

# The rationality of Stark-Heegner points over genus fields of real quadratic fields

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## Introduction

Let  $E$  be an elliptic curve over  $\mathbb{Q}$  of conductor  $N$ , and let  $f$  be the normalised cusp form of weight 2 on  $\Gamma_0(N)$  attached to  $E$ . Suppose that there is an odd

prime  $p \parallel N$  of multiplicative reduction for  $E$ , and let  $K$  be a real quadratic field satisfying the following modified Heegner hypothesis:

1. the prime  $p$  is inert in  $K$ ;
2. all the primes dividing  $M := N/p$  are split in  $K$ .

Fix an embedding of  $K$  and its  $p$ -adic completion  $K_p$  into  $\mathbb{C}_p := \widehat{\mathbb{Q}_p}$ , and let  $\mathcal{H}_p = \mathbb{C}_p - \mathbb{Q}_p$  denote the  $p$ -adic upper half plane. Since  $p$  is inert in  $K$ , note that  $\mathcal{H}_p \cap K$  is non-empty.

The first section of this paper briefly recalls the main construction of [Da01], which associates to the cusp form  $f$  and to any  $\tau \in \mathcal{H}_p \cap K$  a so-called *Stark-Heegner point*  $P_\tau \in E(K_p)$ . Conjecture 5.9 of [Da01] predicts that some integer multiple of this point is defined over a ring class field of  $K$  depending on  $\tau$ , and gives an explicit description, analogous to the Shimura reciprocity law, for the action of  $\text{Gal}(K^{\text{ab}}/K)$  on the collection of Stark-Heegner points attached to  $K$ .

The main result of the present article (Theorem 1) gives some evidence for Conjecture 5.9 of [Da01] by showing that certain integral linear combinations of Stark-Heegner points are global points on  $E$  defined over the expected abelian extension of  $K$ . The non-vanishing of these points is also related to the first derivative at  $s = 1$  of the Hasse-Weil  $L$ -series of  $E/K$  in the spirit of the Gross-Zagier formula, lending support for Conjecture 5.15 of [Da01].

Before stating Theorem 1 precisely, some further notations are required. Let  $M_2(\mathbb{Z}[1/p])$  denote the ring of  $2 \times 2$  matrices with entries in  $\mathbb{Z}[1/p]$ , and let  $R \subset M_2(\mathbb{Z}[1/p])$  denote the subring of matrices which are upper-triangular modulo  $M$ . The order associated to  $\tau \in \mathcal{H}_p \cap K$  is defined to be

$$\mathcal{O}_\tau = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R \text{ such that } a\tau + b = c\tau^2 + d\tau \right\} \subset K,$$

the inclusion on the right being the one that sends the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  to the element  $c\tau + d$ . Via this inclusion,  $\mathcal{O}_\tau$  is identified with a  $\mathbb{Z}[1/p]$ -order of  $K$ . Let  $D$  be the discriminant of a (not necessarily maximal) order of  $K$ , and let  $\mathcal{O}_D$  denote the  $\mathbb{Z}[1/p]$ -order of  $K$  of that discriminant. Set

$$\mathcal{H}_p^D = \{ \tau \in \mathcal{H}_p \cap K \text{ such that } \mathcal{O}_\tau = \mathcal{O}_D \}.$$

The group

$$\Gamma := \{ \gamma \in R^\times \text{ such that } \det(\gamma) = 1 \} \tag{1}$$

acts on  $\mathcal{H}_p$  by Möbius transformations, and preserves  $\mathcal{H}_p^D$  for any  $D$ .

Let  $G_D := \text{Pic}^+(\mathcal{O}_D)$  denote the Picard group of oriented  $\mathcal{O}_D$ -modules, or equivalently, the group of  $\mathbf{SL}_2(\mathbb{Z})$ -equivalence classes of primitive integral binary quadratic forms of discriminant  $D$  equipped with the group law given by Gaussian composition. Class field theory defines an isomorphism

$$\text{rec} : G_D \longrightarrow \text{Gal}(H_D/K),$$

where  $H_D$  is the so-called *narrow ring class field* attached to  $\mathcal{O}_D$ . The fact that  $p$  is inert in  $K$  implies that the prime ideal  $p\mathcal{O}_K$  splits completely in  $H_D/K$ . Choose a prime of  $H_D$  above  $p$ . This choice determines an extension to  $H_D$  of the chosen embedding  $K \longrightarrow \mathbb{C}_p$ .

The quotient  $\mathcal{H}_p^D/\Gamma$  is equipped with a natural action of  $G_D$  whose definition is recalled in Section 1.4, and which is written  $(g, \tau) \mapsto \tau^g$ , for  $g \in G_D$  and  $\tau \in \mathcal{H}_p^D/\Gamma$ .

Conjecture 5.9 of [Da01] predicts that some fixed multiple of  $P_\tau$  is a global point in  $E(H_D)$ , so that  $P_\tau$  belongs to  $E(H_D) \otimes \mathbb{Q}$ , and that

$$P_{\tau^g} = \text{rec}(g)^{-1}(P_\tau) \quad \text{for all } g \in G_D. \quad (2)$$

(Note that this compatibility does not depend on the choice of embedding of  $H_D$  into  $\mathbb{C}_p$  that was made.)

Suppose now that  $D$  is the discriminant of  $K$ . A *genus character* of  $K$  is a quadratic unramified character of  $\text{Gal}(\bar{K}/K)$ . Such a genus character  $\chi$  cuts out a biquadratic (or quadratic, in the special case where  $\chi$  is the trivial character) extension of  $\mathbb{Q}$ , denoted  $H_\chi$ :

$$H_\chi = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}), \quad \text{where } D = D_1 D_2.$$

Let  $\chi_1$ ,  $\chi_2$ , and  $\epsilon_K$  be the Dirichlet characters associated to the quadratic fields  $\mathbb{Q}(\sqrt{D_1})$ ,  $\mathbb{Q}(\sqrt{D_2})$ , and  $K$  respectively. Note that  $\chi_1 \chi_2 = \epsilon_K$ . The genus characters are in bijection with the factorisations of  $D$  into a product of two relatively prime fundamental discriminants, or, equivalently, with the unordered pairs of primitive quadratic Dirichlet characters  $(\chi_1, \chi_2)$  of coprime conductors satisfying  $\chi_1 \chi_2 = \epsilon_K$ . (The trivial character  $\chi$  corresponds to the factorisation  $D = 1 \cdot D$ .) Let  $E(H_\chi)^\chi$  denote the submodule of the Mordell-Weil group  $E(H_\chi)$  on which  $\text{Gal}(H_D/K)$  acts via the character  $\chi$ .

Define the point

$$P_\chi = \sum_{g \in G_D} \chi(g) P_{\tau^g} \in E(K_p).$$

Then Conjectures 5.9 and 5.15 of [Da01] imply that  $P_\chi$  is a global point in  $E(H_\chi)^\times$ .

Let  $q \in p\mathbb{Z}_p$  be the Tate period attached to  $E$ , and write

$$\Phi_{\text{Tate}} : K_p^\times / q^\mathbb{Z} \longrightarrow E(K_p)$$

for the Tate uniformisation (which is defined since  $p$  is inert in  $K$  so that  $E$  acquires split multiplicative reduction over  $K_p$ ). Let  $\log_q : K_p^\times \longrightarrow K_p$  denote the branch of the  $p$ -adic logarithm satisfying  $\log_q(q) = 0$ , and define a homomorphism

$$\log_E : E(K_p) \longrightarrow K_p$$

by the rule

$$\log_E(P) := \log_q(\Phi_{\text{Tate}}^{-1}(P)).$$

For each  $m|N$  with  $\gcd(m, N/m) = 1$ , let  $w_m$  denote the sign of the Fricke involution at  $m$  acting on  $f$ . Note that the modified Heegner hypothesis implies that  $\epsilon_K(-M) = 1$ , so that  $\chi_1(-M) = \chi_2(-M)$ . The main result of this article is

**Theorem 1.** *Let  $\chi$  be the genus character attached to the pair of Dirichlet characters  $\chi_1$  and  $\chi_2$ . Suppose that  $E$  has at least two primes of multiplicative reduction, and that  $\chi_1(-M) = -w_M$ .*

1. *There is a global point  $\mathbf{P}_\chi \in E(H_\chi)^\times$  and  $t \in \mathbb{Q}^\times$  such that*

$$\log_E(P_\chi) = t \log_E(\mathbf{P}_\chi). \tag{3}$$

2. *The point  $\mathbf{P}_\chi$  is of infinite order if and only if  $L'(E/K, \chi, 1) \neq 0$ .*

By way of providing a context for the proof of Theorem 1, we note the analogy between the approach that it follows and Kronecker’s “solution to Pell’s equation” in terms of special values of the Dedekind eta-function. (See Chapter IX of [We] for a historical account, and Chapter II.1 of [Sie] for a more detailed treatment.) In the classical setting considered by Kronecker, the fundamental discriminant  $D$  is taken to be *negative* so that it corresponds to an imaginary quadratic subfield  $K$  of  $\mathbb{C}$ . The  $p$ -adic upper half plane is replaced by its archimedean counterpart  $\mathcal{H}$ , and  $\mathcal{H}^D$  is given the obvious meaning (with  $M = 1$ ). This set is preserved under the action of  $\mathbf{SL}_2(\mathbb{Z})$  by Möbius transformations, and the quotient  $\mathcal{H}^D / \mathbf{SL}_2(\mathbb{Z})$  is equipped with a

natural action of the class group  $G_D$  of  $K$ . A quadratic character  $\chi$  of  $G_D$ —assumed to be non-trivial, although the trivial character requires no special consideration in the setting of Stark-Heegner points—corresponds to a pair of Dirichlet characters  $\chi_1$  and  $\chi_2$  which are even and odd respectively, cutting out quadratic extensions  $K_1$  and  $K_2$  of  $\mathbb{Q}$ . Let  $\epsilon_1 > 1$  be the fundamental unit of the real quadratic field  $K_1$ , denote by  $h_j$  (for  $j = 1, 2$ ) the class number of  $K_j$ , and write  $w_2$  for the number of roots of unity in  $K_2$ . After setting

$$\eta^*(\tau) := |D|^{-1/4} \sqrt{2y} |\eta(\tau)|^2,$$

Kronecker shows (cf. Theorem 6 of Chapter 11.1 of [Sie]) that for any  $\tau \in \mathcal{H}^D$ ,

$$\sum_{\sigma \in G_D} \chi(\sigma) \log \eta^*(\tau^\sigma) = -\frac{2h_1 h_2}{w_2} \log(\epsilon_1). \quad (4)$$

This expresses a solution to the Pell equation  $x^2 - D_1 y^2 = 1$  in terms of the function  $\eta^*$  evaluated at suitable quadratic imaginary arguments. Kronecker's proof is obtained by combining the following three ingredients:

1. The *Kronecker limit formula*, which expresses the left hand side of (4) in terms of the  $L$ -series  $\zeta(K, \chi, s) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) N(\mathfrak{a})^{-s}$ , where the sum is taken over all the ideals  $\mathfrak{a}$  of  $K$ :

$$\sum_{\sigma \in G_D} \chi(\sigma) \log \eta^*(\tau^\sigma) = \frac{d}{ds} \zeta(K, \chi, s) \Big|_{s=0}. \quad (5)$$

2. A *factorisation* of  $\zeta(K, \chi, s)$  as a product of the Dirichlet  $L$ -series attached to  $\chi_1$  and  $\chi_2$ :

$$\zeta(K, \chi, s) = L(\chi_1, s) L(\chi_2, s). \quad (6)$$

3. *Dirichlet's class number formula* which asserts that

$$L'(\chi_1, 0) = h_1 \log(\epsilon_1), \quad L(\chi_2, 0) = \frac{2h_2}{w_2}. \quad (7)$$

Kronecker's identity (4) is a direct consequence of (5), (6) and (7). It can also be understood in the framework of the theory of complex multiplication, which relates the individual quantities  $\eta^*(\tau^\sigma)$  to elliptic units defined over

$H_D$ . Kronecker’s approach is noteworthy in that it *makes no use* of the theory of complex multiplication. This represents an advantage in the setting of Stark-Heegner points attached to real quadratic fields, where no analogue of the theory of complex multiplication is known, and where the algebraicity of the individual quantities  $P_\tau$  remains conjectural.

The proof of Theorem 1 is explained in Section 4. At this stage we limit ourselves to some general remarks on the counterparts to steps 1, 2, and 3 above in our approach.

1. The role of Kronecker’s limit formula is played by Theorem 4.1, which relates the Stark-Heegner point  $P_\chi$  to the leading term of a *Hida  $p$ -adic  $L$ -function*  $L_p(f_\infty/K, \chi, k)$  attached to the datum of a Hida family  $\{f_\infty\}$  interpolating  $f$  in weight two. The relation that emerges between periods of Hida families and Stark-Heegner points represents a new insight that was suggested by combining the calculations in [Das] and [DD] with the main result of [BD98]. It is hoped that Theorem 4.1 may be of some independent interest beyond its role in the proof of Theorem 1.
2. The Hida  $L$ -function  $L_p(f_\infty/K, \chi, k)$  interpolates the central critical values  $L(f_k/K, \chi, k/2)$  of the weight  $k$  specialisations of  $f_\infty$ . This interpolation property is a direct consequence of a formula of Popa [Po] expressing these values in terms of certain “geodesic cycle integrals” attached to  $f$  and  $K$ , à la Shintani. This interpolation property is the key to expressing  $L_p(f/K, \chi, k)$  as a product of two Mazur-Kitagawa  $p$ -adic  $L$ -functions  $L_p(f_\infty, \chi_j, k, s)$  attached to  $\{f_\infty\}$  and the Dirichlet characters  $\chi_1$  and  $\chi_2$ , restricted to the central critical line  $s = k/2$ .
3. One is finally reduced to expressing the leading term in a neighbourhood of  $k = 2$  of  $L_p(f_\infty, \chi_j, k, k/2)$  in terms of rational quantities and logarithms of global points. This last ingredient is supplied by Theorem 5.4 of [BD05], whose precise formulation is recalled in Section 4, and whose proof relies on a  $p$ -adic analytic construction of (classical) Heegner points coming from Shimura curve parametrisations, via the Cerednik-Drinfeld theory of  $p$ -adic uniformisation of these curves. It is this reliance on parametrisations by Shimura curves over  $\mathbb{Q}$  which forces the assumption in Theorem 1 that  $E$  has at least two primes of multiplicative reduction.

# 1 A review of Stark-Heegner points

This section reviews the definition of Stark-Heegner points that is given in [Da01], presenting it in a way that is adapted to the subsequent proof of Theorem 1.

## 1.1 Modular symbols

Let  $g \in S_k(\Gamma_0(N))$  be a normalised cusp form of even weight  $k \geq 2$  on  $\Gamma_0(N)$ , and let  $K_g$  denote the finite extension of  $\mathbb{Q}$  generated by its Fourier coefficients  $a_n(g)$  ( $n \geq 1$ ). We view  $K_g$  as a subfield both of  $\mathbb{C}$  and  $\mathbb{C}_p$ , by fixing complex and  $p$ -adic embeddings of  $K_g$ .

The space  $\mathcal{P}_k(\mathbb{Q})$  of homogenous polynomials in two variables of degree  $k - 2$  with rational coefficients is equipped with a right action of  $\mathbf{GL}_2(\mathbb{Q})$  given by the rule

$$(P|\gamma)(x, y) := P(ax + by, cx + dy), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (8)$$

Let  $V_k(\mathbb{Q})$  denote the  $\mathbb{Q}$ -linear dual of  $\mathcal{P}_k(\mathbb{Q})$ . The field  $\mathbb{Q}$  can be replaced by any field  $F$  in these definitions. Denote by  $\mathcal{P}_k(F)$  the  $F$ -vector space of homogenous polynomials of degree  $k - 2$  with coefficients in  $F$ , and by  $V_k(F)$  its  $F$ -linear dual.

A *modular symbol* with values in an abelian group  $G$  is a function

$$I : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow G, \quad \text{denoted } (r, s) \mapsto I\{r \rightarrow s\},$$

satisfying

$$I\{r \rightarrow s\} + I\{s \rightarrow t\} = I\{r \rightarrow t\}, \quad \text{for all } r, s, t \in \mathbb{P}_1(\mathbb{Q}).$$

The periods of the form  $g$  are encoded in the basic  $V_k(\mathbb{C})$ -valued modular symbol  $\tilde{I}_g$  defined by

$$\tilde{I}_g\{r \rightarrow s\}(P) := 2\pi i \int_r^s g(z)P(z, 1)dz.$$

This symbol is invariant under  $\Gamma_0(N)$ , in the sense that

$$\tilde{I}_g\{\gamma r \rightarrow \gamma s\}(P|\gamma^{-1}) = \tilde{I}_g\{r \rightarrow s\}(P), \quad \text{for all } \gamma \in \Gamma_0(N).$$

The matrix  $c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  normalises  $\Gamma_0(N)$  and hence induces an involution on the space  $MS(V_k(\mathbb{C}))^{\Gamma_0(N)}$  of  $\Gamma_0(N)$ -invariant  $V_k(\mathbb{C})$ -valued modular symbols. Let  $\tilde{I}_g^+$  and  $\tilde{I}_g^-$  denote the plus and minus eigencomponents of  $\tilde{I}_g$  for this involution.

**Proposition 1.1.** *There exist complex periods  $\Omega_g^+$  and  $\Omega_g^-$  with the property that the modular symbols*

$$I_g^+ := (\Omega_g^+)^{-1} \tilde{I}_g^+, \quad I_g^- := (\Omega_g^-)^{-1} \tilde{I}_g^-$$

*belong to  $MS(V_k(K_g))$ . These periods can be chosen to satisfy*

$$\Omega_g^+ \Omega_g^- = \langle g, g \rangle,$$

*where  $\langle g, g \rangle$  is the Petersson scalar product of  $g$  with itself.*

*Proof.* The proof is explained, for example, in Section 1.1 of [KZ]. (See in particular the first corollary and the third theorem in that section).  $\square$

Choose a “sign at infinity”  $w_\infty \in \{+1, -1\}$ , and set

$$\Omega_g := \begin{cases} \Omega_g^+ & \text{if } w_\infty = +1; \\ \Omega_g^- & \text{if } w_\infty = -1; \end{cases} \quad I_g := \begin{cases} I_g^+ & \text{if } w_\infty = +1; \\ I_g^- & \text{if } w_\infty = -1. \end{cases}$$

Note that the modular symbol  $I_g$  can be viewed as an element of  $MS(V_k(\mathbb{C}_p))$  thanks to the chosen embedding of  $K_g$  into  $\mathbb{C}_p$ .

## 1.2 Double integrals

In the case where  $f$  is a modular form of weight 2 on  $\Gamma_0(N)$  with rational Fourier coefficients, the modular symbol  $I_f$  is  $\mathbb{Q}$ -valued, and can even be rescaled so that it takes values in  $\mathbb{Z}$ . Assume that this has been done from now on.

A measure on  $\mathbb{P}_1(\mathbb{Q}_p)$  is an element of the continuous  $\mathbb{Q}_p$ -linear dual of the space of continuous  $\mathbb{Q}_p$ -valued functions on  $\mathbb{P}_1(\mathbb{Q}_p)$ , equipped with the topology of uniform convergence. Given a measure  $\mu$ , and a compact open subset  $U$  of  $\mathbb{P}_1(\mathbb{Q}_p)$ , set  $\mu(U) := \mu(\chi_U)$ , where  $\chi_U$  denotes the characteristic function of  $U$ . The function  $U \mapsto \mu(U)$  is a bounded, finitely additive  $\mathbb{Q}_p$ -valued function on the set of compact open subsets of  $\mathbb{P}_1(\mathbb{Q}_p)$ . Conversely,



any such function gives rise to a  $\mathbb{Q}_p$ -valued measure on  $\mathbb{P}_1(\mathbb{Q}_p)$ . The measure  $\mu$  is said to be *integral*, or  *$\mathbb{Z}$ -valued*, if  $\mu(U)$  belongs to  $\mathbb{Z}$  for all compact open  $U \subset \mathbb{P}_1(\mathbb{Q}_p)$ .

Recall the group  $\Gamma$  of equation (1) in the introduction. Fix a subset  $\mathbb{P}$  of  $\mathbb{P}_1(\mathbb{Q})$  on which  $\Gamma$  acts transitively by Möbius transformations.

The following elementary proposition is key to the definition of Stark-Heegner points attached to real quadratic fields.

**Proposition 1.2.** *There exists a unique system of  $\mathbb{Z}$ -valued measures on  $\mathbb{P}_1(\mathbb{Q}_p)$ , indexed by  $r, s \in \mathbb{P}$  and denoted  $\mu_f\{r \rightarrow s\}$ , satisfying the following properties.*

1. For all  $r, s \in \mathbb{P}$ ,

$$\mu_f\{r \rightarrow s\}(\mathbb{P}_1(\mathbb{Q}_p)) = 0, \quad \mu_f\{r \rightarrow s\}(\mathbb{Z}_p) = I_f\{r \rightarrow s\}.$$

2. For all  $\gamma \in \Gamma$ , and all compact open  $U \subset \mathbb{P}_1(\mathbb{Q}_p)$ ,

$$\mu_f\{\gamma r \rightarrow \gamma s\}(\gamma U) = \mu_f\{r \rightarrow s\}(U).$$

*Proof.* The proof of this proposition is identical to that of Proposition 2.6 of [DD], which considered the case where the newform  $f$  is replaced by the logarithmic derivative of a modular unit, a weight two Eisenstein series. The main property of this Eisenstein series that is used is the fact that it is fixed by the Hecke operator  $U_p^2$ . Since this is also true of  $f$ , the proof of Proposition 2.6 can be adapted to the setting at hand with essentially no modifications.  $\square$

**Remark 1.3.** We have chosen to consider modular symbols defined on  $\mathbb{P} \times \mathbb{P}$ , largely for convenience: for example this will guarantee the uniqueness of the “indefinite integral” of Proposition 1.4.

The measures  $\mu_f$  can be used to define a “double multiplicative integral” attached to  $\tau_1, \tau_2 \in \mathcal{H}_p$  and  $r, s \in \mathbb{P}$  as in equation (71) of [Da01], by setting

$$\int_{\tau_1}^{\tau_2} \int_r^s \omega_f := \int_{\mathbb{P}_1(\mathbb{Q}_p)} \left( \frac{t - \tau_2}{t - \tau_1} \right) d\mu_f\{r \rightarrow s\}(t). \quad (9)$$

The “multiplicative integral” notation appearing on the right indicates that a limit of Riemann products is being taken, rather than a limit of Riemann

sums, i.e., that the integration is relative to the multiplicative structure on  $\mathbb{C}_p^\times$ . Such a definition is made possible by the fact that for each  $r, s \in \mathbb{P}$ , the measure  $\mu_f\{r \rightarrow s\}$  is  $\mathbb{Z}$ -valued, and that the integrand is a continuous  $K_p^\times$ -valued function on  $\mathbb{P}_1(\mathbb{Q}_p)$  relative to the natural topology on  $K_p^\times$ . (For a more detailed discussion of this multiplicative integral, and its basic properties, see the discussion following Lemma 1.10 in [Da01].)

Recall the Tate period  $q \in p\mathbb{Z}_p$  attached to  $E/\mathbb{Q}_p$ , and the branch  $\log_q$  of the  $p$ -adic logarithm sending  $q$  to 0. Define

$$\int_{\tau_1}^{\tau_2} \int_r^s \omega_f := \log_q \left( \int_{\tau_1}^{\tau_2} \int_r^s \omega_f \right). \quad (10)$$

Note that the definition of this ‘‘additive integral’’ differs somewhat from the definition that is given in [Da01], where the Iwasawa branch of the  $p$ -adic logarithm satisfying  $\log(p) = 0$  is used. (Cf. equation (73) of [Da01].)

### 1.3 Indefinite integrals

The following result justifies the choice of branch of  $p$ -adic logarithm that was made in (10).

**Proposition 1.4.** *There is a unique function from  $\mathcal{H}_p \times \mathbb{P} \times \mathbb{P}$  to  $\mathbb{C}_p$ , denoted*

$$(\tau, r, s) \mapsto \int_{\tau}^{\tau} \int_r^s \omega_f,$$

*satisfying*

1. For all  $\gamma \in \Gamma$ ,

$$\int_{\gamma\tau}^{\gamma\tau} \int_{\gamma r}^{\gamma s} \omega_f = \int_{\tau}^{\tau} \int_r^s \omega_f.$$

2. For all  $\tau_1, \tau_2 \in \mathcal{H}$ ,

$$\int_{\tau_2}^{\tau_2} \int_r^s \omega_f - \int_{\tau_1}^{\tau_1} \int_r^s \omega_f = \int_{\tau_1}^{\tau_2} \int_r^s \omega_f.$$

3. For all  $r, s, t \in \mathbb{P}$ ,

$$\int_{\tau}^{\tau} \int_r^t \omega_f + \int_{\tau}^{\tau} \int_t^s \omega_f = \int_{\tau}^{\tau} \int_r^s \omega_f.$$

*Proof.* The proof of this proposition is explained in Section 3.1 of [Da01], where it is reduced to the exceptional zero conjecture of Mazur, Tate and Teitelbaum proved by Greenberg and Stevens. (In the notations of Section 3.1 of [Da01], and in particular of equation (162), we have

$$\int_r^\tau \int_r^s \omega_f = \log_q(\eta_{f,\tau}\{r \rightarrow s\}).$$

A more direct proof, albeit one whose main idea can still be traced to the calculations of Greenberg and Stevens, can be obtained by specialising the approach described in [BDI] to weight 2 modular forms.  $\square$

The function which is characterised indirectly in Proposition 1.4 is called the *indefinite integral* attached to  $f$ . The articles [DG] and [DP] explain how Proposition 1.4 can be used to produce efficient algorithms for the numerical evaluation of the indefinite integral. Section 2.3 gives a direct formula for it in terms of the periods of the Hida family interpolating  $f$ , which is better adapted to the general calculations of this paper.

**Remark 1.5.** It is the *existence* of the indefinite integral that relies crucially on the branch of  $p$ -adic logarithm chosen in (10). Its *uniqueness* then follows by noting that the difference

$$\delta\{r \rightarrow s\} := \int_r^\tau \int_r^s \omega_f^{(1)} - \int_r^\tau \int_r^s \omega_f^{(2)}$$

of any two functions satisfying properties 1, 2 and 3 of Proposition 1.4 is independent of the choice of  $\tau$ , and hence defines a  $\Gamma$ -invariant  $\mathbb{C}_p$ -valued modular symbol on  $\mathbb{P} \times \mathbb{P}$ . Since  $\Gamma$  acts transitively on  $\mathbb{P}$ , such a symbol is determined by the homomorphism  $\varphi_\delta : \Gamma \rightarrow \mathbb{C}_p$  defined by choosing a base point  $r \in \mathbb{P}$  and setting  $\varphi_\delta(\gamma) = \delta\{r \rightarrow \gamma r\}$ . But this homomorphism is necessarily trivial, since  $\Gamma$  has finite abelianisation and  $\mathbb{C}_p$  is torsion-free.

**Remark 1.6.** Proposition 1.4 implies the existence of a lattice  $Q \subset \mathbb{C}_p^\times$  which is commensurable with  $q^\mathbb{Z}$ , and an “indefinite multiplicative integral”

$$\int_r^\tau \int_r^s \omega_f \in \mathbb{C}_p^\times / Q$$

satisfying the obvious multiplicative analogues of the properties listed in Proposition 1.4. The lattice  $Q$  appears as the obstruction to splitting a

two-cocycle in  $H^2(\Gamma, \mathbb{C}_p^\times)$  constructed in terms of the double multiplicative integral. Note that  $\log_q(Q) = 0$ , and that we can write

$$\int \int_r^\tau \omega_f = \log_q \left( \int \int_r^\tau \omega_f \right).$$

#### 1.4 Stark-Heegner points

Given  $\tau \in \mathcal{H}_p \cap K$ , let  $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  denote the unique generator for the stabiliser of  $\tau$  in  $\Gamma$  satisfying

$$c\tau + d > 1. \quad (11)$$

(Where in this inequality, we have made use of the fixed real embedding of  $K$ .) Associate to  $\tau$  a multiplicative and additive period by choosing any base point  $r \in \mathbb{P}$  and setting

$$J_\tau^\times := \int \int_r^\tau \omega_f \in K_p^\times / Q, \quad J_\tau := \log_q(J_\tau^\times) = \int \int_r^\tau \omega_f. \quad (12)$$

Since  $\Phi_{\text{Tate}}(Q)$  is contained in  $E(K_p)_{\text{tors}}$ , the image of  $J_\tau^\times$  under  $\Phi_{\text{Tate}}$  is well-defined in  $E(K_p) \otimes \mathbb{Q}$ , and is called the *Stark-Heegner point* attached to  $\tau$  and  $f$ :

$$P_\tau := \Phi_{\text{Tate}}(J_\tau^\times), \quad \text{so that } \log_E(P_\tau) = J_\tau. \quad (13)$$

Let  $D$  be the discriminant of  $K$ . The Heegner hypothesis imposed on  $K$  in the introduction implies the existence of an element  $\delta \in \mathbb{Z}/M\mathbb{Z}$  satisfying

$$\delta^2 \equiv D \pmod{M}.$$

Fix such a  $\delta$  once and for all. Let  $\mathcal{F}^D$  denote the set of primitive binary quadratic forms  $Ax^2 + Bxy + Cy^2$  of discriminant  $D$  satisfying

$$M|A, \quad B \equiv \delta \pmod{M}.$$

The set  $\mathcal{F}^D$  is preserved under the natural action of  $\Gamma_0(M)$ . Recall the class group  $G_D$  of  $\mathbf{SL}_2(\mathbb{Z})$ -equivalence classes of primitive binary quadratic forms of discriminant  $D$ . The natural map

$$\mathcal{F}^D / \Gamma_0(M) \longrightarrow G_D$$

obtained by sending the class of the quadratic form  $Q$  to its corresponding  $\mathbf{SL}_2(\mathbb{Z})$ -equivalence class is readily seen to be a bijection, so that  $\mathcal{F}^D/\Gamma_0(M)$  is endowed with the structure of a principal homogenous space under  $G_D$ . If  $Q(x, y)$  belongs to  $\mathcal{F}^D/\Gamma_0(M)$ , and  $\sigma$  to  $G_D$ , write  $Q^\sigma$  for the image of  $Q$  by  $\sigma$ . Let  $H_D$  denote the narrow Hilbert class field of  $K$ , whose Galois group is identified with  $G_D$ , and let

$$\text{rec} : G_D \longrightarrow \text{Gal}(H_D/K)$$

denote the isomorphism arising from the reciprocity law of global class field theory.

Given  $Q = Ax^2 + Bxy + Cy^2 \in \mathcal{F}^D/\Gamma_0(M)$ , let

$$\tau_Q := \frac{-B + \sqrt{D}}{2A} \tag{14}$$

be a fixed root of the quadratic polynomial  $Q(x, 1)$ . Note that  $\tau_Q$  belongs (via the fixed  $p$ -adic embedding of  $K$ ) to  $\mathcal{H}_p^D$ , and that its image in  $\mathcal{H}_p^D/\Gamma$  is well-defined. Given  $\sigma \in G_D$ , write  $\tau^\sigma \in \mathcal{H}_p^D/\Gamma$  for the root of any quadratic form in the  $\Gamma_0(M)$ -equivalence class of  $Q^\sigma$ . This definition gives a precise meaning to the conjectural equation (2) of the Introduction.

Since the definition of Stark-Heegner points is purely  $p$ -adic analytic, little can be said about the action of  $\text{Gal}(H_D/K)$  on these points independently of assertion (2) in the introduction. However, something unconditional can be asserted about the action of the Frobenius element at  $p$ , denoted  $\tau_p \in \text{Gal}(H_D/\mathbb{Q})$ . Since the prime  $p$  is inert in  $K$ , the element  $\tau_p$ , which is only defined up to conjugation, corresponds to a reflection in the generalised dihedral group  $\text{Gal}(H_D/\mathbb{Q})$ . Proposition 5.10 of [Da01] asserts the existence of an element  $\sigma_\tau$  in  $G_D$  satisfying

$$\tau_p(J_\tau) = -w_M J_{\tau\sigma_\tau}, \quad \tau_p(P_\tau) = w_N P_{\tau\sigma_\tau}. \tag{15}$$

Note the sign difference in the two equations, which arises from the fact that  $\tau_p$  does not commute with  $\Phi_{\text{Tate}}$  in general, but rather satisfies

$$\tau_p \Phi_{\text{Tate}} \tau_p = a_p \Phi_{\text{Tate}} = -w_p \Phi_{\text{Tate}}.$$

The element  $\sigma_\tau$  is denoted by  $\sigma$  in Proposition 5.10 of [Da01], but we have denoted it here by  $\sigma_\tau$  to emphasize its dependence on  $\tau$ . Indeed, replacing  $\tau$  by  $\tau^\alpha$ , for some  $\alpha \in G_D$ , one can see that

$$\sigma_{\tau^\alpha} = \sigma_\tau \alpha^{-2}. \tag{16}$$

This identity is a consequence of the fact that  $\tau_p$  does not commute with the elements of  $G_D$ , but rather satisfies

$$\tau_p \sigma = \sigma^{-1} \tau_p, \quad \text{for all } \sigma \in G_D.$$

Equation (16) shows that the image of  $\sigma_\tau$  in  $G_D/G_D^2$  is independent of  $\tau \in \mathcal{H}_p^D/\Gamma$ . Denote this element by  $\sigma$ , in keeping with the notations of Proposition 5.10 of [Da01].

It will be necessary to have a precise description of  $\sigma$ . To do this, we give a formula for  $\chi(\sigma)$ , as  $\chi$  runs over the characters of  $G_D/G_D^2$ . These characters are precisely the genus characters attached to the discriminant  $D$ , and correspond to pairs  $(\chi_1, \chi_2)$  of primitive quadratic Dirichlet characters of coprime conductor satisfying  $\chi_1 \chi_2 = \epsilon_K$ . (Cf. the discussion in the Introduction preceding the statement of Theorem 1.) The pair  $(\chi_1, \chi_2)$  is characterised by the property

$$\chi(\text{frob}_\lambda) = \chi(\text{frob}_{\bar{\lambda}}) = \chi_1(\ell) = \chi_2(\ell), \quad (17)$$

for any rational prime  $\ell$  which splits completely as a product  $\ell = \lambda \bar{\lambda}$  of prime ideals of  $K$  of norm  $\ell$ .

**Proposition 1.7.** *For any genus character  $\chi$  attached to the discriminant  $D$ , corresponding to the pair  $(\chi_1, \chi_2)$  of quadratic Dirichlet characters,*

$$\chi(\sigma) = \chi_1(-M).$$

*Proof.* In the notations of the proof of Proposition 5.10 of [Da01], the element  $\sigma$  corresponds to the class in  $\text{Pic}^+(\mathcal{O})$  of the element  $\mathfrak{c} = \infty \lambda_1^{e_1} \cdots \lambda_t^{e_t}$ , where  $M = \ell_1^{e_1} \cdots \ell_t^{e_t}$  is the factorisation of  $M$  into a product of distinct prime powers, and  $\lambda_j$  is some ideal of  $K$  above  $\ell_j$ . (More precisely,  $\lambda_j = (\ell_j, \sqrt{D} - \delta)$ , but note that the value of  $\mathfrak{c}$  modulo squares in  $G_D$  is independent of the exact value of the  $\lambda_j$ 's.) By equation (17),

$$\chi(\sigma) = \chi(\mathfrak{c}) = \chi_1(-M),$$

as was to be shown. □

## 2 Hida theory

### 2.1 Hida families

Let

$$\tilde{\Lambda} := \mathbb{Z}_p[[\mathbb{Z}_p^\times]], \quad \Lambda = \mathbb{Z}_p[[ (1 + p\mathbb{Z}_p)^\times ]]$$

denote the usual Iwasawa algebras, and let

$$\mathcal{X} = \text{hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) = \text{hom}(\tilde{\Lambda}, \mathbb{Z}_p) \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$$

be the space of continuous  $p$ -adic characters of  $\mathbb{Z}_p^\times$ , equipped with its natural topology. The space  $\mathcal{X}$  contains  $\mathbb{Z}$  as a dense subset by sending  $k \in \mathbb{Z}$  to the character  $\theta_k(x) := x^{k-2}$ . Note the shift by two in our convention, which means that  $k = 2$  corresponds to the augmentation map on  $\tilde{\Lambda}$ .

Following the discussion that is made in Section 1.2 of [BD05], we associate to  $f$  a so-called *Hida family*

$$f_\infty := \sum_{n=1}^{\infty} a_n(k)q^n.$$

This is a formal  $q$ -expansion with coefficients in the ring  $\mathcal{A}(U)$  of  $p$ -adic analytic functions on  $U$ , where  $U$  is an appropriate neighbourhood of  $2 \in \mathcal{X}$ . Assume for simplicity that  $U$  is contained in the residue disc of 2 modulo  $p-1$ , and let  $\mathbb{Z}^{\geq 2}$  denote the set of integers which are  $\geq 2$ . The formal  $q$ -expansion  $f_\infty$  is characterised by the following properties:

1. If  $k$  belongs to  $U \cap \mathbb{Z}^{\geq 2}$ , the  $q$ -expansion

$$f_k := \sum_{n=1}^{\infty} a_n(k)q^n$$

is a normalised eigenform of weight  $k$  on  $\Gamma_0(N)$ . For this reason it is referred to as the *weight  $k$  specialisation* of  $f_\infty$ .

2.  $f_2 = f$ .

Note in particular that the field  $K_{f_k}$  generated by the Fourier coefficients of the normalised eigenform  $f_k$  is a finite extension of  $\mathbb{Q}$ . For each  $k \in U \cap \mathbb{Z}^{\geq 2}$ , we fix the Shimura periods  $\Omega_k^+ := \Omega_{f_k}^+$  and  $\Omega_k^- := \Omega_{f_k}^-$  as in Proposition 1.1. This choice of periods allows us to talk about the  $V_k(\mathbb{C}_p)$ -valued modular symbols  $I_{f_k}^+$  and  $I_{f_k}^-$  associated to each  $f_k$ . The modular symbol  $I_{f_k}$  will be taken to be  $I_{f_k}^+$  or  $I_{f_k}^-$  depending on the choice of  $w_\infty$  that was made.

## 2.2 Periods attached to Hida families

Let  $L_* := \mathbb{Z}_p^2$  denote the standard  $\mathbb{Z}_p$ -lattice in  $\mathbb{Q}_p^2$ , and let  $L'_*$  denote its set of *primitive vectors*, i.e., the vectors in  $L_*$  which are not divisible by  $p$ . Let  $\mathbb{D}$  denote the space of compactly supported  $p$ -adic measures on  $\mathcal{W} := \mathbb{Q}_p^2 - \{0\}$ , and let  $\mathbb{D}_*$  denote the subspace of measures that are supported on  $L'_*$ . The action of the group  $\mathbb{Z}_p^\times$  on  $\mathcal{W}$  and  $L'_*$  given by  $\lambda(x, y) = (\lambda x, \lambda y)$  gives rise to  $\tilde{\Lambda}$  and  $\Lambda$  module structures on  $\mathbb{D}$  and  $\mathbb{D}_*$ . The module  $\mathbb{D}$  is also equipped with a right  $\tilde{\Lambda}$ -linear action of  $\mathbf{GL}_2(\mathbb{Q}_p)$  defined by the rule extending (8)

$$\int_{\mathcal{W}} F d(\mu|\gamma) = \int_{\mathcal{W}} (F|\gamma^{-1}) d\mu, \quad (18)$$

where  $\mathbf{GL}_2(\mathbb{Q}_p)$  operates on the continuous functions on  $\mathcal{W}$  by the rule

$$(F|\gamma)(x, y) := F(ax + by, cx + dy), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (19)$$

Denote by  $\Gamma_0(p\mathbb{Z}_p)$  the group of matrices in  $\mathbf{GL}_2(\mathbb{Z}_p)$  which are upper triangular modulo  $p$ . Our interest in the space  $\mathbb{D}_*$  lies in the fact that it is equipped, for all  $k \in \mathbb{Z}^{\geq 2}$ , with a  $\Gamma_0(p\mathbb{Z}_p)$ -equivariant homomorphism

$$\rho_k : \mathbb{D}_* \longrightarrow V_k$$

defined by

$$\rho_k(\mu)(P) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x, y) d\mu(x, y).$$

(Note that  $\rho_k$  does not respect the full action of  $\mathbf{GL}_2(\mathbb{Z}_p)$ , because the domain of integration that appears in its definition is only preserved by  $\Gamma_0(p\mathbb{Z}_p)$ .) The homomorphism  $\rho_k$  gives rise to a homomorphism, denoted by the same letter by abuse of notation:

$$\rho_k : MS_{\Gamma_0(M)}(\mathbb{D}_*) \longrightarrow MS_{\Gamma_0(N)}(V_k(\mathbb{C}_p)).$$

The space  $MS_{\Gamma_0(M)}(\mathbb{D}_*)$  is equipped with a natural action of the Hecke operators, and of the operator  $U_p$  in particular. Let  $MS_{\Gamma_0(M)}^{\text{ord}}(\mathbb{D}_*)$  denote the *ordinary subspace* of  $MS_{\Gamma_0(M)}(\mathbb{D}_*)$ .

Proposition (6.1) of [GS] asserts that this module is free and of finite rank over  $\Lambda$ .



Let  $\Lambda^\dagger$  denote the ring of functions which can be represented by a convergent power series expansion in some neighbourhood of  $2 \in \mathcal{X}$ , and set

$$\mathbb{D}_*^\dagger := \mathbb{D}_* \otimes_\Lambda \Lambda^\dagger, \quad MS_{\Gamma_0(M)}^{\text{ord}}(\mathbb{D}_*)^\dagger := MS_{\Gamma_0(M)}^{\text{ord}}(\mathbb{D}_*) \otimes_\Lambda \Lambda^\dagger.$$

Similar notations are adopted, with the obvious meanings, when  $\mathbb{D}_*$  is replaced by  $\mathbb{D}$ . If  $\mu = \lambda_1\mu_1 + \cdots + \lambda_t\mu_t$  is any element of  $\mathbb{D}^\dagger$ , then there exists a neighbourhood  $U_\mu$  of  $2 \in \mathcal{X}$  on which all the  $\lambda_j$  are defined. Call such a region a *neighbourhood of regularity* for  $\mu$ . Given  $k \in \mathbb{Z}$ , a function  $F$  on  $\mathcal{W}$  is said to be *homogeneous of degree  $k$*  if  $F(\lambda x, \lambda y) = \lambda^k F(x, y)$ . Observe that for any  $k \in U_\mu \cap \mathbb{Z}^{\geq 2}$ , and any homogenous function  $F(x, y)$  of degree  $k - 2$ , one can integrate  $F$  against  $\mu$  on any compact open region  $X \subset \mathcal{W}$  by the rule

$$\int_X F d\mu := \lambda_1(k) \int_X F d\mu_1 + \cdots + \lambda_t(k) \int_X F d\mu_t.$$

The following result of Greenberg and Stevens plays a key role in the constructions of this section.

**Theorem 2.1.** *There exists a  $\mathbb{D}_*^\dagger$ -valued modular symbol  $\mu_* \in MS_{\Gamma_0(M)}^{\text{ord}}(\mathbb{D}_*)^\dagger$  such that*

1.  $\rho_2(\mu_*) = I_f$ ;
2. For all  $k \in U_{\mu_*} \cap \mathbb{Z}^{\geq 2}$ , there exists a scalar  $\lambda(k) \in \mathbb{C}_p$  such that

$$\rho_k(\mu_*) = \lambda(k) I_{f_k}.$$

*Proof.* This is Theorem 1.5 of [BD05], which follows from Theorem (5.13) of [GS] and whose proof is explained in Section 6 of that paper. See also [BDI] where extensions of this result to modular forms that are not necessarily ordinary are discussed.  $\square$

Since  $R^\times$  acts transitively on the set of  $\mathbb{Z}_p$ -lattices in  $\mathbb{Q}_p^2$ , and the stabiliser of  $L_*$  for this action is precisely  $\Gamma_0(M)$ , we may define a collection of  $\mathbb{D}^\dagger$ -valued modular symbols  $\mu_L$  indexed by the  $\mathbb{Z}_p$ -lattices in  $\mathbb{Q}_p^2$  as in Proposition 1.8 of [BD05], by imposing the rules:

$$\mu_{L_*} = \mu_*, \quad \int_{\gamma X} (F|\gamma^{-1}) d\mu_{\gamma L} \{\gamma r \rightarrow \gamma s\} = \int_X F d\mu_L \{r \rightarrow s\},$$

for all  $\gamma \in \Gamma$ , for all lattices  $L \subset \mathbb{Q}_p^2$ , and all compact open regions  $X \subset \mathcal{W}$ .

Let

$$U := U_{\mu_*} = U_{\mu_L}$$

denote a region of regularity for the measures  $\mu_L$ . The measures  $\mu_L\{r \rightarrow s\}$  are supported on the compact subsets  $L'$  of  $\mathcal{W}$ , and they satisfy the following property:

**Lemma 2.2.** *Suppose that  $L_2 \subset L_1$  is a sublattice of index  $p$  in  $L_1$ . For all  $k \in U \cap \mathbb{Z}^{\geq 2}$ , for all homogenous polynomials  $\tilde{P}(x, y)$  of degree  $k-2$ , and for all  $r, s \in \mathbb{P}$ , we have*

$$\int_{L'_1 \cap L'_2} \tilde{P} d\mu_{L_2}\{r \rightarrow s\} = a_p(k) \int_{L'_1 \cap L'_2} \tilde{P} d\mu_{L_1}\{r \rightarrow s\}.$$

*Proof.* See Lemma 1.10 of [BD05]. □

The basic property of the measures  $\mu_*\{r \rightarrow s\}$  given in Theorem 2.1 can be re-written as

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x, y) d\mu_*\{r \rightarrow s\}(x, y) = \lambda(k) I_{f_k}\{r \rightarrow s\}(P), \quad (20)$$

for all  $P \in \mathcal{P}_k(\mathbb{Q})$ , and for all  $r, s \in \mathbb{P}$ . It will be useful to understand the value of the the integral appearing on the left in (20), when the region of integration is taken to be the full  $L'_* = (\mathbb{Z}_p^2)'$  instead of the subset  $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ .

To write down such a formula, we first remark that for each integer  $k > 2$ , the form  $f_k$  is old at  $p$ , and there is a unique normalised eigenform  $f_k^\sharp$  of weight  $k$  on  $\Gamma_0(M)$  satisfying

$$f_k(z) = f_k^\sharp(z) - p^{k-1} a_p(k)^{-1} f_k^\sharp(pz).$$

(By convention, we set  $f_2^\sharp := 0$ .) Let  $I_{f_k^\sharp}$  be the modular symbol attached to  $f_k^\sharp$  via the choice of complex period  $\Omega_k$ . This modular symbol satisfies the relation

$$I_{f_k}\{r \rightarrow s\}(P) = I_{f_k^\sharp}\{r \rightarrow s\}(P) - p^{k-2} a_p(k)^{-1} I_{f_k^\sharp}\{r/p \rightarrow s/p\}(P(x, y/p)).$$

**Proposition 2.3.** *For all  $k \in U \cap \mathbb{Z}^{\geq 2}$ , for all  $P \in \mathcal{P}_k(\mathbb{C}_p)$  and for all  $r, s \in \mathbb{P}$ ,*

$$\int_{(\mathbb{Z}_p^2)'} P(x, y) d\mu_*\{r \rightarrow s\}(x, y) = \lambda(k) (1 - a_p(k)^{-2} p^{k-2}) I_{f_k^\sharp}\{r \rightarrow s\}(P).$$

*Proof.* Let  $L_*$  and  $L_\infty$  denote the lattices

$$L_* = \mathbb{Z}_p^2, \quad L_\infty = \mathbb{Z}_p \oplus p\mathbb{Z}_p,$$

so that

$$(L'_* \cap L'_\infty) = \mathbb{Z}_p^\times \times p\mathbb{Z}_p, \quad \left( L'_* \cap \frac{1}{p}L'_\infty \right) = \mathbb{Z}_p \times \mathbb{Z}_p^\times. \quad (21)$$

Let  $\theta \in R^\times$  be any matrix of determinant  $p$  satisfying

$$\theta(L_*) = L_\infty, \quad \theta(L_\infty) = pL_*, \quad \text{so that } \theta(\mathbb{Z}_p^\times \times p\mathbb{Z}_p) = p(\mathbb{Z}_p \times \mathbb{Z}_p^\times). \quad (22)$$

Observe that  $L'_*$  can be written as a disjoint union of the two regions appearing in (21). Hence we may write

$$\int_{(\mathbb{Z}_p^2)'} P(x, y) d\mu_* \{r \rightarrow s\}(x, y) = J_1 + J_2,$$

where

$$\begin{aligned} J_1 &= \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P d\mu_f \{r \rightarrow s\} = \lambda(k) I_{f_k} \{r \rightarrow s\}(P), \\ J_2 &= \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} \tilde{P} d\mu_f \{r \rightarrow s\} \\ &= \int_{p(\mathbb{Z}_p \times \mathbb{Z}_p^\times)} (\tilde{P}|\theta^{-1}) d\mu_{L_\infty} \{\theta r \rightarrow \theta s\} \\ &= p^{k-2} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} (\tilde{P}|\theta^{-1}) d\mu_{\frac{1}{p}L_\infty} \{\theta r \rightarrow \theta s\} \\ &= a_p(k)^{-1} p^{k-2} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} (\tilde{P}|\theta^{-1}) d\mu_{L_*} \{\theta r \rightarrow \theta s\} \\ &= a_p(k)^{-1} p^{k-2} \lambda(k) I_{f_k} \{\theta r \rightarrow \theta s\}(P|\theta^{-1}), \end{aligned}$$

and the penultimate equality follows from Lemma 2.2. To evaluate the contributions  $J_1$  and  $J_2$  in terms of the form  $f_k^\sharp$ , note that, for any choice of  $\theta$  satisfying (22),

$$f_k(z) = f_k^\sharp(z) - p^{k-1} a_p(k)^{-1} (f_k^\sharp|\theta)(z).$$

A direct calculation, using change of variables, then shows that

$$I_{(f_k^\sharp|\theta)} \{r \rightarrow s\}(P) = p^{-1} I_{f_k^\sharp} \{\theta r \rightarrow \theta s\}(P|\theta^{-1}).$$

Hence

$$\begin{aligned} J_1 &= \lambda(k) \left( I_{f_k^\#} \{r \rightarrow s\}(P) - p^{k-1} a_p(k)^{-1} I_{(f_k^\#|\theta)} \{r \rightarrow s\}(P) \right) \\ &= \lambda(k) \left( I_{f_k^\#} \{r \rightarrow s\}(P) - p^{k-2} a_p(k)^{-1} I_{f_k^\#} \{\theta r \rightarrow \theta s\}(P|\theta^{-1}) \right), \end{aligned}$$

while

$$\begin{aligned} J_2 &= \lambda(k) a_p(k)^{-1} p^{k-2} \left( I_{f_k^\#} \{\theta r \rightarrow \theta s\}(P|\theta^{-1}) \right. \\ &\quad \left. - p^{k-2} a_p(k)^{-1} I_{f_k^\#} \{r \rightarrow s\}(P|\theta^{-2}) \right). \end{aligned}$$

Note that  $\theta^{-2} = p^{-1}$ , so that  $P|\theta^{-2} = p^{-k+2}P$ . Therefore

$$J_1 + J_2 = \lambda(k) (1 - a_p(k)^{-2} p^{k-2}) I_{f_k^\#} \{r \rightarrow s\}(P).$$

The proposition follows. □

### 2.3 Indefinite integrals revisited

Recall the indefinite integral that was defined in Section 1.3. The relevance of Hida families to Stark-Heegner points can be explained by the fact that the system of distribution-valued modular symbols  $\mu_L \{r \rightarrow s\}$  that was introduced in Section 2.2 can be used to give a direct formula for this indefinite integral.

We content ourselves with doing this when  $\tau$  belongs to  $\mathcal{H}_p \cap K_p$ , so that it is defined over a quadratic unramified extension of  $\mathbb{Q}_p$ . In that case, the function

$$(x, y) \mapsto x - \tau y$$

identifies  $\mathbb{Q}_p^2$  with  $K_p$ . Let  $L_\tau$  be the  $\mathbb{Z}_p$ -lattice in  $\mathbb{Q}_p^2$  defined by

$$L_\tau = \{(x, y) \text{ such that } x - \tau y \text{ belongs to } \mathcal{O}_K \otimes \mathbb{Z}_p\}.$$

**Theorem 2.4.** *For all  $\tau \in \mathcal{H}_p \cap K_p$ , and for all  $r, s \in \mathbb{P}$ ,*

$$\int_r^\tau \int_r^s \omega_f = \int_{L_\tau'} \log(x - \tau y) d\mu_{L_\tau} \{r \rightarrow s\}(x, y),$$

where  $\log : K_p^\times \rightarrow K_p$  is any branch of the  $p$ -adic logarithm.

*Proof.* Note that the expression  $(x - \tau y)$  belongs to  $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times$  for any  $(x, y) \in L'_\tau$ , so that the integrand on the right is independent of the branch of the  $p$ -adic logarithm that was chosen to define it. To prove Theorem 2.4, it suffices to check that the three defining properties of the indefinite integral listed in Proposition 1.4 are satisfied by the expression appearing on the right in Theorem 2.4. The invariance under  $\Gamma$  stated as property 1 is a consequence of Proposition 4.6 of [BDI]. As for property 2, it follows from Proposition 4.7 of [BDI], which holds for any branch of the  $p$ -adic logarithm, including  $\log_q$  for which the extra term appearing in Proposition 4.7 of [BDI] vanishes. Finally, property 3 is a direct consequence of the definitions.  $\square$

**Corollary 2.5.** *The Stark-Heegner point  $P_\tau$  associated to  $Q \in \mathcal{F}^D/\Gamma_0(M)$  satisfies*

$$\log_E P_\tau = J_\tau = \int_{(\mathbb{Z}_p^2)'} \log(x - \tau y) d\mu_*\{r \rightarrow \gamma_\tau r\}(x, y).$$

*Proof.* This follows from (13) and Theorem 2.4, after noting that  $L_\tau = \mathbb{Z}_p^2$  when  $\tau = \tau_Q$  and  $Q$  is an element of  $\mathcal{F}^D$ .  $\square$

### 3 $p$ -adic $L$ -functions

#### 3.1 The Mazur-Kitagawa $p$ -adic $L$ -function

Let  $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \{\pm 1\}$  be a primitive quadratic Dirichlet character of conductor  $m$  with  $\chi(-1) = w_\infty$ , and let

$$\tau(\chi) := \sum_{a=1}^m \chi(a) e^{2\pi i a/m}$$

denote the Gauss sum attached to  $\chi$ . For each  $k \in U \cap \mathbb{Z}^{\geq 2}$ , and  $1 \leq j \leq k-1$  with  $j$  odd, the expression

$$L^*(f_k, \chi, j) := \frac{(j-1)! \tau(\chi)}{(-2\pi i)^{j-1} \Omega_k} L(f_k, \chi, j) \tag{23}$$

belongs to  $K_{f_k}$ ; it is called the *algebraic part* of the special value  $L(f_k, \chi, j)$ . Recall that the period  $\Omega_k$  that appears in the definition of  $L^*(f_k, \chi, j)$  was

chosen at the end of Section 2.1 and depends on the sign  $w_\infty$  that was fixed in that section, and therefore on the parity of  $\chi$ .

One defines  $L^*(f_k^\sharp, \chi, j)$  similarly, by replacing  $f_k$  by  $f_k^\sharp$  in the definition above, so that

$$L^*(f_k, \chi, j) = (1 - \chi(p)a_p(k)^{-1}p^{k-1-j})L^*(f_k^\sharp, \chi, j).$$

The measures  $\mu_*\{r \rightarrow s\}$  can be used to define the *Mazur-Kitagawa two-variable  $p$ -adic  $L$ -function* of  $(k, s) \in U \times \mathcal{X}$  by the rule:

$$L_p(f_\infty, \chi, k, s) = \sum_{a=1}^m \chi(pa) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \left(x - \frac{pa}{m}y\right)^{s-1} y^{k-s-1} d\mu_* \left\{ \infty \rightarrow \frac{pa}{m} \right\}.$$

This function satisfies the following interpolation property with respect to special values of the classical  $L$ -functions  $L^*(f_k, \chi, j)$ .

**Theorem 3.1.** *Suppose that  $k \in U \cap \mathbb{Z}^{\geq 2}$ , and that  $1 \leq j \leq k-1$  satisfies  $\chi(-1) = (-1)^{j-1}w_\infty$ . Then*

$$L_p(f_\infty, \chi, k, j) = \lambda(k)(1 - \chi(p)a_p(k)^{-1}p^{j-1})L^*(f_k, \chi, j).$$

*Proof.* See Theorem 1.12 of [BD05]. □

It will be useful to have a formula expressing  $L_p(f_\infty, \chi, k, j)$  in terms of the special value  $L^*(f_k^\sharp, \chi, j)$ . Theorem 3.1 implies that

$$L_p(f_\infty, \chi, k, j) = \lambda(k)(1 - \chi(p)a_p(k)^{-1}p^{j-1})(1 - \chi(p)a_p(k)^{-1}p^{k-1-j})L^*(f_k^\sharp, \chi, j).$$

In particular, specialising at  $j = k/2$  one obtains

**Corollary 3.2.** *Suppose that  $\chi$  satisfies*

$$\chi(-1) = (-1)^{k/2-1}w_\infty.$$

*Then for all  $k \in U \cap \mathbb{Z}^{\geq 2}$ ,*

$$L_p(f_\infty, \chi, k, k/2) = \lambda(k)(1 - \chi(p)a_p(k)^{-1}p^{\frac{k}{2}-1})^2 L^*(f_k^\sharp, \chi, j).$$

Note that the Euler factor appearing in this last expression is a perfect square.

### 3.2 $p$ -adic $L$ -functions attached to real quadratic fields

Given  $Q = Ax^2 + Bxy + Cy^2$  in  $\mathcal{F}^D$ , let  $\tau = \tau_Q$  and  $\bar{\tau}$  be the roots of the quadratic polynomial  $Q(x, 1)$ , ordered as in (14). These belong to  $K$ , and can be viewed as elements of  $\mathbb{C}_p$  via the chosen embedding of  $K$  into  $\mathbb{C}_p$ .

The quadratic form  $Q$  has a stabiliser in  $\Gamma_0(M)$  of rank one, generated by the element  $\gamma_\tau \in \Gamma_0(M)$  normalised as in (11). Let  $\epsilon = c\tau + d$  denote the corresponding fundamental unit of  $K$ . By analogy with the definition of the Mazur-Kitagawa  $p$ -adic  $L$ -function, it is tempting to associate to  $f_\infty$  and  $Q$  a “two-variable  $p$ -adic  $L$ -function” by the rule:

$$\mathcal{L}_p(f_\infty, Q, k, s) := A^{\frac{k}{2}-1} \int_{(\mathbb{Z}_p^2)'} (x - \tau y)^{s-1} (x - \bar{\tau} y)^{k-s-1} d\mu_*\{r \rightarrow \gamma_\tau r\}(x, y).$$

Note that this expression depends on the choice of base point  $r$  in a crucial way: more precisely, replacing  $r$  by  $r' \in \mathbb{P}$  has the effect of modifying  $\mathcal{L}_p(f_\infty, Q, k, s)$  by the term

$$A^{\frac{k}{2}-1} (1 - \epsilon^{2s-k}) \int_{(\mathbb{Z}_p^2)'} (x - \tau y)^{s-1} (x - \bar{\tau} y)^{k-s-1} d\mu_*\{r \rightarrow r'\}(x, y).$$

It follows that the restriction of  $\mathcal{L}_p(f_\infty, Q, k, s)$  to the *central critical line*  $s = k/2$  is independent of the choice of  $r$ . This motivates the following definitions.

**Definition 3.3.** Let  $r \in \mathbb{P}$  be any base point. The *partial square root  $p$ -adic  $L$ -function* attached to  $f_\infty$  and  $Q$  is the function of  $k \in U$  defined by

$$\mathcal{L}_p(f_\infty, Q, k) := \int_{(\mathbb{Z}_p^2)'} Q(x, y)^{\frac{k-2}{2}} d\mu_*\{r \rightarrow \gamma_\tau r\}(x, y).$$

Let  $\chi$  be a fixed genus character of  $G_D$ . This character is said to be *even* (resp. *odd*) if it cuts out a totally real, resp. imaginary, quadratic extension of  $K$ . If  $\chi$  is even (resp. odd), then the associated Dirichlet characters  $\chi_1$  and  $\chi_2$  are both even (resp. odd). Recall the sign at infinity  $w_\infty$  that was chosen in defining the modular symbols  $I_{f_k}$  and in choosing the Shimura period  $\Omega_k$ .

**Definition 3.4.** Let  $\chi$  be a genus character. Assume that  $w_\infty = 1$  if  $\chi$  is even, and that  $w_\infty = -1$  if  $\chi$  is odd.

1. The *square root  $p$ -adic  $L$ -function* attached to  $f_\infty$  and  $\chi$  is the function of  $k \in U$  defined by

$$\mathcal{L}_p(f_\infty/K, \chi, k) := \sum_{\sigma \in G_D} \chi(\sigma) \mathcal{L}_p(f_\infty, Q^\sigma, k).$$

2. The  *$p$ -adic  $L$ -function* attached to  $f_\infty$  and  $\chi$  is the function of  $k \in U$  defined by

$$L_p(f_\infty/K, \chi, k) := \mathcal{L}_p(f_\infty/K, \chi, k)^2.$$

We now prove the interpolation property for  $L_p(f_\infty/K, \chi, k)$  which justifies its designation as a  $p$ -adic  $L$ -function.

**Theorem 3.5.** *For all  $k \in U \cap \mathbb{Z}^{\geq 2}$ ,*

$$L_p(f_\infty/K, \chi, k) = \lambda(k)^2 (1 - a_p(k)^{-2} p^{k-2})^2 D^{\frac{k-2}{2}} L^*(f_k^\# / K, \chi, k/2),$$

where

$$L^*(f_k^\# / K, \chi, k/2) = \frac{(\frac{k}{2} - 1)!^2 \sqrt{D}}{(2\pi i)^{k-2} \Omega_k^2} L(f_k^\# / K, \chi, k/2). \quad (24)$$

*Proof.* By definition, we have:

$$\begin{aligned} \mathcal{L}_p(f_\infty/K, Q, k) &= \int_{(\mathbb{Z}_p^2)'} Q(x, y)^{\frac{k-2}{2}} d\mu_* \{r \rightarrow \gamma_Q r\}(x, y) \\ &= \lambda(k) (1 - a_p(k)^{-2} p^{k-2}) I_{f_k^\#} \{r \rightarrow \gamma_Q r\}(Q^{\frac{k-2}{2}}), \end{aligned}$$

where the last equality follows from Proposition 2.3. Hence

$$\begin{aligned} L_p(f_\infty/K, \chi, k) &= \lambda(k)^2 (1 - a_p(k)^{-2} p^{k-2})^2 \left( \sum_{\sigma \in G_D} \chi(\sigma) I_{f_k^\#} \{r \rightarrow \gamma_{Q^\sigma} r\} ((Q^\sigma)^{\frac{k-2}{2}}) \right)^2 \\ &= \lambda(k)^2 (1 - a_p(k)^{-2} p^{k-2})^2 (2\pi i)^2 \Omega_k^{-2} \mathbb{L}_\chi, \end{aligned} \quad (25)$$

where

$$\mathbb{L}_\chi = \left( \sum_{\sigma \in G_D} \chi(\sigma) \int_{z_0}^{\gamma_Q z_0} f_k^\#(z) Q^\sigma(z, 1)^{\frac{k-2}{2}} dz \right)^2.$$



The crucial ingredient in the proof of Theorem 3.5 is Theorem 6.3.1 of [Po], which asserts that

$$\mathbb{L}_\chi = D^{\frac{k-1}{2}} (2\pi)^{-k} \left( \frac{k-2}{2} \right)!^2 L(f_k^\sharp/K, \chi, k/2). \quad (26)$$

It follows readily from (25) and (26) that

$$L_p(f_\infty/K, \chi, k) = \lambda(k)^2 (1 - a_p(k)^{-2} p^{k-2})^2 D^{\frac{k-2}{2}} L^*(f_k^\sharp/K, \chi, k/2),$$

as was to be shown.  $\square$

### 3.3 A factorisation formula

We come to the following factorisation relating the Mazur-Kitagawa  $p$ -adic  $L$ -functions (more precisely, their restriction to the central critical line) with the  $p$ -adic  $L$ -functions attached to the real quadratic field  $K$  in the previous section.

**Theorem 3.6.** *For all  $k \in U$ ,*

$$L_p(f_\infty/K, \chi, k) = D^{\frac{k-2}{2}} L_p(f_\infty, \chi_1, k, k/2) L_p(f_\infty, \chi_2, k, k/2).$$

*Proof.* By comparing definitions (23) and (24), we see that

$$L^*(f_k^\sharp/K, \chi, k/2) = L^*(f_k^\sharp, \chi_1, k/2) L^*(f_k^\sharp, \chi_2, k/2).$$

Also, because  $p$  is inert in  $K$ , the Dirichlet characters  $\chi_1$  and  $\chi_2$  satisfy  $\chi_1(p) = -\chi_2(p)$ , and hence the product of the Euler factors appearing in Corollary 3.2 with  $\chi$  replaced by  $\chi_1$  and  $\chi_2$  is equal to the Euler factor appearing in Theorem 3.5. Hence Theorem 3.5 and Corollary 3.2 imply that for all  $k \in U \cap \mathbb{Z}^{\geq 2}$ ,

$$L_p(f_\infty/K, \chi, k) = D^{\frac{k-2}{2}} L_p(f_\infty, \chi_1, k, k/2) L_p(f_\infty, \chi_2, k, k/2). \quad (27)$$

Since  $U \cap \mathbb{Z}^{\geq 2}$  is dense in  $U$ , and the two sides of (27) are continuous on  $U$ , they necessarily agree on this region.  $\square$

## 4 Proof of the main result

We begin by noting the following connection between the Stark-Heegner point  $P_\chi$  and the leading term of the  $p$ -adic  $L$ -function that was introduced in Section 3.2. This relation can be viewed as being in the spirit of a Gross-Zagier formula since it relates Stark-Heegner points to derivatives of  $L$ -series. It is even more closely in the spirit of the main Theorem of [BD98] and its extension to Hida families stated in Theorem 4.9 of [BD05].

**Theorem 4.1.** *For all genus characters  $\chi$  of  $G_D$ ,*

$$\frac{d}{dk} \mathcal{L}_p(f_\infty/K, \chi, k)_{k=2} = \frac{1}{2} (1 - \chi_1(-M)w_M) \log_E(P_\chi).$$

*Proof.* By definition for each  $Q \in \mathcal{F}^D/\Gamma_0(M)$  with associated roots  $\tau$  and  $\bar{\tau}$ , we have

$$\begin{aligned} \frac{d}{dk} \mathcal{L}_p(f_\infty, Q, k)_{k=2} &= \frac{1}{2} \int_{(\mathbb{Z}_p^2)'} (\log(x - \tau y) + \log(x - \bar{\tau} y)) d\mu_*\{r \rightarrow \gamma_\tau r\} \\ &= \frac{1}{2} (J_\tau + \tau_p J_{\bar{\tau}}) \\ &= \frac{1}{2} (J_\tau - w_M J_{\tau\sigma_\tau}), \end{aligned}$$

where the last equality is a consequence of equation (15). It follows from Proposition 1.7 that

$$\frac{d}{dk} \mathcal{L}_p(f_\infty/K, \chi, k)_{k=2} = \frac{1}{2} (1 - \chi_1(-M)w_M) \left( \sum_{\sigma \in G_D} \chi(\sigma) J_{\tau^\sigma} \right),$$

as was to be proved. □

**Corollary 4.2.** *For all genus characters  $\chi$  of  $G_D$ ,*

$$\frac{d^2}{dk^2} L_p(f_\infty/K, \chi, k)_{k=2} = \begin{cases} 2 \log_E^2(P_\chi) & \text{if } \chi_1(-M) = -w_M \\ 0 & \text{if } \chi_1(-M) = w_M. \end{cases}$$

We are now ready to prove Theorem 1 of the introduction.

**Theorem 4.3.** *Let  $\chi$  be the genus character attached to the pair of Dirichlet characters  $\chi_1$  and  $\chi_2$ . Suppose that  $E$  has at least two primes of multiplicative reduction, and that  $\chi_1(-M) = -w_M$ .*

1. There is a point  $\mathbf{P}_\chi \in E(H_\chi)^\times$  and  $t \in \mathbb{Q}^\times$  such that

$$\log_E(P_\chi) = t \log_E(\mathbf{P}_\chi).$$

2. The point  $\mathbf{P}_\chi$  is of infinite order if and only if  $L'(E/K, \chi, 1) \neq 0$ .

*Proof.* The proof proceeds in three stages, which are parallel to Kronecker's "solution to Pell's equation" described in the last paragraph of the introduction.

1. Corollary 4.2 expresses the logarithm of the Stark-Heegner point  $P_\chi$  in terms of special values of  $L$ -series:

$$\log_E^2(P_\chi) = \frac{1}{2} \frac{d^2}{dk^2} L_p(f_\infty/K, \chi, k)_{k=2}. \quad (28)$$

2. Theorem 3.6 asserts that

$$L_p(f_\infty/K, \chi, k) = D^{\frac{k-2}{2}} L_p(f_\infty, \chi_1, k, k/2) L_p(f_\infty, \chi_2, k, k/2), \quad (29)$$

so that we are now reduced to understanding the leading terms of the Mazur-Kitagawa  $p$ -adic  $L$ -functions attached to  $\chi_1$  and  $\chi_2$  in a neighbourhood of  $k = 2$ .

3. Let

$$\text{sign}(E, \chi_j) := -w_N \chi_j(-N)$$

denote the sign in the functional equation for the complex  $L$ -series  $L(E, \chi_j, s)$ . Since  $\chi_1(-N)\chi_2(-N) = \epsilon_K(-N) = -1$ , it follows that  $\text{sign}(E, \chi_1)$  and  $\text{sign}(E, \chi_2)$  are *opposite*. Order  $\chi_1$  and  $\chi_2$  in such a way that

$$\text{sign}(E, \chi_1) = -1, \quad \text{sign}(E, \chi_2) = 1. \quad (30)$$

Then  $\chi_1(-N) = w_N$ , and the running hypothesis that  $\chi_1(-M) = -w_M$  implies that

$$\chi_1(p) = -w_p = a_p. \quad (31)$$

Therefore the Mazur-Kitagawa  $p$ -adic  $L$ -function  $L_p(f, \chi_1, k, s)$  has an exceptional zero at  $(k, s) = (2, 1)$ . Conditions (30) and (31) imply that we are in the situation where Theorem 5.4 of [BD05] can be applied to  $L_p(f, \chi_1, k, k/2)$ . Hence, this  $p$ -adic analytic function vanishes to order

$\geq 2$  at  $k = 2$ , and there is a global point  $\mathbf{P}_{\chi_1} \in E(\mathbb{Q}(\sqrt{D_1}))$  and a rational number  $\ell_1 \in \mathbb{Q}^\times$  satisfying the following three properties:

A.

$$\frac{d^2}{dk^2} L_p(f_\infty, \chi_1, k, k/2)_{k=2} = \ell_1 \log^2(\mathbf{P}_{\chi_1}). \quad (32)$$

B. The point  $\mathbf{P}_{\chi_1}$  is of infinite order if and only if  $L'(E, \chi_1, 1) \neq 0$ .

C. The rational number  $\ell_1$  satisfies

$$\ell_1 \equiv L^*(f, \psi, 1) \pmod{(\mathbb{Q}^\times)^2}, \quad (33)$$

for *any* primitive quadratic Dirichlet character  $\psi$  for which  $L(f, \psi, 1) \neq 0$  and such that

$$\psi(\ell) = \chi_1(\ell) \text{ for } \ell|M, \quad \psi(p) = -\chi_1(p).$$

On the other hand, the quantity

$$L_p(f_\infty, \chi_2, 2, 1) = 2L^*(E, \chi_2, 1) =: 2\ell_2 \quad (34)$$

is a rational number, which is non-zero if and only if  $L(E, \chi_2, 1)$  does not vanish. Note that  $\ell_1 \ell_2^{-1}$  is a rational square by (33). Choose  $t \in \mathbb{Q}^\times$  in such a way that

$$t^2 = \begin{cases} \ell_1 \ell_2^{-1} & \text{if } \ell_2 \neq 0; \\ 1 & \text{otherwise,} \end{cases}$$

and let

$$\mathbf{P}_\chi := \begin{cases} \mathbf{P}_{\chi_1} & \text{if } L'(E/K, \chi, 1) = L'(E, \chi_1, 1)L(E, \chi_2, 1) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

Equations (28), (29), (32) and (34) imply Theorem 4.3, after possibly adjusting the sign of  $t$ .  $\square$

**Remark 4.4.** The condition  $\chi_1(-M) = -w_M$  that is imposed in Theorem 4.3 is needed both in the first and third steps of the argument. When  $\chi_1(-M) = w_M$ , the signs in the functional equations for both  $L_p(f_k, \chi_1, s)$  and  $L_p(f_k, \chi_2, s)$  are  $-1$ .

1. In the case of  $L_p(f_k, \chi_1, s)$ , this arises from the fact that the sign in the classical functional equation for  $L(f_k, \chi_1, s)$  is  $-1$ , while  $L_p(f_k, \chi_1, s)$  does not have an exceptional zero, because  $\chi_1(p) = w_p = -a_p$ .
2. For  $L_p(f_\infty, \chi_2, k, s)$ , the classical  $L$ -function  $L(f_k, \chi_2, s)$  vanishes to even order, but its  $p$ -adic counterpart has an exceptional zero and therefore vanishes to odd order at the central critical point. (This latter situation is precisely the one that was studied by Greenberg and Stevens in [GS], where the vanishing of  $L_p(f_\infty, \chi_2, k, s)$  on the central critical line was used to prove the “exceptional zero conjecture” of Mazur, Tate and Teitelbaum.)

Thus, in the setting where  $\chi_1(-M) = w_M$ , equation (29) implies that the  $p$ -adic  $L$ -function  $\mathcal{L}_p(f_\infty/K, \chi, k)$  vanishes identically. Hence no arithmetic information is to be extracted from this function, and a proof of Theorem 1 would seem to require a different approach. (An eventual extension of equation (32) to the setting of Hilbert modular forms attached to the real quadratic field  $K$  seems a promising avenue.)

**Remark 4.5.** In the case where  $\chi$  is not quadratic, the crucial factorisation (29) ceases to be available. This is a more serious obstacle, and the ideas explored in his paper appear to shed no light on the algebraicity of the individual Stark-Heegner points  $P_\tau$  when  $G_D$  is not of exponent 2.

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