Euler systems and the Birch and Swinnerton-Dyer conjecture

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(joint work with Massimo Bertolini, Victor Rotger)

The Birch and Swinnerton-Dyer conjecture for an elliptic curve $E/\mathbb{Q}$ asserts that
\[ \text{ord}_{s=1} L(E, s) = \text{rank}(E(\mathbb{Q})) , \]
where $L(E, s)$ is the Hasse-Weil $L$-function attached to $E$. The scope of the conjecture can be broadened somewhat by introducing an Artin representation
\[ \varrho : G_{\mathbb{Q}} \rightarrow \text{Aut}(V_{\varrho}) \simeq \text{GL}_n(\mathbb{C}) , \]
and studying the Hasse-Weil-Artin $L$-function $L(E, \varrho, s)$, namely, the $L$-function attached to $H^1_{\text{et}}(E_{\mathbb{Q}}, \mathbb{Q}_p) \otimes V_{\varrho}$, viewed as a (compatible system of) $p$-adic representations. The “equivariant Birch and Swinnerton-Dyer conjecture” states that
\[ \text{ord}_{s=1} L(E, \varrho, s) = \dim_{\mathbb{C}} \text{hom}_{G_{\mathbb{Q}}}(V_{\varrho}, E(H) \otimes \mathbb{C}) , \]
where $H$ is a finite extension of $\mathbb{Q}$ through which $\varrho$ factors. Denote by BSD$_r(E, \varrho)$ the assertion that the right-hand side of (3) is equal to $r$ when the same is true of the left-hand side. Virtually nothing is known about BSD$_r(E, \varrho)$ when $r > 1$. For $r \leq 1$, there are the following somewhat fragmentary results, listed in roughly chronological order:

**Theorem** (Gross-Zagier 1984, Kolyvagin 1989) If $\varrho$ is induced from a ring class character of an imaginary quadratic field, and $r \leq 1$, then BSD$_r(E, \varrho)$ holds.

**Theorem A** (Kato, 1990) If $\varrho$ is abelian (i.e., corresponds to a Dirichlet character), then BSD$_0(E, \varrho)$ holds.

**Theorem B** (Bertolini-Darmon-Rotger, 2011) If $\varrho$ is an odd, irreducible, two-dimensional representation whose conductor is relatively prime to the conductor of $E$, then BSD$_0(E, \varrho)$ holds.

**Theorem C** (Darmon-Rotger, 2012) If $\varrho = \varrho_1 \otimes \varrho_2$, where $\varrho_1$ and $\varrho_2$ are odd, irreducible, two-dimensional representations of $G_{\mathbb{Q}}$ satisfying:

1. $\det(\varrho_1) = \det(\varrho_2)^{-1}$, so that $\varrho$ is isomorphic to its contragredient representation;
2. $\varrho$ is regular, i.e., there is a $\sigma \in G_{\mathbb{Q}}$ for which $\varrho(\sigma)$ has distinct eigenvalues;
3. the conductor of $\varrho$ is prime to that of $E$;
then BSD$_0(E, \varrho)$ holds.

This lecture endeavoured to explain the proofs of Theorems A, B, and C, emphasising the fundamental unity of ideas underlying all three.

The key ingredients are certain global cohomology classes
\[ \kappa(f, g, h) \in H^1(\mathbb{Q}, V_f \otimes V_g \otimes V_h(c)) \]
attached to triples $(f, g, h)$ of modular forms of respective weights $(k, \ell, m)$; here $V_f$, $V_h$ and $V_g$ denote the Serre-Deligne representations attached to $f$, $g$ and $h$, and it is assumed that the triple tensor product of Galois representations admits
a Kummer-self-dual Tate twist, denoted $V_f \otimes V_g \otimes V_h(c)$. (This is true when the product of nebentype characters associated to $f$, $g$ and $h$ is trivial.)

When $f$, $g$ and $h$ are all of weight two and level dividing $N$, and $f$ is cuspidal, associated to an elliptic curve $E$, say, the class $\kappa(f, g, h)$ admits a geometric construction via $p$-adic étale regulators/Abel-Jacobi images of

1. Beilinson-Kato elements in the higher Chow group $\text{CH}^2(X_1(N), 2)$ of the modular curve $X_1(N)$, when $g$ and $h$ are Eisenstein series of weight two arising as logarithmic derivatives of suitable Siegel units;
2. Beilinson-Flach elements in the higher Chow group $\text{CH}^2(X_1(N)^2, 1)$ when $g$ is cuspidal and $h$ is an Eisenstein series;
3. Gross-Kudla-Schoen diagonal cycles in the Chow group $\text{CH}^2(X_1(N)^3)$, when all forms are cuspidal.

When $g$ and $h$ are of weight one rather than two, and hence, are associated to certain (possibly reducible) odd two-dimensional Artin representations, the construction of $\kappa(f, g, h)$ via $K$-theory and algebraic cycles ceases to be available. The class $\kappa(f, g, h)$ is obtained instead by a process of $p$-adic analytic continuation, interpolating the geometric constructions at all classical weight two points of Hida families passing through $g$ and $h$ in weight one, and then specialising to this weight. The resulting $\kappa(f, g, h)$ is called the \textit{generalised Kato class} attached to the triple $(f, g, h)$ of modular forms of weights $(2, 1, 1)$.

The generalised Kato classes arising from $(p$-adic limits of) Beilinson-Kato elements, Beilinson-Flach elements, and Gross-Kudla-Schoen cycles are germane to the proofs of Theorems A, B and C respectively. The key point in all three proofs is an explicit reciprocity law which asserts that the global class $\kappa(f, g, h)$ is non-cristalline at $p$ precisely when the classical central critical value $L(f, g, h, 1) = L(E, g, 1)$ is non-zero. The non-cristalline classes attached to $(f, g, h)$ (of which there are actually four, attached to various choices of ordinary $p$-stabilisations of $g$ and $h$) can then be used (by a standard argument involving local and global Tate duality) to conclude that the natural inclusion of $E(H)$ into $E(H \otimes Q_p)$ becomes zero when restricted to $\mathfrak{g}_g \otimes \mathfrak{g}_h$-isotypic components, and hence, that $\text{hom}_{G_Q}(V_{\mathfrak{g}_g}, E(H) \otimes \mathbb{C})$ is trivial when $L(E, g, 1) \neq 0$.

The lecture strived to set the stage for the two that immediately followed, which were both devoted to further developments arising from these ideas:

1. Victor Rotger’s lecture studied the generalised Kato classes $\kappa(f, g, h)$ when $L(f, g, h, 1) = 0$. In that case, they belong to the Selmer group of $E/H$, and can be viewed as $p$-adic avatars of $L''(E, g, 1)$;
2. Sarah Zerbes’ lecture reported on [LLZ1], [LLZ2], [KLZ] in which the study of Beilinson-Flach elements undertaken in [BDR] is generalised, extended and refined. By making more systematic use of the Euler system properties of Beilinson-Flach elements, notably the possibility of “tame deformations” at primes $\ell \neq p$, the article [KLZ] is also able to establish strong finiteness results for the relevant $p$-isotypic parts of the Shafarevich-Tate group of $E$ over $H$, in the setting of Theorem B.
REFERENCES


