189-457B: Algebra 4

Assignment 2

Due: Wednesday, February 8

1. Convert the following matrices with entries in $\mathbf{Z}$ into Smith normal form, i.e., for each such M find a diagonal matrix D with entries $d_1|d_2|d_3$ satisfying $D = AMB$, where A and B are both *invertible* matrices with entries in $\mathbf{Z}$.

$$M = \begin{pmatrix} 26 & 44 & 18 \\ 8 & 14 & 6 \\ 12 & 12 & 12 \end{pmatrix}, \qquad M = \begin{pmatrix} 2 & 7 & 2 \\ 3 & 13 & 3 \\ 5 & 20 & 5 \end{pmatrix}.$$

2. Let M_1 and M_2 be two $n \times n$ matrices with coefficients in $\mathbf{Z}$, satisfying

$$\det(M_1) = \det(M_2) = p_1 p_2 \dots p_n, \tag{1}$$

where the p_j are *distinct* prime numbers. Show that there exist two invertible matrices A and B with entries in $\mathbf{Z}$, satisfying

$$M_1 = AM_2B.$$

Give an example to show that this conclusion need not hold if the assumption on the distinctness of the primes in (1) is dropped.

3. Reduce the following matrices

$$M = \begin{pmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix}, \quad M = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix}$$

with entries in the field $\mathbf{Z}/2\mathbf{Z}[x]$ in Smith normal form. What does your calculation tell you about the minimal and characteristic polynomials of the matrix $M - xI$? Explain.

4. If R is a principal ideal domain, show that every finitely generated torsion-free R-module is free. (You are allowed to invoke any theorem proved in class, in showing this!) Given an example of a finitely generated, torsion free module over the ring $R = \mathbf{Z}[x]$ of polynomials over $\mathbf{Z}$ which is finitely generated, torsion-free, but not free.

5. Let R be the ring $\mathbf{Z}[\sqrt{-5}]$ and let M be subset of R consisting of the elements of the form $a + b\sqrt{-5}$ where $a + b$ is even. Show that M is a torsion-free R-module which is not free over R.

6. Let F be a field. Show that the ring

$$F((t)) := \left\{ \sum_{i=-N}^{\infty} a_i t^i, \quad a_i \in F \right\}$$

of Laurent series in a variable t is a field. (I.e., that every non-zero element in $F((t))$ has a multiplicative inverse.)

7. Let F be a field of characteristic p. Show that the function $\varphi : F \longrightarrow F$ given by $\varphi(x) = x^p$ is a ring homomorphism. Show that this homomorphism is an isomorphism when F is of finite cardinality.

8. Give an example of a field F of characteristic p for which the homomorphism φ of question 7 is not surjective, and describe the subfield $K := \varphi(F)$ of F precisely.

9. Show that the polynomial $p(x) = x^4 - 2$ has precisely two roots in in the field $F = \mathbf{Q}[2^{1/4}]$ by writing down the factorisation of $p(x)$ into irreducible factors in $F[x]$. What is the degree of the splitting field of $p(x)$ over $\mathbf{Q}$?

10. Show that the degree p polynomial $f(x) = x^p - x - t$ with coefficients

in the field $\mathbf{K} := \mathbf{F}_p(t)$ is irreducible in $\mathbf{K}[x]$. (Hint: use the fact that $f(x) = f(x+j)$ for all $j \in \mathbf{Z}/p\mathbf{Z}$ to deduce an action of the additive group $G = \mathbf{Z}/p\mathbf{Z}$ on the set of irreducible factors of $f(x)$. Conclude that $f(x)$ is reducible if and only if it has a root, and then show that such a root does not exist.) Write down the factorisation of the polynomial $f(x)$ as a product of p linear factors in the extension field $\mathbf{K}' := \mathbf{K}[s]/(s^p - s - t)$ of $\mathbf{K}$.