Honours Algebra 2, Assignment 4

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Question 1. *Find the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 15 & -4 & -2 \\ 27 & -8 & -3 \\ 58 & -14 & -9 \end{pmatrix}$ *with entries in* $\mathbb{R}$. *Show that* A *is diagonalisable. What if the entries of* A *were taken to be in the field* $\mathbb{Z}_p$*, where* p *is a prime number?*

Solution We compute

$$\det \begin{pmatrix} 15-\lambda & -4 & -2 \\ 27 & -8-\lambda & -3 \\ 58 & -14 & -9-\lambda \end{pmatrix} = -\lambda^3 - 2\lambda^2 + \lambda + 2 = -(\lambda+2)(\lambda+1)(\lambda-1),$$

so the eigenvalues are $\{\lambda_1, \lambda_2, \lambda_3\} = \{-2, -1, 1\}$. Since the characteristic polynomial splits completely into distinct linear factors, this tells us immediately that A is diagonalisable. Solving the equations $Ax = \lambda_i x$ for x (by a method of your choice) gives

$$\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

as eigenvectors. Now the relations $Av_i = \lambda_i v_i$ will still hold after reducing modulo p, so we will still have three eigenvectors over $\mathbb{Z}_p$. It remains to check that they are linearly independent over $\mathbb{Z}_p$, but

$$\det \begin{pmatrix} 0 & 1 & 1 \\ -1 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} = -1 \not\equiv 0 \pmod{p}$$

for any prime p. So the eigenvectors remain linearly independent and thus a basis over $\mathbb{Z}_p$ for any prime p; since we can always find a basis consisting of eigenvectors of A, A is always diagonalisable.

Question 2. *Suppose* T_1 *and* T_2 *are commuting linear transformations on an* F*-vector space* V *of dimension* n. *Suppose that* T_2 *is diagonalisable and has* n distinct *eigenvalues. Show that* T_1 *is also diagonalisable. Show with an example that the assumption on the eigenvalues of* T_2 *being distinct is essential for the statement to be true.*

Solution This is a good question to remember, as it demonstrates (the beginning of) how important a property commutativity (of matrices) is. Let $\{\lambda_1, \dots, \lambda_n\}$ be eigenvalues of T_2 with corresponding eigenvectors $\{v_1, \dots, v_n\}$.

Note that eigenvectors for distinct eigenvalues are always linearly independent. In this case, we are told that T_2 is diagonalizable so the eigenvectors span a basis and must be linearly independent by assumption. Now, observe that

$$T_2(T_1 v_i) = T_1(T_2 v_i) = T_1(\lambda_i v_i) = \lambda_i(T_1 v_i)$$

so if v_i is an eigenvector of T_2, then so is $T_1 v_i$. The eigenspace corresponding to v_i is T_1-invariant (if v is an eigenvector, $T_1 v$ is an eigenvector). Since each of the eigenspaces is one-dimensional, it must be that $T_1 v_i$ is a scalar multiple of v_i, that is, v_i is an eigenvector for T_1. Thus, T_1 shares a basis of eigenvectors with T_2, so by definition T_1 is diagonalisable.

Question 3. *Three brands (denoted A,B, and C) of a common consumer product with a weekly purchase cycle vie for shares of a fixed-size market. It has been observed that among the consumers who purchase brand A in a given week, 70% remain loyal to that brand the following week, while 20% swich over to brand B and 10% to Brand C. Among the purchasers of Brand B, only 40% remain loyal to their brand in the following week, the remaining 60% switchin in equal proportions to Brands A and C. Half of the purchasers of Brand C remain loyal to their brand, 20% urchase Brand A, and 30% opt for Brand B. This pattern of consumer behaviour remains constant from one week to the next. This (somewhat reductive, and simplistic!) model of consumer behaviour relative to brands A, B and C is usefully summarised in the so-called* brand transition matrix

$$M = \begin{pmatrix} 0.7 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{pmatrix}.$$

a. *Show that if x, y and z denote the number of consumers purchasing brand A, B and C in the initial (zeroth) week, the number of consumers purchasing those brands in the nth week will be (x_n, y_n, z_n), where*

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

b. *Show that the matrix M is diagonalisable. Compute its eigenvalues and eigenvectors.*

c. *Find a diagonal matrix D and an invertible matrix P with the property that $M = PDP^{-1}$. Use this calculation to show that the entries of the matrix M^n converge to certain limits in $\mathbb{R}$.*

d. *Show that, as time passes, the shares of Brands A, B and C will stabilise, regardless of the numbers of consumers purchasing those brands in the initial week. Compute the values of the stable market shares.*

Solution a. Based on the problem description, we have the system

$$\begin{aligned} x_{n+1} &= 0.7x_n + 0.3y_n + 0.2z_n \\ y_{n+1} &= 0.2x_n + 0.4y_n + 0.3z_n \\ z_{n+1} &= 0.1x_n + 0.3y_n + 0.5z_n, \end{aligned}$$

or equivalently $\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = M \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$. Furthermore we clearly have $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so assume inductively that

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = M \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = M^{n+1} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

so by induction the claim is proven.

b. The solution to the problem is not pretty; fear not. We compute:

$$\begin{aligned} \det \begin{pmatrix} 0.7-\lambda & 0.3 & 0.2 \\ 0.2 & 0.4-\lambda & 0.3 \\ 0.1 & 0.3 & 0.5-\lambda \end{pmatrix} &= -\lambda^3 + 1.6\lambda^2 - 0.66\lambda + 0.06 \\ &= -(\lambda-1)(\lambda - \frac{3-\sqrt{3}}{10})(\lambda - \frac{3+\sqrt{3}}{10}). \end{aligned}$$

Since the characteristic equation splits completely into distinct linear factors, M is diagonalisable with eigenvalues 1, $\frac{3-\sqrt{3}}{10}$ and $\frac{3+\sqrt{3}}{10}$. By solving some more linear systems we find the respective eigenvectors:

$$\begin{pmatrix} 21 \\ 13 \\ 12 \end{pmatrix}, \begin{pmatrix} 1 \\ -2-\sqrt{3} \\ \sqrt{3}+1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{3}-2 \\ 1-\sqrt{3} \end{pmatrix}$$

(it's OK to use a computer algebra system for this).

c. Now we use the matrix of eigenvectors to diagonalize M; *i.e.*

$$P = \begin{pmatrix} 21 & 1 & 1 \\ 13 & -2-\sqrt{3} & \sqrt{3}-2 \\ 12 & \sqrt{3}+1 & 1-\sqrt{3} \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3-\sqrt{3}}{10} & 0 \\ 0 & 0 & \frac{3+\sqrt{3}}{10} \end{pmatrix}.$$

This is, of course, equivalent to changing from the standard basis to a basis composed of the eigenvectors of M. Then $M^n = PD^nP^{-1}$. We have

$$D^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{3-\sqrt{3}}{10}\right)^n & 0 \\ 0 & 0 & \left(\frac{3+\sqrt{3}}{10}\right)^n \end{pmatrix},$$

and since $\left|\frac{3+\sqrt{3}}{10}\right| < 1$ and $\left|\frac{3-\sqrt{3}}{10}\right| < 1$, as n tends to infinity, D^n tends to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so M^n tends to

$$P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} \frac{21}{46} & \frac{21}{46} & \frac{21}{46} \\ \frac{13}{46} & \frac{13}{46} & \frac{13}{46} \\ \frac{12}{46} & \frac{12}{46} & \frac{12}{46} \end{pmatrix}.$$

Denote this matrix by B.

d. Since $\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and as n tends to infinity M^n tends to B, the stable market shares are given by

$$B \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{21}{46}(x+y+z) \\ \frac{13}{46}(x+y+z) \\ \frac{12}{46}(x+y+z) \end{pmatrix} = \begin{pmatrix} \frac{21}{46} \\ \frac{13}{46} \\ \frac{12}{46} \end{pmatrix}$$

since $x + y + z = 1$.

Question 4. *Alice, Bob, Charles, Dan and Eve share a five bedroom apartment in the McGill ghetto. The electrician who was hired to renovate the apartment has completely messed up the wiring in the light switches, and as a result:*

1. *When Alice switches the light in her room on or off, this also switches the light settings in Bob's, Dan's and Eve's rooms.*

2. *When Bob switches his light on or off, this also switches the light settings in Dan's room.*

3. *When Charles switches his light on or off, this also switches the light settings in Alice and Dan's rooms.*

4. *When Dan switches his light on or off, this also switches the light settings in Alice and Eve's rooms.*

5. *When Eve switches her light on or off, this also switches the light settings in Alice, Charles and Dan's rooms.*

The five roommates all wake up at 7:30 (to be on time for their 8:30 algebra class). Given that all the lights are off at that time, which roomates should flip their switches so that the lights in all five rooms are turned on?

Solution Consider the vector space $V = \mathbb{Z}_2^5$. Let each vector in V correspond to a light configuration, where v^i (the i^{th} coordinate of v) corresponds to the lightswitch of the i^{th} person, with 1 meaning 'on' and 0 meaning 'off' (duh). Let (A, B, C, D, E) be the correspondance of coordinates to people (duh). The question boils down to solving $Ax = (1, 1, 1, 1, 1)$, where A is determined by acting each of the standard basis vectors e_i, $i = 1, \ldots, 5$ on A by the given rules; e.g., condition (1) corresponds to Alice $= e_1$ turning on her light, and so $Ae_1 = (1, 1, 0, 1, 1) =$ (switch, switch, stay, switch, switch). Then

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Row-reducing $Ax = (1, 1, 1, 1, 1)$, we get that $x = (1, 0, 0, 1, 1)$. Thus, A, D and E should turn their lights on (that is, Adam, Dan and Eve).