Honours Algebra 2, Assignment 2

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Question 1. *Let $\mathbb{C}$ be the field of complex numbers and let V be the $\mathbb{C}$-vector space $\mathbb{C}^3$. Find the coordinates of the vector $(1,0,1)$ in the basis (v_1, v_2, v_3), where*

$$v_1 = (2i, 1, 0), v_2 = (2, -1, 1), v_3 = (0, 1+i, 1-i).$$

Solution Essentially this amounts to inverting the change of basis matrix

$$T = \begin{pmatrix} 2i & 2 & 0 \\ 1 & -1 & 1+i \\ 0 & 1 & 1-i \end{pmatrix}$$

which can be done by the adjugate matrix method (or Gauss-Jordan elimination, or whatever):

$$\begin{aligned} T^{-1} &= \frac{1}{|T|} \begin{pmatrix} -(1-i)-(1+i) & -(1-i) & 1 \\ -2(1-i) & 2i(1-i) & -(2i-2) \\ 2(1+i) & -2i(1+i) & -2i-2 \end{pmatrix}^T \\ &= \frac{1}{2i(-1+i-1-i)-(2-2i)} \begin{pmatrix} -2 & -1+i & 1 \\ -2+2i & 2+2i & 2-2i \\ 2+2i & 2-2i & -2-2i \end{pmatrix}^T \\ &= \frac{1}{-2-2i} \begin{pmatrix} -2 & -2+2i & 2+2i \\ -1+i & 2+2i & 2-2i \\ 1 & 2-2i & -2-2i \end{pmatrix}, \end{aligned}$$

so

$$T^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-2}{-2-2i} + \frac{2+2i}{-2-2i} \\ \frac{-1+i}{-2-2i} + \frac{2-2i}{-2-2i} \\ \frac{1}{-2-2i} + \frac{-2-2i}{-2-2i} \end{pmatrix} = \begin{pmatrix} \frac{2i(-2+2i)}{8} \\ \frac{(1-i)(-2+2i)}{8} \\ \frac{(-1-2i)(-2+2i)}{8} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} - \frac{i}{2} \\ \frac{i}{2} \\ \frac{3}{4} + \frac{i}{4} \end{pmatrix}.$$

So with respect to the basis (v_1, v_2, v_3), our vector $(1,0,1)$ has coordinates $(-\frac{1}{2} - \frac{i}{2}, \frac{i}{2}, \frac{3}{4} + \frac{i}{4})$.

Question 2. *Let V denote the $\mathbb{Q}$-vector space of 2×2 matrices with entries in $\mathbb{Q}$. Let A be the matrix $\begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix}$. Show that the function*

$$T : V \longrightarrow V \quad \textit{defined by } T(X) = AX - XA$$

is a linear transformation from V to V. Choose a basis for V, and write down the matrix of T with respect to your basis.

Solution The linearity is pretty routine; it follows right from the fact that matrices form a ring, so addition is associative and commutative, and multiplication distributes over addition on the left and right (and scalars commute with matrices; indeed a ring with this extra structure of commutative multiplication by scalars in another ring is called an *algebra*):

$$\begin{aligned} T(aX+bY) = A(aX+bY)-(aX+bY)A &= aAX+bAY-aXA-bYA \\ &= a(AX-XA)+b(AY-YA) \\ &= aT(X)+bT(Y), \end{aligned}$$

for $a, b \in \mathbb{Q}$, $X, Y \in V$. You can choose whatever basis you like (so long as it's consistent) but most of the solutions from the last assignment used a "canonical" choice for the space of matrices: Let

$$B = \left\{ v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be a basis. Then the columns of the matrix of T with respect to this basis will correspond to the action of T on the basis elements. Computing;

$$\begin{aligned} T(v_1) = \begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix} &= \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} 2 & -5 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 5 \\ 3 & 0 \end{pmatrix} = 5v_2 + 3v_3 = \begin{pmatrix} 0 \\ 5 \\ 3 \\ 0 \end{pmatrix}; \end{aligned}$$

$$\begin{aligned} T(v_2) = \begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix} &= \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 17 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -15 \\ 0 & 3 \end{pmatrix} = -3v_1 - 15v_2 + 3v_4 = \begin{pmatrix} -3 \\ -15 \\ 0 \\ 3 \end{pmatrix}; \end{aligned}$$

$$\begin{aligned} T(v_3) = \begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix} &= \begin{pmatrix} -5 & 0 \\ 17 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & -5 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 0 \\ 15 & 5 \end{pmatrix} = -5v_1 + 15v_3 + 5v_4 = \begin{pmatrix} -5 \\ 0 \\ 15 \\ 5 \end{pmatrix}; \end{aligned}$$

$$\begin{aligned} T(v_4) = \begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix} &= \begin{pmatrix} 0 & -5 \\ 0 & 17 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 3 & 17 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -5 \\ -3 & 0 \end{pmatrix} = 5v_2 + 3v_3 = \begin{pmatrix} 0 \\ -5 \\ -3 \\ 0 \end{pmatrix}. \end{aligned}$$

Thus we have

$$T = \begin{pmatrix} 0 & -3 & -5 & 0 \\ 5 & -15 & 0 & -5 \\ 3 & 0 & 15 & -3 \\ 0 & 3 & 5 & 0 \end{pmatrix}.$$

Question 3. *Let V be a finite-dimensional vector space. Show that any linear transformation $\varphi\colon V \to V$ is injective if and only if it is surjective. Show that neither property implies the other if the condition that V be finite-dimensional is dropped.*

a. First, let's note that this could be proven very easily by a dimension argument. If we consider that $V = W + V/W$ (since $V/\ker(\varphi) \cong \operatorname{im}(\varphi)$), we must have $\dim(V) = \dim(\ker(\varphi)) + \dim(\operatorname{im}(\varphi))$.

However, we do this a different way to illustrate the basic 'linear algebra' way of doing things. The style of argument used here illustrates a recurring theme in the course, and is really at the heart of linear algebra: specify a basis, and you've got everything.

We will show that φ is surjective $\Leftrightarrow$ it sends a basis to a basis $\Leftrightarrow$ it is injective. We suppose first that φ sends a basis to a basis, so let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V. If $\{\varphi(v_1), \ldots, \varphi(v_n)\}$ is a basis, for any $v \in V$,

$$\begin{aligned} v &= \sum_{i=1}^{n} \alpha_i \varphi(v_i) \\ &= \varphi\left(\sum_{i=1}^{n} \alpha_i (v_i)\right) \end{aligned}$$

so $w = \sum_{i=1}^{n} \alpha_i v_i$ is the desired vector such that $\varphi(w) = v$. Conversely, if φ is not surjective, then $\{\varphi(v_1), \ldots, \varphi(v_n)\}$ cannot be a basis (as every vector in V is a linear combination of basis vectors).

Now, suppose that φ is not injective. There is v_1 such that $\varphi(v_1) = 0$ (since $Tx = Ty \Leftrightarrow T(x - y) = 0$, so take $v_1 = x - y$). Then in particular, $\operatorname{Span}\{v_1\} \subset \ker \varphi$, as $\varphi(\alpha v_1) = \alpha\varphi(v_1) = 0$ for any α, by linearity of φ. Thus, we can extend $\{v_1\}$ to a basis $\{v_1, v_2, \ldots, v_n\}$ (it is important to understand why you can do this). Then φ doesn't take a basis to a basis, as

$$\begin{aligned} \alpha_1\varphi(v_1) + 0\varphi(v_2) + \cdots + 0\varphi(v_n) &= 0 \\ = \alpha_1 \cdot 0 &= 0 \end{aligned}$$

is a non-trivial linear combination of the $\varphi(v_i)$ for any $\alpha_1 \neq 0$, and a basis is linearly independent by definition.

Since injectivity and surjectivity are both equivalent to taking a basis to a basis (for a finite dimensional vector space), it follows that they are also equivalent to each other.

(Exercise: cut the proof down by going straight from one to the other, in a similar way.)

b. If V is not finite dimensional, you can consider the example of differentiation from assignment 1, question 6. The linear transfomation $D\colon \mathcal{C}^\infty \to \mathcal{C}^\infty$ is surjective by the fundamental theorem of calculus, yet not injective since $\frac{d}{dx}c = 0$ for any constant $c \in \mathbb{R}$. One example of a transformation that is injective yet not surjective is the shift operator on the infinite-dimensional vector space of all bounded sequences, $\ell^\infty = \{x = (x_1, x_2, x_3, \ldots, x_n, \ldots) \mid |x_i| \leq M \in \mathbb{R},\ \ \forall\ \ i \in \mathbb{N}\}$, defined by

$$\mathcal{S}(x) = \mathcal{S}(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots).$$

$\mathcal{S}$ is a linear operator and injective (check it), but cannot be surjective as

$$\{x \in \ell^\infty \mid x_1 \neq 0\} \not\subset \mathrm{im}\mathcal{S}.$$

To see that this style of operator isn't so unnatural, consider the vector space $\mathbb{R}[x]$ and view the coefficients of polynomials as defining infinite-dimensional vectors. What does the integration operator do?

Question 4. *Let V and W be finite-dimensional of dimension n and m respectively, over the field $\mathbb{Z}_2$. How many functions are there from V to W? How many linear transformations? What are the dimensions of these spaces of functions and linear transformations, respectively?*

Solution The essence of the question is combinatorial, and basic combinatorics are mostly a matter of asking yourself the question in the right way. The below attemps to provide some kind of explanation, and different ways you might determine the answer. If you just want the answers, there are 2^{nm} transformations and 2^{m2^n} functions, of dimensions nm and $m2^n$ respectively.

First, determine the number of elements in each vector space. You know that an $n-$dimensional vector space has vectors with n entries:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

Since we have two choices for each entry, there are $2 \times 2 \times \cdots \times 2 = 2^n$ choices for different vectors, and no more. Alternatively, given a basis of n vectors (recall that by definition, $\dim(V) = \#$ elements in $\mathcal{B}$, where $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for V), we have a distinct vector n for each linear combination $\sum_{i=1}^n \alpha_i v_i$. There are 2^n unique linear combinations, so there are 2^n elements in V. Similarly, we have 2^m elements for W.

We first tackle the linear transformations. An important claim here is that every matrix defines a linear transformation (relative to a basis) and vice-versa. In the first assignment, question 4., you have already shown that the dimension of the space of all $m \times n$ matrices is nm. It follows from this observation that the space of all linear transformations *also* has dimension nm, and a basis is the same as the one given in the previous assignment. (Note that I've only written this argument out, but it is iron-clad. In particular, it is an important exercise to show the isomorphism of linear transformations and matrices [and $\mathrm{Hom}_F(V, W)$, which I also defined in the solutions to the previous assignment] for finite-dimensional vector spaces.)

From the above observation, it is easy to determine the number of linear transformations: simply consider the number of choices you have in each entry of a matrix:

$$\begin{pmatrix} \square & \square & \dots & \square \\ \square & \square & \dots & \square \\ \vdots & & \vdots & \\ \square & \square & \dots & \square \end{pmatrix}$$

Each $\square$ has two choices, and you have nm boxes, so there are $2 \times 2 \times 2 \times \cdots \times 2 = 2^{nm}$ choices for each linear transformation; hence, there are 2^{nm} linear transformations.

An alternative (yet equivalent) way of thinking about this is as follows. The key fact to note is that a linear transformation is *entirely defined by what it does to a basis*. Once I've done that, I can do nothing else – in fact, if I specified my transformation on all my basis vectors and do it on one more vector, I'm very likely to screw up the linearity of my transformation (try it!). So it suffices to specify where $\{v_1, \dots, v_n\}$ go.

So the important question is: where can they go? Well, I'm free to send them anywhere; I can choose the 0 transformation if I want, and send all my basis vectors to 0, just as I could send them all to any other element. I'm *free* to do what I want on my basis vectors. For each vector, I can send them to # elements in $W = 2^m$, so I have $2^m \times 2^m \times \cdots \times 2^m = (2^m)^n = 2^{nm}$ choices, which agrees which what we found above.

If you've followed this so far, the rest is cake. First we determine the number of functions from V to W. The important distinction to make is that now, not only can I send any element in V to any element in W, specifying where element $v_1 \in V$ goes in no way affects where $v_2 \in V$ can go. This means that I can specify where each element in V goes individually. There are 2^m choices for each of the 2^n elements, so I have $2^m \times 2^m \times \cdots \times 2^m$, 2^n *times*, i.e. $(2^m)^{2^n} = 2^{m2^n}$ functions.

What about the dimension? Well, one way of proceeding is to suppose you have some basis for the functions,

$$\{f_1, f_2, \dots, f_k\}.$$

Since each linear combination determines a different function (of which we know there are 2^{m2^n}, and there are 2^k linear combinations, we must have $2^k = 2^{m2^n}$, so $k = m2^n$. Another way of proceeding is to explicitly construct a basis. Consider

$$\mathcal{B} = \{f_1, \dots, f_{ij}, \dots, f_k \mid 1 \leq i \leq 2^n, 1 \leq j \leq m\}$$

where

$$f_{ij}v_l = \begin{cases} (e_j)_W & \text{if } l = k \\ 0_W & \text{if } l \neq k \end{cases}.$$

This is indeed a basis, and it has $\#i \times \#j = 2^n \times m = m2^n$ elements. I could write out why, but this is long enough as it is, and probably the best way to understand why is to give yourself a low-dimensional working example (no pain, no gain).

Question 5. *Let V be the set of real-valued sequences $(a_n)_{n\geq 0}$ satisfying*

$$a_{n+1} = a_n + a_{n-1} \quad \textit{for all } n \geq 1.$$

a. *Show that V, equipped with the usual internal addition and scalar multiplication on sequences, is a vector space over $\mathbb{R}$.*

b. *What is the dimension of V over $\mathbb{R}$?*

c. *A* geometric progression *is a sequence of the form $a_n = \alpha^n$ for some $\alpha \in \mathbb{R}$. Show that V has a basis consisting of geometric progressions, by producing such a basis.*

d. *The* Fibonacci sequence *is the unique sequence $(a_n) \in V$ satisfying*

$$a_0 = a_1 = 1, \quad \textit{so that} \quad a_2 = 2, a_3 = 3, a_4 = 5, a_6 = 15, \quad \textit{etc.}$$

Express this sequence as a linear combination of the basis vectors obtained in part c. Deduce from this a closed form expression for the nth term of the Fibonacci sequence.

Solution a. This should be feeling pretty routine by now; get used to it, because I've found sequence spaces are popular in Analysis II and III so you might as well be comfortable writing this stuff out.

i. The additive identity is the zero sequence (0). $(0) \in V$ since $0 + 0 = 0$, and $(a_n + 0) = (a_n) = (0 + a_n)$ for all $(a_n) \in V$.

ii. The additive inverse of (a_n) is $(-a_n)$. This sequence is in V since $a_{n+1} = a_n + a_{n-1}$ implies $-a_{n+1} = -a_n + (-a_{n-1})$ and $(a_n - a_n) = (0)$.

iii. Associativity of addition:

$$\begin{aligned}((a_n) + (b_n)) + (c_n) = (a_n + b_n) + (c_n) &= ((a_n + b_n) + c_n) \\ &= (a_n + (b_n + c_n)) \\ &= (a_n) + (b_n + c_n) \\ &= (a_n) + ((b_n) + (c_n))\end{aligned}$$

by associativity of addition in $\mathbb{R}$.

iv. Commutativity of addition:

$$(a_n) + (b_n) = (a_n + b_n) = (b_n + a_n) = (b_n) + (a_n)$$

by commutativity of addition in $\mathbb{R}$.

v. Closure: Let $(a_n), (b_n) \in V$. Since $a_{n+1} = a_n + a_{n-1}$ and $b_{n+1} = b_n + b_{n-1}$ for all n, then adding these relations gives

$$(a_{n+1} + b_{n+1}) = (a_n + b_n) + (a_{n-1} + b_{n-1})$$

hence $(a_n) + (b_n) \in V$.

vi. Distributivity of scalars over vectors: let $\alpha, \beta \in \mathbb{R}$ and $(a_n) \in V$. Then

$$(\alpha+\beta)(a_n) = ((\alpha+\beta)a_n) = (\alpha a_n+\beta a_n) = (\alpha a_n)+(\beta a_n) = \alpha(a_n)+\beta(a_n)$$

since multiplication is right distributive in $\mathbb{R}$.

vii. Distributivity of vectors over scalars: Let $(a_n), (b_n) \in V$ and $\alpha \in \mathbb{R}$. Then

$$\alpha((a_n)+(b_n)) = \alpha(a_n+b_n) = (\alpha(a_n+b_n)) = (\alpha a_n+\alpha b_n) = (\alpha a_n)+(\alpha b_n) = \alpha(a_n)+\alpha(b_n)$$

since multiplication is left distributive in $\mathbb{R}$.

viii. Respecting multiplicative structure: Let $\alpha, \beta \in \mathbb{R}$ and $(a_n) \in V$. Then

$$(\alpha\beta)(a_n) = ((\alpha\beta)a_n) = (\alpha(\beta a_n)) = \alpha(\beta a_n) = \alpha(\beta(a_n))$$

since multiplication is associative in $\mathbb{R}$.

ix. Multiplicative identity: $1(a_n) = (1a_n) = (a_n)$, as necessary.

x. Finally, we check that scalar multiplication does indeed map V to itself: Let $\alpha \in \mathbb{R}$, $(a_n) \in V$. Then since $a_{n+1} = a_n + a_{n-1}$ for all n, multiplying by α gives $\alpha a_{n+1} = \alpha a_n + \alpha a_{n-1}$ so $\alpha(a_n) \in V$.

So we've written the whole thing out in gory detail, but it was extremely routine and probably very familiar; a better solution (which is more motivated and suggests a deeper understanding of the subject) would be to notice that all the basic axioms will hold because the set of $\mathbb{R}$-valued sequences with these operations already forms a vector space. Thus the only things that really need checking (and the only parts of the proof that aren't identical to the case of all $\mathbb{R}$-valued sequences) are closure under addition and scalar multiplication in order to verify we have a subspace.

b. V is 2-dimensional; the most obvious way to see this is that once you've specified the first two terms of a sequence, the others are determined by the recurrence relation. More formally, we can let $v_1 = (1, 0, \ldots)$ and $v_2 = (0, 1, \ldots)$ (where the rest of the terms given by the recurrence). To show that $\{v_1, v_2\}$ is a spanning set, let $(a_n) \in V$ be given. We claim that $(a_n) = a_0 v_1 + a_1 v_2$. This is proven by strong induction; we note that $a_0 v_1 + a_1 v_2 = (a_0, 0, \ldots) + (0, a_1, \ldots) = (a_0, a_1, \ldots)$ so our linear combination agrees with (a_n) on the first two terms. Now suppose $n \geq 3$ and that all the terms up to $n-1$ agree. Then since our linear combination is V, its terms satisfy the recurrence, so the nth term is given by $a_{n-2} + a_{n-1} = a_n$, so our linear combination is indeed equal to (a_n). To show that $\{v_1, v_2\}$ is linearly independent, suppose $\alpha v_1 + \beta v_2 = 0$ for some $\alpha, \beta \in \mathbb{R}$. Then considering only the first two terms we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

so multiplying gives $\alpha = \beta = 0$, proving linear independence. Thus V has a basis of cardinality 2, so is 2-dimensional.

c. Suppose $(\alpha^n) \in V$. Then considering only the first 3 terms of this sequence we must have $1 + \alpha = \alpha^2$, so solving this quadratic equation gives $\alpha = \varphi = \frac{1+\sqrt{5}}{2}$ or $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$. Indeed, considering the first two terms of each, since

$$\det \begin{pmatrix} 1 & 1 \\ \overline{\varphi} & \varphi \end{pmatrix} = \sqrt{5} \neq 0,$$

we see (φ^n) and $(\overline{\varphi}^n)$ are linearly independent, hence form a basis.

d. This amounts to solving the system

$$\begin{pmatrix} 1 & 1 \\ \overline{\varphi} & \varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which can be solved by elimination (or adjugate matrices, or whatever):

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ \overline{\varphi} & \varphi & 1 \end{array}\right) \xrightarrow{L_2 = L_2 - \overline{\varphi} L_1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & \varphi - \overline{\varphi} & 1 - \overline{\varphi} \end{array}\right)$$
$$\xrightarrow{L_2 = \frac{1}{\sqrt{5}} L_2} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1-\overline{\varphi}}{\sqrt{5}} \end{array}\right)$$
$$\xrightarrow{L_1 = L_1 - L_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{\sqrt{5}-1+\overline{\varphi}}{\sqrt{5}} \\ 0 & 1 & \frac{1-\overline{\varphi}}{\sqrt{5}} \end{array}\right)$$

so we have $x = \frac{\sqrt{5}-1+\overline{\varphi}}{\sqrt{5}} = \frac{2\sqrt{5}-2+1-\sqrt{5}}{2\sqrt{5}} = \frac{1}{2} - \frac{1}{2\sqrt{5}}$ and $y = \frac{1-\overline{\varphi}}{\sqrt{5}} = \frac{1}{\sqrt{5}} - \frac{1-\sqrt{5}}{2\sqrt{5}} = \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{1}{2} + \frac{1}{2\sqrt{5}}$ and thus we recover the famous equation:

$$f_n = \frac{1}{2}\varphi^n + \frac{1}{2\sqrt{5}}\varphi^n + \frac{1}{2}\overline{\varphi}^n - \frac{1}{2\sqrt{5}}\overline{\varphi}^n = \frac{\varphi^{n+1} - \overline{\varphi}^{n+1}}{\sqrt{5}}$$