definition 2.4. Hence it is K-isogenous to the CM abelian variety B_ψ constructed in Section 2.2. In particular, the modular parametrisation $\Phi_\psi:=\Phi_{\theta_\psi}$ can be viewed as a surjective morphism of abelian varieties over K

$$\Phi_\psi: J_1(N)\longrightarrow B_\psi. \tag{3.19}$$

Given a good pair (ψ,χ), recall the Heegner divisor $\Delta\in J_1(N)(H_{c,\mathfrak{N}})$ that was constructed in Section 3.3, and the Heegner point

$$P_\psi(\chi):=P_{\theta_\psi}(\chi)=\sum_{\sigma\in G}\chi^{-1}(\sigma)\Phi_\psi(\Delta^\sigma)\quad\in B_\psi(H_\chi)\otimes_{T_\psi}E_{\psi,\chi} \tag{3.20}$$

that was defined in equation (3.10) of that section with $f=\theta_\psi$. Recall also that ω_ψ is an E_ψ-vector space generator of $\Omega^1(B_\psi/E_\psi)^{T_\psi}$. Viewing the point $P_\psi(\chi)$ as a formal linear combination of elements of $B_\psi(H_\chi)$ with coefficients in $E_{\psi,\chi}$, we define the expression $\log_{\omega_\psi}(P_\psi(\chi))$ by $E_{\psi,\chi}$-linearity.

In the rest of this section, we will denote by $E'_{\psi,\chi}$ the subfield of $\bar{\mathbb{Q}}$ generated by E_ψ, E_χ, and the abelian extension H'_χ of K cut out by the finite order characters χ and χ^*. The motivation for singling out good pairs for a special definition lies in the following proposition.

Proposition 3.21. *For any pair (ψ,χ) which is good for ν,*

$$\mathscr{L}_{p,\mathfrak{c}}(\nu^*)=\Omega_p(\nu^*)^{-1}\log^2_{\omega_\psi}(P_\psi(\chi))\pmod{(E'_{\psi,\chi})^\times}, \tag{3.21}$$

where $\Omega_p(\nu^*)$ is the p-adic period from Definition 2.13.

Proof. By Theorem 3.12 applied to $f=\theta_\psi$,

$$L_p(\theta_\psi,\chi\mathbf{N}_K)=\log^2_{\omega_\psi}(P_\psi(\chi))\pmod{E^\times_{\psi,\chi}}. \tag{3.22}$$

On the other hand, since $E'_{\psi,\chi}$ contains $E_{\psi,\chi,\varepsilon}$, Corollary 3.17 implies that

$$\begin{aligned}L_p(\theta_\psi,\chi\mathbf{N}_K)&=\mathscr{L}_{p,\mathfrak{c}}(\psi^{-1}\chi\mathbf{N}_K)\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi\mathbf{N}_K)\pmod{(E'_{\psi,\chi})^\times}\\ &=\mathscr{L}_{p,\mathfrak{c}}(\nu^*)\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi\mathbf{N}_K)\pmod{(E'_{\psi,\chi})^\times},\end{aligned} \tag{3.23}$$

where the second equality follows from condition 3 in the definition of a good pair. The value $\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi\mathbf{N}_K)$ is non-zero by condition 4 in the definition of a good pair. Therefore, by Cor. 3.3,

$$\mathscr{L}_{p,\mathfrak{c}M}(\psi^{*-1}\chi\mathbf{N}_K)=\Omega_p(\psi^{-1}\chi^*\mathbf{N}_K)\pmod{E^\times_{\psi,\chi}}. \tag{3.24}$$

Since $\psi\chi^{-1}=\nu$, we have

$$\Omega_p(\psi^{-1}\chi^*\mathbf{N}_K)=\Omega_p(\nu^{-1}\chi^{-1}\chi^*\mathbf{N}_K)=\Omega_p(\nu^*\cdot\chi^*/\chi)=\Omega_p(\nu^*)\pmod{(E'_{\psi,\chi})^\times}, \tag{3.25}$$

where the last equality follows from from Lemma 2.14. The proposition now follows by combining the equations (3.22) through (3.25). □

To go further, we will analyse the expression $\log_{\omega_\psi}(P_\psi(\chi))$ and relate it to quantities depending solely on ν and not on the good pair (ψ,χ). It will be useful to view the point $P_\psi(\chi)$ appearing in (3.21) as an element of $B^0_{\psi,\chi}(H_{c,\mathfrak{N}})$ or as a K-rational point on the abelian variety $B_{\psi,\chi}$ that was introduced in Section 2.2. More precisely, after setting

$$\mathbf{P}_\psi(\chi):=\sum_{\sigma\in G}\Phi_\psi(\Delta^\sigma)\otimes\chi^{-1}(\sigma)\quad\in B_\psi(\bar{K})\otimes_{T_\psi}T_{\psi,\chi}=B^0_{\psi,\chi}(\bar{K}), \tag{3.26}$$

we observe that, for all $\tau \in \mathrm{Gal}(\bar{K}/K)$,

$$\begin{aligned} \tau *_0 \mathbf{P}_\psi(\chi) &= \sum_{\sigma\in G} \Phi_\psi(\Delta^{\tau\sigma}) \otimes \chi^{-1}(\sigma) \\ &= \sum_{\sigma\in G} \Phi_\psi(\Delta^{\sigma}) \otimes \chi^{-1}(\sigma\tau^{-1}) = \mathbf{P}_\psi(\chi)\chi(\tau). \end{aligned}$$

The point $\mathbf{P}_\psi(\chi)$ therefore belongs to $B_{\psi,\chi}(K)$ by (2.9). For the following lemmas, recall the differentials $\omega^0_{\psi,\chi} \in \Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$ and $\omega_{\psi,\chi} \in \Omega^1(B_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$.

Lemma 3.22. *For all good pairs (ψ,χ) attached to $\nu = \psi\chi^{-1}$,*

$$\log_{\omega_\psi}(P_\psi(\chi)) = \log_{\omega^0_{\psi,\chi}}(\mathbf{P}_\psi(\chi)).$$

Proof. Let $G = \mathrm{Gal}(H_{c,\mathfrak{N}}/K)$ and let $P = \Phi_\psi(\Delta)$. Also, let i be the map defined in (2.4). Then

$$\begin{aligned} \log_{\omega_\psi}(P_\psi(\chi)) &= \sum_{\sigma\in G} \chi(\sigma)^{-1} \log_{\omega_\psi}(P^\sigma) = \sum_{\sigma\in G} \chi(\sigma)^{-1} \log_{i^*(\omega^0_{\psi,\chi})}(P^\sigma) \\ &= \sum_{\sigma\in G} \chi(\sigma)^{-1} \log_{\omega^0_{\psi,\chi}}(P^\sigma \otimes 1) = \sum_{\sigma\in G} \log_{\chi(\sigma)^{-1}\omega^0_{\psi,\chi}}(P^\sigma \otimes 1) \\ &= \sum_{\sigma\in G} \log_{\omega^0_{\psi,\chi}}(P^\sigma \otimes \chi(\sigma)^{-1}) = \log_{\omega^0_{\psi,\chi}}\left(\sum_{\sigma\in G} P^\sigma \otimes \chi(\sigma)^{-1}\right) = \log_{\omega^0_{\psi,\chi}}(\mathbf{P}_{\psi,\chi}). \end{aligned}$$

□

Lemma 3.23.

$$\log_{\omega^0_{\psi,\chi}}(\mathbf{P}_\psi(\chi)) = \log_{\omega_{\psi,\chi}}(\mathbf{P}_\psi(\chi)) \pmod{(E'_{\psi,\chi})^\times},$$

Proof. This follows from Lemma 2.8 since the Gauss sum $\mathfrak{g}(\chi)$ lies in $(E'_{\psi,\chi})^\times$. □

Lemma 3.24. *There exists $P_\nu \in B_\nu(K)$ and $\omega_\nu \in \Omega^1(B_\nu/E_\nu)^{T_\nu}$ such that*

$$\log_{\omega_{\psi,\chi}}(\mathbf{P}_\psi(\chi)) = \log_{\omega_\nu}(P_\nu) \pmod{(E'_{\psi,\chi})^\times}.$$

Proof. Recall from Lemma 2.9 that there is a K-rational isogeny

$$B_\nu \otimes_{T_\nu} T_{\psi,\chi} \longrightarrow B_{\psi,\chi}.$$

Composing it with the natural morphism $B_\nu \longrightarrow B_\nu \otimes_{T_\nu} T_{\psi,\chi}$, we obtain a T_ν-equivariant morphism $j : B_\nu \longrightarrow B_{\psi,\chi}$ defined over K with finite kernel. The fact that $L(\nu,s)$ has a simple zero at $s=1$ implies that $B_\nu(K)\otimes\mathbb{Q}$ is one-dimensional over E_ν, and therefore that $B_{\psi,\chi}(K)\otimes\mathbb{Q}$ is one-dimensional over $E_{\psi,\chi}$. In particular, if P_ν is any generator of $B_\nu(K)\otimes\mathbb{Q}$, we may write

$$\mathbf{P}_\psi(\chi) = \lambda j(P_\nu)$$

for some non-zero scalar $\lambda \in E^\times_{\psi,\chi}$. But letting

$$\omega_\nu = j^*(\omega_{\psi,\chi}) \in \Omega^1(B_\nu/E'_{\psi,\chi})^{T_\nu},$$

we have

$$\begin{aligned} \log_{\omega_{\psi,\chi}}(\mathbf{P}_\psi(\chi)) &= \log_{\omega_{\psi,\chi}}(\lambda j(P_\nu)) = \log_{\lambda^*\omega_{\psi,\chi}}(j(P_\nu)) = \lambda \log_{\omega_{\psi,\chi}}(j(P_\nu)) \\ &= \lambda \log_{j^*\omega_{\psi,\chi}}(P_\nu) = \lambda \log_{\omega_\nu}(P_\nu). \end{aligned}$$

The lemma now follows after multiplying ω_ν by an appropriate scalar in $(E'_{\psi,\chi})^\times$ so that it belongs to $\Omega^1(B_\nu/E_\nu)^{T_\nu}$. □

Proposition 3.25. *There exists $\omega_\nu \in \Omega^1(B_\nu/E_\nu)^{T_\nu}$ and $P_\nu \in B_\nu(K)$ such that*

$$\mathscr{L}_{p,\mathfrak{c}}(\nu^*) = \Omega_p(\nu^*)^{-1} \log^2_{\omega_\nu}(P_\nu) \pmod{(E'_{\psi,\chi})^\times}, \tag{3.27}$$

for all good pairs (ψ,χ) attached to ν.

Proof. This follows immediately from Proposition 3.21 and Lemmas 3.22 through 3.24. □

While Proposition 3.25 brings us close to Theorem 2 of the Introduction, it is somewhat more vague in that both sides of the purported equality may differ *a priori* by a non-zero element of the typically larger field $E'_{\psi,\chi}$. The alert reader will also notice that this proposition is potentially vacuous for now, because the existence of a good pair for ν has not yet been established! The next proposition repairs this omission, and directly implies Theorem 2 of the Introduction.

Proposition 3.26. *The set S_ν of pairs (ψ, χ) that are good for ν is non-empty. Furthermore,*

$$\bigcap_{(\psi,\chi)\in S_\nu} E'_{\psi,\chi} = E_\nu. \tag{3.28}$$

The proof of Proposition 3.26 rests crucially on a non-vanishing result of Rohrlich and Greenberg ([Ro], [Gre]) for the central critical values of Hecke L-series. In order to state it, we fix a rational prime ℓ which is split in K and let

$$K_\infty^- = \cup_{n\geq 0} K_n^-$$

be the so-called *anti-cyclotomic* $\mathbb{Z}_\ell$ *extension* of K; it is the unique $\mathbb{Z}_\ell$-extension of K which is Galois over $\mathbb{Q}$ and for which $\mathrm{Gal}(K_\infty^-/\mathbb{Q}) = \mathbb{Z}_\ell \rtimes (\mathbb{Z}/2\mathbb{Z})$ is a generalised dihedral group.

Lemma 3.27 (Greenberg, Rohrlich)**.** *Let ψ_0 be a self-dual Hecke character of K of infinity type $(1,0)$. Assume that the sign w_{ψ_0} in the functional equation of $L(\psi_0, s)$ is equal to 1. Then there are infinitely many finite-order characters χ of $\mathrm{Gal}(K_\infty^-/K)$ for which $L(\psi_0\chi, 1) \neq 0$.*

Proof. Let $\mathfrak{c}'$ be the conductor of ψ_0. In light of the hypothesis that $w_{\psi_0} = 1$, Theorem 1 of [Gre] implies that the Katz p-adic L-function (with $p = \ell$) does not vanish identically on any open ℓ-adic neighbourhood of ψ_0 in $\Sigma_{\mathrm{sd}}(\mathfrak{c}')$. (Cf. the discussion in the first paragraph of the proof of Proposition 1 on p. 93 of [Gre].) If U is any sufficiently small such neighbourhood, then

(1) The restriction to U of the Katz p-adic L-function is described by a power series with p-adically bounded coefficients, and therefore admits only finitely many zeros by the Weierstrass preparation theorem.
(2) The region U contains a dense subset of points of the form $\psi_0\chi$, where χ is a finite order character of $\mathrm{Gal}(K_\infty^-/K)$.

Lemma 3.27 follows directly from these two facts. □

Proof of Proposition 3.26. Let $\bar{S}_\nu \supset S_\nu$ be the set of pairs satisfying conditions 1-3 in the definition of a good pair, but without necessarily requiring the more subtle fourth condition. The proof of Proposition 3.26 will be broken down into four steps.

Step 1. The set $\bar{S}_\nu$ is non-empty.

To see this, let ψ be any Hecke character of K of infinity type $(1,0)$ and conductor $\mathfrak{m}$, where $\mathfrak{m}$ is a cyclic $\mathcal{O}_K$-ideal prime to $\mathfrak{c}$. Setting $\chi = \psi\nu^{-1}$, the pair (ψ, χ) satisfies conditions 1 and 3 by construction, and 2 as well on account of Remark 3.19. Therefore, the pair (ψ, χ) belongs to $\bar{S}_\nu$.

Step 2. Given $(\psi, \chi) \in \bar{S}_\nu$, there exist (ψ_1, χ_1) and $(\psi_2, \chi_2) \in S_\nu$ with $E'_{\psi_1,\chi_1} \cap E'_{\psi_2,\chi_2} \subset E'_{\psi,\chi}$.

To see this, let $\ell = \lambda\bar{\lambda}$ be a rational prime which splits in K and is relatively prime to the class number of K and the conductors of ψ and χ, and which is unramified in $E'_{\psi,\chi}/\mathbb{Q}$. For such a prime, let

$$K_\infty = \cup_{n\geq 0} K_n, \qquad K'_\infty = \cup_{n\geq 0} K'_n$$

be the unique $\mathbb{Z}_\ell$-extensions of K which are unramified outside of λ and $\bar{\lambda}$ respectively, with $[K_n : K] = \ell^n$ and likewise for K'_n. The condition that ℓ does not divide the class number of K implies that the fields K_n and K'_n are totally ramified at λ and $\bar{\lambda}$ respectively. If α is any character of $\mathrm{Gal}(K_\infty/K)$, the pair $(\psi_1, \chi_1) := (\psi\alpha, \chi\alpha)$ still belongs to $\bar{S}_\nu$, with $\mathfrak{m}$ in condition 1 replaced by $\mathfrak{m}\lambda^n$ for a suitable $n \geq 0$. Furthermore,

$$L(\psi_1^*\chi_1^{-1}\mathbf{N}_K^{-1}, 0) = L(\psi^*\chi^{-1}\mathbf{N}_K^{-1} \cdot (\alpha^*/\alpha), 0). \tag{3.29}$$

The character α^*/α is an anticyclotomic character of K of ℓ-power order and conductor, and all such characters can be obtained by choosing α appropriately. The fact that (ψ, χ) satisfies conditions (1),(2) and (3) of a good pair implies (see Remark 3.20) that the sign $w_{\psi^*\chi^{-1}}$ is equal to $+1$. Hence, by Lemma

3.27, there exists a choice of α for which the L-value appearing on the right of (3.29) is non-vanishing. The corresponding pair (ψ_1, χ_1) belongs to S_ν and satisfies

$$E'_{\psi_1,\chi_1} \subset E'_{\psi,\chi,\ell,n} := E'_{\psi,\chi}\mathbb{Q}(\zeta_{\ell^n})K_nK'_n$$

for some n. Note that the extension $E'_{\psi,\chi,\ell,n}/E'_{\psi,\chi}$ has degree dividing $\ell^\infty(\ell-1)$. Repeating the same construction with a different rational prime ℓ' in place of ℓ such that $\ell'-1$ is prime to ℓ yields a second pair $(\psi_2,\chi_2) \in S_\nu$ and a corresponding extension $E'_{\psi,\chi,\ell',n'}$, whose degree over $E'_{\psi,\chi}$ divides $\ell'^\infty(\ell'-1)$, and such that

$$E'_{\psi_2,\chi_2} \subset E'_{\psi,\chi,\ell',n'}.$$

Let

$$E'' := E'_{\psi,\chi,\ell,n} \cap E'_{\psi,\chi,\ell',n'}.$$

We see then using degrees that $E''/E'_{\psi,\chi}$ has degree dividing $(\ell-1)$, hence $E'' \subseteq E'_{\psi,\chi}\mathbb{Q}(\zeta_\ell)$. Since ℓ is unramified in $E'_{\psi,\chi}/\mathbb{Q}$, the extension $E''/E'_{\psi,\chi}$ must be totally ramified at the primes above ℓ. On the other hand being a subextension of $E'_{\psi,\chi,\ell',n'}/E'_{\psi,\chi}$, it is also unramified at the primes above ℓ, hence $E'' = E'_{\psi,\chi}$. It follows that $E'_{\psi_1,\chi_1} \cap E'_{\psi_2,\chi_2} \subset E'_{\psi,\chi}$.

Thanks to Step 2, we are reduced to showing that

$$\bigcap_{(\psi,\chi)\in\bar{S}_\nu} E'_{\psi,\chi} = E_\nu. \tag{3.30}$$

The next step shows that the fields $E'_{\psi,\chi}$ can be replaced by $E_{\psi,\chi}$ in this equality.

Step 3. For all $(\psi,\chi) \in \bar{S}_\nu$, there exists a finite order character α of G_K such that the pair $(\psi\alpha,\chi\alpha)$ belongs to $\bar{S}_\nu$ and

$$E'_{\psi,\chi} \cap E'_{\psi\alpha,\chi\alpha} \subseteq E_{\psi,\chi}. \tag{3.31}$$

To see this, note that the finite order character χ has cyclic image, isomorphic to $\mathbb{Z}/n\mathbb{Z}$ say. Pick α such that conditions (i)-(iii) below are satisfied:

(i) α has order n and is ramified at a single prime λ of K which lies over a rational prime ℓ that is split in K.
(ii) λ is prime to the conductors of χ and χ^*.
(iii) ℓ is unramified in $E'_{\psi,\chi}/\mathbb{Q}$.

Conditions (i) and (ii) imply:

(iv) The field $H_{\chi\alpha}/K$ is totally ramified at λ and unramified at λ^* while $H_{\chi^*\alpha^*}$ is unramified at λ and totally ramified at λ^*.

Taking

$$L = E_{\psi,\chi}, \quad M_1 = H_\chi H_{\chi^*}, \quad M_2 = H_{\chi\alpha}H_{\chi^*\alpha^*},$$

we see from (iii) and (iv) that

(v) LM_1/L is unramified at all primes above ℓ, and

(vi) Any subextension of LM_2/L is ramified at some prime above λ or λ^*.

Thus, $LM_1 \cap LM_2 = L$. But $LM_1 = E_{\psi,\chi}H_\chi H_{\chi^*} = E'_{\psi,\chi}$. Also since α has order n, we have $E_{\psi',\chi'} = E_{\psi,\chi}$ and

$$LM_2 = E_{\psi,\chi}H_{\chi\alpha}H_{\chi^*\alpha^*} = E_{\psi',\chi'}H_{\chi\alpha}H_{\chi^*\alpha^*} = E'_{\psi',\chi'},$$

so (3.31) follows.

Step 4. We are now reduced to showing

$$\bigcap_{(\psi,\chi)\in\bar{S}_\nu} E_{\psi,\chi} = E_\nu. \tag{3.32}$$

We will do this by showing

(3.33) There exists a pair $(\psi,\chi) \in \bar{S}_\nu$ such that $E_{\psi,\chi} = E_\nu$.

We begin by choosing an ideal $\mathfrak{m}_0$ of $\mathcal{O}_K$ with the property that $\mathcal{O}_K/\mathfrak{m}_0 = \mathbb{Z}/M\mathbb{Z}$ is cyclic, and an odd quadratic Dirichlet character ε_M of conductor dividing M. Let ψ_0 be any Hecke character satisfying

$$\psi_0((a)) = \varepsilon_M(a \bmod \mathfrak{m}_0)a$$

on principal ideals (a) of K. Such a ψ_0 satisfies condition 1 in Definition 3.18, and therefore, after letting χ_0 be the finite order character satisfying

$$\nu^* = \psi_0^{-1}\chi_0\mathrm{N}_K,$$

it follows that (ψ_0, χ_0) belongs to $\bar{S}_\nu$. Furthermore, the restriction of ψ_0 to the group of principal ideals of K takes values in K, and therefore

$$\chi_0(\sigma) \in E_\nu, \qquad \text{for all } \sigma \in G_H := \mathrm{Gal}(\bar{K}/H). \tag{3.34}$$

The character ψ_0 itself takes values in a CM field of degree $[H : K]$, denoted E_0, which need not be contained in E_ν in general. To remedy this problem, let H_0 be the abelian extension of the Hilbert class field H cut out by the character χ_0. Next, let H_0' be any abelian extension of K containing H such that

(1) There is an isomorphism $u : \mathrm{Gal}(H_0'/K) \longrightarrow \mathrm{Gal}(H_0/K)$ of abstract groups such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Gal}(H_0'/H) & \longrightarrow & \mathrm{Gal}(H_0'/K) & \longrightarrow & \mathrm{Gal}(H/K) & \longrightarrow & 0\ , \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{Gal}(H_0/H) & \longrightarrow & \mathrm{Gal}(H_0/K) & \longrightarrow & \mathrm{Gal}(H/K) & \longrightarrow & 0 \end{array} \tag{3.35}$$

commutes, where the dotted arrows indicate the isomorphisms induced by u and the other arrows are the canonical maps of Galois theory.

(2) The relative discriminant of H_0' over K is relatively prime to its conjugate (and therefore to the discriminant of K, in particular).

If the bottom exact sequence of groups in (3.35) is split, then the extension H_0' is readily produced, using class field theory. To handle the general case, we follow an approach that is suggested by the proof of Prop. 2.1.7 in [Se]. Let $\tilde{\Phi} := \mathrm{Gal}(H_0/K)$ and let $\Psi : G_K \longrightarrow \tilde{\Phi}$ be the homomorphism attached to the extension H_0. Since H is everywhere unramified over K, the restriction Ψ_v of Ψ to a decomposition group at any prime v of K maps the inertia subgroup I_v to $\mathfrak{C} := \mathrm{Gal}(H_0/H)$. After viewing $\mathfrak{C}$ as a module of finite cardinality endowed with the trivial action of G_K, let $H^1_S(K, \mathfrak{C}) := \mathrm{Hom}(G_{K,S}, \mathfrak{C})$ denote the group of homomorphisms from G_K to $\mathfrak{C}$ which are unramified outside a given finite set S of primes of K, and let $H^1_{[S]}(K, \mathfrak{C}^*)$ denote the *dual Selmer group* attached to $H^1_S(K, \mathfrak{C})$ in the sense of Theorem 2.18 of [DDT] for example. Here $\mathfrak{C}^* := \mathrm{Hom}(\mathfrak{C}, \mathbf{G}_m)$ is the *Kummer dual* of $\mathfrak{C}$, which is isomorphic to μ_n when $\mathfrak{C} = \mathbb{Z}/n\mathbb{Z}$ is cyclic of order n. Kummer theory (along with the non-degeneracy of the local Tate pairing) identifies $H^1_{[S]}(K, \mu_n)$ with the subgroup of $K^\times/(K^\times)^n$ consisting of elements α for which

$$\mathrm{ord}_v(\alpha) = 0 \pmod n \text{ for all } v, \qquad \mathrm{res}_v(\alpha) \in (K_v^\times)^n \text{ for all } v \in S.$$

Let S be any finite set of primes of K at which Ψ is unramified, satisfying the further conditions

$$v \in S \Rightarrow \bar{v} \notin S, \qquad \text{and } H^1_{[S]}(K, \mathfrak{C}^*) = 0. \tag{3.36}$$

The existence of such a set S follows from the statement that for any $\alpha \in K^\times - (K^\times)^n$, there is a set of primes v of K of positive Dirichlet density for which the image of α in $K_v^\times$ is not an n-th power. (This statement follows in turn from the Chebotarev density theorem applied to the extension $K(\mu_n, \alpha^{1/n})$.) Now let T be any finite set of places which is disjoint from S. Comparing the statement of Theorem 2.18 of [DDT] in the case $M = \mathfrak{C}$ and $\mathscr{L} = S$ and $\mathscr{L} = S \cup T$ respectively, and noting that both $H^1_{[S]}(K, \mathfrak{C}^*)$ and (a fortiori) $H^1_{[S\cup T]}(K, \mathfrak{C}^*)$ are trivial, gives

$$\frac{\# H^1_{S\cup T}(K, \mathfrak{C})}{\# H^1_S(K, \mathfrak{C})} = \prod_{v \in T} \frac{\# H^1(K_v, \mathfrak{C})}{\# \mathfrak{C}} = \prod_v \# \mathrm{Hom}(I_v, \mathfrak{C}).$$

It follows that the rightmost arrow in the tautological exact sequence

$$0 \longrightarrow H^1_S(K, \mathfrak{C}) \longrightarrow H^1_{S\cup T}(K, \mathfrak{C}) \longrightarrow \prod_{c \in T} \mathrm{Hom}(I_v, \mathfrak{C})$$

is surjective. Letting T be the set of places at which Ψ is ramified, it follows that there is a homomorphism $\epsilon : G_K \longrightarrow \mathfrak{C}$ satisfying

$$\epsilon_v = \Psi_v \quad \text{on } I_v, \qquad \text{for all } v \notin S.$$

After possibly enlarging the set S satisfying (3.36) and translating ϵ by a suitable homomorphism unramified outside S, we may further assume that the homomorphism $\Psi\epsilon^{-1}$ maps G_K surjectively onto $\tilde{\Phi}$; the field H_0' can then be obtained as the fixed field of the kernel of the homomorphism $\Psi\epsilon^{-1}$.

With the extension H_0' in hand, let $\alpha : \mathrm{Gal}(H_0'/K) \longrightarrow E_\chi^\times$ be the finite order Hecke character given by

$$\alpha(\sigma) = \chi_0(u(\sigma))^{-1},$$

and set $(\psi, \chi) = (\psi_0\alpha, \chi_0\alpha)$. By construction, (ψ, χ) belongs to $\bar{S}_\nu$. We claim that χ and ψ take values in E_ν. Since $\nu^* = \psi^{-1}\chi\mathrm{N}_K$, it is enough to prove this statement for χ. Observe that, for all integral ideals $\mathfrak{a}$ prime to the conductors of χ_0, χ, and ψ, we have

$$\chi(\mathfrak{a}) = \chi_0(\sigma_\mathfrak{a})/\chi_0(u(\sigma_\mathfrak{a})) = \chi_0(\sigma_\mathfrak{a} u(\sigma_\mathfrak{a})^{-1}).$$

But the element $\sigma_\mathfrak{a} u(\sigma_\mathfrak{a})^{-1}$ belongs to $\mathrm{Gal}(H_0/H)$ by construction, and hence $\chi_0(\sigma_\mathfrak{a}^{-1} u(\sigma_\mathfrak{a}))$ belongs to E_ν by (3.34). It follows that ψ and χ are E_ν–valued, and therefore $E_{\psi,\chi} = E_\nu$, as claimed in (3.33). ☐

3.6. **Elliptic curves with complex multiplication.** Theorem 2 of the Introduction admits an alternate formulation involving algebraic points on elliptic curves with complex multiplication rather than K-rational points on the CM abelian varieties B_ν of Theorem 2.5. The goal of this section is to describe this variant. As in the introduction, we just write $\mathscr{L}_p$ for the p-adic L-function $\mathscr{L}_{p,\mathfrak{c}}$, where $\mathfrak{c}$ is the conductor of ν.

We begin by reviewing the explicit construction of B_ν in terms of CM elliptic curves. The reader is referred to §4 of [GS], whose treatment we largely follow, for a more detailed exposition. Let F be any abelian extension of K for which

$$\nu_F := \nu \circ \mathrm{N}_{F/K} \tag{3.37}$$

becomes K-valued. There exists an elliptic curve A/F with complex multiplication by $\mathcal{O}_K$ whose associated Grossencharacter is ν_F. (Cf. Thm. 6 of [Shi] and its corollary on p. 512.) Let

$$B := \mathrm{Res}_{F/K}(A).$$

It is an abelian variety over K of dimension $d := [F : K]$. Let $G := \mathrm{Gal}(F/K) = \mathrm{Hom}_K(F, \bar{\mathbb{Q}})$, where the natural identification between these two sets arises from the distinguished embedding of F into $\bar{\mathbb{Q}}$ that was fixed from the outset. By definition of the restriction of scalars functor, there are natural isomorphisms

$$B_{/F} = \prod_{\sigma \in G} A^\sigma, \qquad B(\bar{K}) = A(\bar{K} \otimes_K F) = \prod_{\sigma \in G} A^\sigma(\bar{K})$$

of algebraic groups over F and abelian groups respectively. In particular, a point of $B(\bar{K})$ is described by a d-tuple $(P_\tau)_{\tau \in G}$, with $P_\tau \in A^\tau(\bar{K})$. Relative to this identification, the Galois group G_K acts on $B(\bar{K})$ on the left by the rule

$$\xi(P_\tau)_\tau = (\xi P_\tau)_{\xi\tau}, \qquad \text{for all } \xi \in G_K.$$

Consider the "twisted group ring"

$$T := \oplus_{\sigma \in G} \mathrm{Hom}_F(A, A^\sigma) = \left\{ \sum_{\sigma \in G} a_\sigma \sigma, \text{ with } a_\sigma \in \mathrm{Hom}_F(A, A^\sigma) \right\}, \tag{3.38}$$

with multiplication given by

$$(a_\sigma \sigma)(a_\tau \tau) = a_\sigma a_\tau^\sigma \sigma\tau, \tag{3.39}$$

where the isogeny a_τ^σ belongs to $\mathrm{Hom}_F(A^\sigma, A^{\sigma\tau})$ and the composition of isogenies in (3.39) is to be taken from left to right. The right action of T on $B(\bar{K})$ defined by

$$(P_\tau)_\tau * (a_\sigma \sigma) := (a_\sigma^\tau(P_\tau))_{\tau\sigma} \tag{3.40}$$

commutes with the Galois action described in (3.6), and corresponds to a natural inclusion $T \hookrightarrow \mathrm{End}_K(B)$. The K-algebra $E := T \otimes_\mathbb{Z} \mathbb{Q}$ is isomorphic to a finite product

$$E = \prod_i E_i$$

of CM fields, and $\dim_K(E) = \dim(B)$. Therefore, the pair (B,E) is a CM abelian variety in the sense of Definition 2.4. The compatible system of ℓ-adic Galois representations attached to (B,E) corresponds to an E-valued algebraic Hecke character $\tilde{\nu}$ in the sense of Definition 2.3, satisfying the relation

$$\sigma_{\mathfrak{a}}(P) = P * \tilde{\nu}(\mathfrak{a}), \quad \text{for all } \mathfrak{a} \in I_{\mathfrak{f}\ell} \text{ and } P \in B(\bar{K})_{\ell^\infty}, \tag{3.41}$$

where $\sigma_{\mathfrak{a}} \in G_K^{\mathrm{ab}}$ denotes as before the Artin symbol attached to $\mathfrak{a} \in I_{\mathfrak{f}\ell}$.

The element $\tilde{\nu}(\mathfrak{a}) \in T$ is of the form $\varphi_{\mathfrak{a}}\sigma_{\mathfrak{a}}$, where

$$\varphi_{\mathfrak{a}} : A \to A^{\sigma_{\mathfrak{a}}}, \tag{3.42}$$

is an isogeny of degree $\mathrm{N}\mathfrak{a}$ satisfying

$$\varphi_{\mathfrak{a}}(P) = P^{\sigma_{\mathfrak{a}}}, \tag{3.43}$$

for any $P \in A[\mathfrak{g}]$ with $(\mathfrak{g},\mathfrak{a}) = 1$. Note that the isogenies $\varphi_{\mathfrak{a}}$ satisfy the following cocycle condition:

$$\varphi_{\mathfrak{a}\mathfrak{b}} = \varphi_{\mathfrak{b}}^{\sigma_{\mathfrak{a}}} \circ \varphi_{\mathfrak{a}}. \tag{3.44}$$

The following proposition relates the Hecke characters $\tilde{\nu}$ and ν.

Proposition 3.28. *Given any homomorphism $j \in \mathrm{Hom}_K(E,\mathbb{C})$, let $\nu_j := j \circ \tilde{\nu}$ be the corresponding $\mathbb{C}$-valued Hecke character of K of infinity type $(1,0)$. The assignment $j \mapsto \nu_j$ gives a bijection from $\mathrm{Hom}_K(E,\mathbb{C})$ to the set $\Sigma_{\nu,F}$ of Hecke characters ν' of K (of infinity type $(1,0)$) satisfying*

$$\nu' \circ N_{F/K} = \nu \circ N_{F/K}.$$

Proposition 3.28 implies that there is a unique homomorphism $j_\nu \in \mathrm{Hom}_K(E,\mathbb{C})$ satisfying $j_\nu \circ \tilde{\nu} = \nu$. In particular, j_ν maps E to E_ν and T to a finite index subring of T_ν. The abelian variety B_ν attached to ν in Theorem 2.5 can now be defined as the quotient $B \otimes_{T,j_\nu} T_\nu$. In subsequent constructions, it turns out to be more useful to realise B_ν as a subvariety of B, which can be done by setting

$$B_\nu := B[\ker j_\nu]. \tag{3.45}$$

The natural action of T on B_ν factors through the quotient $T/\ker(j_\nu)$, an integral domain having E_ν as field of fractions.

Consider the inclusion

$$i_\nu : B_\nu(K) \hookrightarrow B(K) = A(F), \tag{3.46}$$

where the last identification arises from the functorial property of the restriction of scalars. The following Proposition gives an explicit description of the image of $(B_\nu(K) \otimes E_\nu)^{T_\nu}$ in $A(F) \otimes_{\mathcal{O}_K} E_\nu$ under the inclusion i_ν obtained from (3.46).

Proposition 3.29. *Let $\tilde{E}$ be any field containing E_ν. The inclusion i_ν of (3.46) identifies $(B_\nu(K) \otimes \tilde{E})^{T_\nu}$ with*

$$(A(F) \otimes_{\mathcal{O}_K} \tilde{E})^{\nu} := \left\{ P \in A(F) \otimes_{\mathcal{O}_K} \tilde{E} \text{ such that } \varphi_{\mathfrak{a}}(P) = \nu(\mathfrak{a}) P^{\sigma_{\mathfrak{a}}}, \text{ for all } \mathfrak{a} \in I_{\mathfrak{f}} \right\}.$$

Proof. It follows from the definitions that $B(K)$ is identified with the set of (P_τ) with $P_\tau \in A^\tau(\bar{K})$ satisfying

$$\xi P_\tau = P_{\xi\tau}, \quad \text{for all } \xi \in G_K. \tag{3.47}$$

Furthermore, if such a (P_τ) belongs to $(B_\nu(K) \otimes E_\nu)^{T_\nu}$, then after setting $\tilde{\nu}(\mathfrak{a}) = \varphi_{\mathfrak{a}}\sigma_{\mathfrak{a}}$ as in (3.42), we also have

$$(\varphi_{\mathfrak{a}}^\tau(P_\tau))_{\tau\sigma_{\mathfrak{a}}} = (P_\tau)_\tau * \tilde{\nu}(\mathfrak{a}) = (\nu(\mathfrak{a})P_\tau)_\tau. \tag{3.48}$$

Equating the $\sigma_{\mathfrak{a}}$-components of these two vectors gives

$$\varphi_{\mathfrak{a}}(P_1) = \nu(\mathfrak{a}) P_{\sigma_{\mathfrak{a}}} = \nu(\mathfrak{a}) \sigma_{\mathfrak{a}} P_1,$$

where 1 is the identity embedding of F and the last equality follows from (3.47). The Proposition follows directly from this, after noting that the identification of $B(K)$ with $A(F)$ is simply the one sending $(P_\tau)_\tau$ to P_1. □

Given a global field F as in (3.37), let F_ν denote the subfield of $\bar{\mathbb{Q}}$ generated by F and E_ν. Recall that $\omega_A \in \Omega^1(A/F)$ is a non-zero differential and that $\Omega_p(A)$ is the associated p-adic period.

Theorem 3.30. *There exists a point $P_{A,\nu} \in (A(F) \otimes_{\mathcal{O}_K} E_\nu)^\nu$ such that*

$$\mathscr{L}_p(\nu^*) = \Omega_p(A)^{-1} \log^2_{\omega_A}(P_{A,\nu}) \pmod{F_\nu^\times}.$$

The point $P_{A,\nu}$ is non-zero if and only if $L'(\nu, 1) \neq 0$.

Proof. Theorem 2 of the Introduction asserts that

$$\mathscr{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1} \log^2_{\omega_\nu}(P_\nu), \tag{3.49}$$

for some point $P_\nu \in B_\nu(K) \otimes \mathbb{Q}$ which is non-trivial if and only if $L'(\nu, 1) \neq 0$. By Lemma 2.14, we find

$$\Omega_p(\nu^*)^{-1} = \Omega_p(A)^{-1} \pmod{F_\nu^\times}. \tag{3.50}$$

Furthermore, by Prop. 3.29, we can view P_ν as a point $P_{A,\nu} \in (A(F) \otimes_{\mathcal{O}_K} E_\nu)^\nu$, and we have

$$\log_{\omega_\nu}(P_\nu) = \log_{\omega_A}(P_{A,\nu}) \pmod{F_\nu^\times}. \tag{3.51}$$

Theorem 3.33 now follows by rewriting (3.49) using (3.50) and (3.51). □

3.7. **A special case.** This section is devoted to a more detailed and precise treatment of Theorem 3.30 under the following special assumptions:

(1) The quadratic imaginary field K has class number one, odd discriminant, and unit group of order two. This implies that $K = \mathbb{Q}(\sqrt{-D})$ where $D := -\operatorname{Disc}(K)$ belongs to the finite set

$$S := \{7, 11, 19, 43, 67, 163\}.$$

(2) ψ_0 is the Hecke character of K of infinity type $(1, 0)$ given by the formula

$$\psi_0((a)) = \varepsilon_K(a \bmod \mathfrak{d}_K)a. \tag{3.52}$$

The character ψ_0 determines (uniquely, up to an isogeny) an elliptic curve $A/\mathbb{Q}$ satisfying

$$\operatorname{End}_K(A) = \mathcal{O}_K, \qquad L(A/\mathbb{Q}, s) = L(\psi_0, s).$$

After fixing A, we will also write ψ_A instead of ψ_0. It can be checked that the conductor of ψ_A is equal to $\mathfrak{d}_K$, and that

$$\psi_A^* = \bar{\psi}_A, \qquad \psi_A \psi_A^* = \mathrm{N}_K, \qquad \varepsilon_{\psi_A} = \varepsilon_K.$$

Remark 3.31. The rather stringent assumptions on K that we have imposed exclude the arithmetically interesting, but somewhat idiosyncratic, cases where $K = \mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(i)$, and $\mathbb{Q}(\sqrt{-2})$.

With the above assumptions, the character ψ_A can be used to give an explicit description of the set $\Sigma_{\mathrm{sd}}(c\mathfrak{d}_K)$:

Lemma 3.32. *Let c be an integer prime to D, and let ν be a Hecke character in $\Sigma_{\mathrm{sd}}(c\mathfrak{d}_K)$. Then ν is of the form*

$$\nu = \psi_A \chi^{-1},$$

where χ is a finite order ring class character of K of conductor c.

Proof. The fact that ν and ψ_A both have central character ε_K implies that χ is a ring class character that is unramified at $\mathfrak{d}_K$, hence has conductor exactly c. □

Given a ring class character χ of conductor c as above with values in a field E_χ, let

$$(A(H_c) \otimes_{\mathcal{O}_K} E_\chi)^\chi := \{P \in A(H_c) \otimes_{\mathcal{O}_K} E_\chi \text{ such that } \sigma P = \chi(\sigma)P, \quad \forall \sigma \in \operatorname{Gal}(H_c/K)\}. \tag{3.53}$$

Finally, choose a nonzero differential $\omega_A \in \Omega^1(A/K)$, and write $\Omega_p(A)$ for the p-adic period attached to this choice as in Section 3.1. Since $A = B_{\psi_0}$ is the abelian variety attached to ψ_0, it follows that $\Omega_p(\psi_A^*) = \Omega_p(A)$.

The following theorem is a more precise variant of Theorem 3.30.

Theorem 3.33. *Let χ be a ring class character of K of conductor prime to $\mathfrak{d}_K$. Then there exists a point $P_A(\chi) \in (A(H_\chi) \otimes_{\mathcal{O}_K} E_\chi)^\chi$ such that*

$$\mathscr{L}_p(\psi_A^* \chi) = \Omega_p(A)^{-1} \mathfrak{g}(\chi) \log^2_{\omega_A}(P_A(\chi)) \pmod{E_\chi^\times}.$$

The point $P_A(\chi)$ is non-zero if and only if $L'(\psi_A \chi^{-1}, 1) \neq 0$.

Proof. By Theorem 2 of the Introduction,

$$\mathscr{L}_p(\psi_A^*\chi) = \mathscr{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1}\log^2_{\omega_\nu}(P_\nu) \pmod{E_\nu^\times}, \tag{3.54}$$

for some point $P_\nu \in B_\nu(K)\otimes\mathbb{Q}$ which is non-trivial if and only if $L'(\psi_A\chi^{-1},1)\neq 0$. Since $\chi^{*-1}=\chi$ and $E_\nu = E_\chi$, we find from Lemma 2.14 that

$$\Omega_p(\nu^*)^{-1} = \Omega_p(\psi_A^*\chi^{*-1})^{-1} = \Omega_p(A)^{-1}\mathfrak{g}(\chi)^{-1} \pmod{E_\chi^\times}. \tag{3.55}$$

After noting that (as in equation (2.7)) $B_\nu = B_{\psi,\chi} = (A\otimes_{\mathcal{O}_K} T_\chi)^{\chi^{-1}}$ as abelian varieties over K, we observe that $\omega_\nu = \omega_{\psi,\chi}$ and that the point $P_\nu \in B_\nu(K)$ can be written as

$$P_\nu = \sum_{\sigma\in G} P^\sigma \otimes \chi^{-1}(\sigma),$$

for some $P\in A(H_c)\otimes\mathbb{Q}$. Letting $P_{A,\chi}$ be the corresponding element in $A(H_c)\otimes_{\mathcal{O}_K} E_\chi$ given by

$$P_{A,\chi} = \sum_{\sigma\in G}\chi^{-1}(\sigma)P^\sigma,$$

we have

$$\log^2_{\omega_\nu}(P_\nu) = \log^2_{\omega_{\psi,\chi}}(P_\nu) = \mathfrak{g}(\chi)^2\log^2_{\omega^0_{\psi,\chi}}(P_\nu) = \mathfrak{g}(\chi)^2\log^2_{\omega_A}(P_{A,\chi}) \pmod{E_\chi^\times}, \tag{3.56}$$

where the second equality follows from Lemma 2.8 and the last from Lemma 3.22. Theorem 3.33 now follows by rewriting (3.54) using (3.55) and (3.56). □

In the special case where χ is a quadratic ring class character of K, cutting out an extension $L = K(\sqrt{a})$ of K, we obtain

$$\mathscr{L}_p(\psi_A^*\chi) = \Omega_p(A)^{-1}\sqrt{a}\log^2_{\omega_A}(P^-_{A,L}) \pmod{K^\times}, \tag{3.57}$$

where $P^-_{A,L}$ is a K-vector space generator of the trace 0 elements in $A(L)\otimes\mathbb{Q}$. Since in this case $\psi_A\chi$ is the Hecke character attached to a CM elliptic curve over $\mathbb{Q}$, from (3.57) one recovers Rubin's Theorem 1 of the Introduction.

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