

189-251B: Algebra 2

Practice Midterm Exam

1. Give the definitions of:
 - a) A vector space over a field F .
 - b) Linear independence of vectors.
 - c) A basis of a vector space.
 - d) The dimension of a vector space.

2. Let V be a vector space over a field F , and let (v_1, \dots, v_n) be a list of vectors in V . Let $T : F^n \rightarrow V$ be the linear transformation defined by

$$T(x_1, \dots, x_n) = x_1v_1 + x_2v_2 + \cdots + x_nv_n.$$

- a) Give a necessary and sufficient condition involving the list (v_1, \dots, v_n) guaranteeing that T is injective.
- b) Give a necessary and sufficient condition involving the list (v_1, \dots, v_n) guaranteeing that T is surjective.

3. Let T be a linear transformation on a finite-dimensional vector space and let $m(x)$ be its minimal polynomial. Show that T is invertible if and only if $m(0) \neq 0$.

4. Let V denote the vector space of 2×2 matrices with entries in F , and let T be the linear transformation that sends a matrix M to its transpose:

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Suppose that $2 \neq 0$ in F . Show that T is diagonalisable, by producing a basis of eigenvectors for T . List the eigenvalues of T , and the dimensions of the associated eigenspaces.

Bonus question 5. Let T be a linear transformation and let $m(x) \in F[x]$ be its minimal polynomial. Show that $g(T)$ is invertible if and only if $\gcd(m(x), g(x)) = 1$.