189-571B: Higher Algebra II

Practice Final Exam

April 2018

Each of the 7 questions below is worth 15 points, and your final grade will be out of 100. No calculators or outside materials are allowed during the exam (nor would they be useful, probably).

1. (a) Show that the ring $\mathbf{Z}[\sqrt{-23}]$ is not a Dedekind ring.

(b) Compute its integral closure, R, and explain why that ring is a Dedekind ring.

(c) Give an explicit factorisation of the ideal $3R$ into a product of prime ideals. Show that the prime divisors of 3 are not principal but that their third powers are.

2. Let R be a (not necessarily commutative) ring with unit and let M be a finitely generated module over R.

(a) Define what it means for the sequence $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ to be a free resolution of M, and show that M always has a free resolution.

(b) Show that if $\cdots \to M_2 \to M_1 \to M_0 \to M \to 0$ is any complex of R-modules, then there exists ring homomorphisms $F_j \to M_j$ for which the obvious diagram commutes.

(c) Recall that a contravariant functor G from the category of R-modules to the category of abelian groups is said to be *left exact* if the sequence

$$0 \to G(M'') \to G(M) \to G(M)$$

is exact in the category of abelian groups whenever

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence. What is the 0-th cohomology of the complex

$$0 \to G(F_0) \to G(F_1) \to \cdots$$

obtained by applying G to a free resolution of M as in (a)?

3. (a) Show that every principal ideal domain is Noetherian and a unique factorisation domain.

(b) Give an example of a Noetherian unique factorisation domain which is not a principal ideal domain.

(c) Let $R = \mathbf{Z}[\sqrt{-5}]$ and let I be the ideal generated by 3 and $(1+\sqrt{-5})$. Show that I is a prime ideal of R which is not principal, but that IR_I is a principal ideal of the localisation R_I of R at I. What is a generator of this ideal? What is the quotient R_I/IR_I isomorphic to?

4. Let D_8 be the dihedral group of order 8 and let Q_8 be the quaternion group of order 8. Show that the group rings $\mathbf{C}[D_8]$ and $\mathbf{C}[Q_8]$ are isomorphic, but that the group rings with real coefficients are not isomorphic for these two groups.

5. Let F be a field and let D be a central division algebra over F.

(a) Show that D has dimension n^2 over F and that it contains a field K/F of degree n.

(b) Suppose that K/F is a Galois extension and that $a \in K$ generates K over F. Show that the endomorphism $r_a : D \to D$ given by $r_a(x) = xa$ is K-linear when D is viewed as a K-vector space under left multiplication.

(c) Show that the endomorphism r_a is diagonalisable and compute its eigenvalues.

6. Give an example of two field extensions K and L of a field F for which:

(a) $K \otimes_F L$ contains nilpotent elements;

(b) $K \otimes_F L$ has zero divisors but no nilpotent elements;

(c) $K \otimes_F L$ is a field.

7. Let R be finite ring with identity, having no zero divisors. Show that R is commutative.