REVIEW OF "LECTURES ON AUTOMORPHIC L-FUNCTIONS" BY JAMES COGDELL, HENRY KIM AND RAM MURTY

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The Langlands Functoriality Conjecture is the number theorist's "grand unified theory". It describes and elucidates the platonic realm of modular forms, L-functions, and motives—a world arguably no less real, in its richness, than the physicist's universe of elusive particles and far-flung galaxies.

The timely volume under review (referred to henceforth as CKM) is made up of three distinct contributions of about 100 pages each:

- **[C]** A survey of L-functions of automorphic forms and converse theorems for GL_n, written by James Cogdell;
- **[K]** An account by Henry Kim of his recent work with Shahidi on certain special cases of functoriality (which will be described more precisely below);
- **[M]** Ram Murty's exposition of some of the applications of these results and of related conjectures to classical questions in analytic number theory.

All three contributions are motivated by the striking work of Kim and Shahidi on functoriality for the symmetric third [KS] and fourth [Kim] powers of the standard representation of $\mathrm{GL}(2)$. Thus CKM makes an ideal introduction to the main techniques—most crucially, the converse theorems of Cogdell and Piatetski-Shapiro, and the Langlands-Shahidi method—that were instrumental in obtaining these results. Because CKM is resolutely pitched

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at the motivated graduate student or the expert in automorphic forms, it is a challenge to convey its contents to a broader readership, reconciling the sometimes competing demands of simplicity and precision. We will sacrifice a bit of the latter to the former, and focus on a single of the many threads that runs through the volume: functoriality for symmetric powers and its relation to the Sato-Tate conjecture, a topic of much current interest in light of the recent breakthroughs of [CHT], [HSBT], and [Ta].

To an elliptic curve E over $\mathbb{Q}$ is associated a collection of *Galois representations* (parametrised by positive integers n)

$$\rho_{E,n} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(E[n]) \simeq \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}),$$

obtained by considering the action of the absolute Galois group of $\mathbb{Q}$, denoted $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, on the n-division points of the divisible group $E(\bar{\mathbb{Q}})$ (a group which is abstractly isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$). Packaging these representations together as n ranges over the powers of a prime ℓ leads to the fundamental ℓ-adic representation

$$\rho_E : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{Z}_\ell). \tag{1}$$

Understanding such naturally occuring representations of $G_{\mathbb{Q}}$ is one of the central questions in number theory. To be more precise about what is meant here by “understanding", we note that the group $G_{\mathbb{Q}}$ carries a plethora of extra structures, most notably a collection of so-called *decomposition subgroups* $G_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ indexed by the rational primes p and arising from the various p-adic completions of $\mathbb{Q}$. The inclusion of G_p in $G_{\mathbb{Q}}$ is obtained by choosing an embedding of the algebraic closure $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ and associating to $\sigma \in G_p$ its restriction to $\bar{\mathbb{Q}}$. The resulting inclusion depends on the choice of embedding, but only up to conjugation in

$G_{\mathbb{Q}}$. The group G_p has a canonical normal subgroup I_p consisting of elements which act trivially on the residue field of $\bar{\mathbb{Q}}_p$, and the quotient G_p/I_p is a procyclic group with a canonical topological generator, the so-called *frobenius element* Frob_p. (It is defined by the condition that it induces the automorphism $x \mapsto x^p$ on the residue field at p.) It is not hard to show that $\rho_E(I_p) = 1$ for all but finitely many p. When this condition is satisfied, one says that ρ_E is *unramified at* p. For such primes, the image $\sigma_p := \rho_E(\mathrm{Frob}_p)$ of Frob_p is a well-defined element of $GL_2(\mathbb{Q}_\ell)$—or, more precisely, a canonical *conjugacy class* (because G_p is only well-defined up to conjugation in $G_{\mathbb{Q}}$). Understanding the behaviour of the classes σ_p as p varies is an important theme in the branch of number theory devoted to *generalised reciprocity laws*. Questions of this type can be traced back to the fundamental law of quadratic reciprocity proved by Gauss. The connection with quadratic reciprocity is that, in the simpler case where

$$\rho : G_{\mathbb{Q}} \longrightarrow \pm 1 \tag{2}$$

is a (continuous) one-dimensional representation of order 2, and K is the quadratic field of discriminant D determined by ρ, (i.e., the fixed field of its kernel), it follows directly from the definition of Frobenius elements that σ_p is 1 or -1 depending on whether D is a square or a non-square modulo p. That this latter condition depends only on the value of p modulo $4D$ is the content of the law of quadratic reciprocity. The periodicity of σ_p for the representation ρ of (2) reveals an a priori unexpected *regularity* of the function $p \mapsto \sigma_p$, and it is this type of pattern one would like to unveil for more complicated sequences of frobenius elements such as those arising from the Galois representation of (1).

Returning to the case of (1), basic facts in the theory of elliptic curves show that the characteristic polynomial of σ_p is of the form

$$x^2 - a_p x + p = (x - \alpha_p)(x - \beta_p),$$

where a_p is an *integer* (which in fact is independent of the choice of $\ell \neq p$) satifying $|a_p| < 2\sqrt{p}$, so that the complex roots α_p and β_p lie on the circle of radius $\sqrt{p}$. The *Hasse-Weil* L-*series*, defined by the infinite product (taken over the primes p for which ρ_E is unramified)

$$L(E, s) := \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1} =: \sum_{n \geq 1} a_n n^{-s},$$

packages the data from the σ_p into an *Euler product* which converges in the right half-plane of $s \in \mathbb{C}$ with $\mathrm{Real}(s) > 3/2$. (This convergence is a direct consequence of the inequality $|a_p| \leq 2\sqrt{p}$.)

The Langlands conjecture, in this special case, is known as the Shimura-Taniyama conjecture. It asserts that the generating series

$$f_E(\tau) := \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau},$$

viewed as an analytic function of the variable τ in the complex upper half-plane, is a *modular form* of weight 2 on a specific (explicitly determined, in terms of E) finite-index subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. Modularity in this setting means that f_E satisfies the deep periodicity

$$f_E\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 f_E(\tau),$$

for all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, together with certain (equally deep) conditions of moderate growth at the boundary of the quotient $\Gamma \backslash \mathcal{H}$.

The proof of the Shimura-Taniyama conjecture was completed in [BCDT] by capitalising on the revolutionary techniques of [W] and [TW] that led to the proof of Fermat's Last Theorem. The Shimura-Taniyama conjecture reveals a pattern satisfied by the σ_p, which, although less easily described than the simple periodicity of quadratic reciprocity, has many of the same desirable consequences for $L(E,s)$. For example, Hecke showed how the modularity of E implies that its L-series has a simple integral representation

$$L(E,s) = (2\pi)^s\Gamma(s)^{-1}\int_0^\infty f_E(iy)y^s\frac{dy}{y},$$

leading to the analytic continuation and functional equation satisfied by $L(E,s)$. These classical topics are recalled, in a treatment that is brief but complete, in the first sections of **[C]**. Cogdell's contribution then turns to a more general framework for Langlands functoriality, where ρ is now taken to be an n-dimensional representation of $G_F := \mathrm{Gal}(\bar{F}/F)$, for some number field F. In this level of generality, the conjectures are most conveniently expressed by replacing classical modular forms by automorphic forms and representations. The automorphic forms considered in **[C]** are special kinds of functions on $\mathrm{GL}_n(\mathbb{A}_F)$, where $\mathbb{A}_F$ denotes the ring of adèles of F, i.e., the restricted product of all the completions of F relative to their maximal compact subrings. It is a requirement of the definition that the form satisfy suitable growth and invariance properties under right translation by elements in a compact subgroup of $\mathrm{GL}_n(\mathbb{A}_F)$, as well as (crucially) being invariant under left translation by the discrete subgroup $\mathrm{GL}_n(F) \subset \mathrm{GL}_n(\mathbb{A}_F)$. An even more flexible (if at first somewhat daunting to the novice) framework for working with automorphic forms is the theory of *automorphic representations*–representations (typically infinite-dimensional) of the adèlic group $\mathrm{GL}_n(\mathbb{A}_F)$ occuring in a suitable space of functions on $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$. The sequence of shifts in point of view that make it possible to pass from classical modular forms, first to automorphic forms, and then to automorphic representations, is well motivated and explained in sections 2 and 3 of **[C]**.

To each automorphic representation π of GL_n over F is associated an L-function $L(\pi, s)$. The precise definition of this L-function would take us a bit far afield, but here are its main features. One first shows that any "reasonably nice" (the technical term being "admissible") representation π can be expressed as a restricted tensor product $\otimes_v \pi_v$, taken over all completions F_v of F, where π_v is a representation of the group $\mathrm{GL}_n(F_v)$. An important subclass of these "local" representations, referred to as *unramified representations*, are parametrized by conjugacy classes in $\mathrm{GL}_n(\mathbb{C})$. For a given π, all but finitely many π_v are unramified in this sense. Hence any automorphic representation gives rise to a collection of conjugacy classes $\sigma_v(\pi)$ in $\mathrm{GL}_n(\mathbb{C})$ indexed by the primes v of F (outside a finite set of exceptions). The $\sigma_v(\pi)$ are called the *Langlands parameters* associated to π. The L-function attached to π, denoted $L(\pi, s)$, is now defined in the same way as the L-function of a Galois representation, but with Frobenius elements replaced by the classes $\sigma_v(\pi)$. One of the predictions of the Langlands conjecture is that, for *any* n-dimensional Galois representation ρ, there is an automorphic representation π_ρ of $\mathrm{GL}_n(F)$ with $L(\rho, s) = L(\pi_\rho, s)$. For such $\pi = \pi_\rho$, the Langlands parameters $\sigma_v(\pi)$ should therefore display *the same sort* of coherence as that which is satisfied by the Frobenius elements σ_v of a Galois representation.

The category of Galois representations of G_F is equipped with the standard panoply of linear algebra constructions (duality, as well tensor, symmetric and alternating products) making it possible to build new representations from old ones. A consequence of functoriality (and this is the way in which functoriality is often exploited in representation theory) is that these constructions should have counterparts on the automorphic side. Such predictions “arising from number theory" are often highly non-trivial, yielding surprising and deep insights into the behaviour of automorphic representations. (The information can also go in the other direction, as in the proof of the Shimura-Taniyama conjecture and Fermat’s Last Theorem, or, even more germane to the present review, the recent progress on the Sato-Tate conjecture.)

There are many ways in which n-dimensional ℓ-adic representations can arise in number theory, typically by considering the étale cohomology groups of varieties over number fields. After the Galois representation ρ_E of (1), the most natural example is perhaps the n-th symmetric power of ρ_E. This is an $(n+1)$-dimensional representation of $G_{\mathbb{Q}}$, denoted

$$\mathrm{Sym}^n \rho_E : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_{n+1}(\mathbb{Q}_\ell). \tag{3}$$

The frobenius elements $\sigma_p^{(n)} := \mathrm{Sym}^n \sigma_p$ associated to this representation have eigenvalues given by

$$\lambda_p^{(i)} := \alpha_p^{n-i}\beta_p^i, \qquad i = 0, 1, \dots, n.$$

The Langlands functoriality conjecture predicts that the representation (3) should, as in the case $n = 1$, be associated to an automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{Q})$. This conjecture implies in particular that the L-series

$$L_n(E, s) := L(\mathrm{Sym}^n \rho_E, s) = \prod_p \prod_{i=0}^{n} (1 - \lambda_p^{(i)} p^{-s})^{-1}$$

admits a functional equation and an analytic continuation to the entire complex plane.

It was already known from work of Gelbart and Jacquet that the Galois representation $\mathrm{Sym}^2 \rho$ is automorphic if ρ is *any* two-dimensional representation of G_F that is itself associated to an automorphic representation of $GL_2(F)$. The breakthrough of Kim and Shahidi is the corresponding statement for the representations $\mathrm{Sym}^3\rho$ [KS] and $\mathrm{Sym}^4\rho$ [Kim].

When combined with the Shimura-Taniyama conjecture, the work of Kim and Shahidi implies that the L-series $L_2(E,s)$, $L_3(E,s)$, and $L_4(E,s)$ admit analytic continuations and functional equations of the standard type. (It also implies, by a technique known as the Rankin-Selberg method, similar analyticity statements for the integers $n \leq 8$, although the L-function $L_n(E,s)$ is not proved to arise from an automorphic form for $n > 4$.)

The basic idea for proving automorphy results of this type is first to relate L-series like $L_n(E,s)$ to the constant terms of certain Eisenstein series; the analytic continuation and functional equation satisfied by the Eisenstein series can then be transferred to the constant term. This powerful method for studying L-series was initiated by Langlands and developped further by Shahidi, and it now goes under the name of the Langlands-Shahidi method. A description of the Langlands-Shahidi method and its use in proving the main results of [KS] and [Kim] is one of the goals of the contribution **[K]**.

The Langlands-Shahidi method explained in **[K]** makes it possible to prove that the L-series $L(\mathrm{Sym}^n\rho, s)$ ($n = 2,3,4$), as well as related L-series obtained by twisting $\mathrm{Sym}^n\rho$ by automorphic representations of lower dimension, behave (from the point of view of their functional equations and analytic properties) *as if* $\mathrm{Sym}^n\rho$ were automorphic. The mechanism for concluding that $\mathrm{Sym}^n\rho_E$ is in fact attached to an automorphic form goes under the rubric of *converse theorems*.

The fundamental work of Hecke (recalled in Section 1 of **[C]**) shows that the L-series attached to classical cusp forms f on $GL(2)$ have analytic continuations and satisfy functional equations of a standard type as well as being bounded in vertical strips. Let us call an L-series *nice* if it satisfies these properties. Hecke's theory shows that the L-series obtained from the *twists* of f by one-dimensional characters are also nice. (We will not go into the precise definition of "nice", or of of twisting, as this would take us too far afield.)

Weil proved a kind of converse to Hecke's statement by showing that if the L-series attached to an admissible representation f of $GL_2(\mathbb{Q})$ and sufficiently many of its twists are nice, then f is is in fact automorphic. Converse theorems have been extended to representations of $GL_n(F)$ and are the main topic discussed in **[C]**.

Knowing that the representations in (3) are automorphic, and hence that the L-series $L_n(E, s)$ possess analytic continuations at $s = 1$, has important applications to the analytic number theory of elliptic curves, which are discussed in Ram Murty's contribution **[M]**. One of the most striking is to the Sato-Tate conjectures, which predicts that (when the elliptic curve E has only the obvious endomorphisms by $\mathbb{Z}$) the complex numbers $\alpha_p/\sqrt{p}$ attached to E are distributed on the unit circle according to a specific density function, the *Sato-Tate distribution*. More precisely, the arguments of α_p should be equidistributed on the interval $[0, \pi]$ according to the density function $\frac{2}{\pi}\sin^2\theta d\theta$. The first Section of **[M]** explains why the analyticity and non-vanishing of $L_n(E, s)$ at $s = 1$ (for all n) implies the Sato-Tate conjecture for E.

The Sato-Tate conjecture (for a fixed elliptic curve E) would follow from a proof of Langlands Functoriality for *all* $\mathrm{Sym}^n\rho$, yielding the analyticity of $L_n(E, s)$ for all n. The result of Kim and Shahidi carries out this program for $n = 3$ and 4, and can be viewed as a significant step towards understanding the Sato-Tate conjecture.

A few months ago, the preprint [Ta], completing the program initiated in [CHT] and [HSBT], succeeded in showing (for a large class of elliptic curves E, essentially those having non-integral j-invariant) the analytic continuation of the L-series $L_n(E, s)$ for *all* n, thereby proving the Sato-Tate conjecture for these E! The proof adapts the methods of [W] and [TW] to the setting of the representations $\mathrm{Sym}^n \rho_E$. The final result is less precise than the results of Kim and Shahidi for $n = 3$ and 4, since it only esablishes *potential modularity* of the representations $\mathrm{Sym}^n \rho_E$: namely, that after restricting $\mathrm{Sym}^n \rho_E$ to the Galois group of some totally real field F (which could depend *a priori* on n) the corresponding L-series is attached to an automorphic form on $\mathrm{GL}_n(F)$. However, even this cruder form of Langlands functoriality is enough to establish the Sato-Tate conjecture, and represents another spectacular success to emerge from the circle of ideas surrounding the “Langlands program".

In conclusion, [CKM] will be valuable both as a reference or textbook for researchers and students interested in a vibrant area of modern mathematics at the intersection of representation theory and number theory which has witnessed a tremendous amount of recent progress.

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