**McGill University** August 2012

**Faculty of Science** Supplemental Final examination

# Abstract Algebra II

## Math 251

???day, August ??, 2012

Time: ?:00 am to ??:00 pm

Examiner: Prof. H. Darmon Associate Examiner: Prof. E.Z.Goren

### INSTRUCTIONS

1. Please write your answers clearly in the exam booklets provided.
2. You may quote any result/theorem seen in the lectures or in the assignments without proving it (unless, of course, it is what the question asks you to prove).
3. This is a closed book exam.
4. Translation dictionary is permitted.
5. Calculators are not permitted.

This exam comprises the cover page and two pages of questions, numbered 1 to 10. Each question is worth 10 points.

1. a) Let $F$ be a field, and let $V$ be the set of $m \times n$ matrices with entries in $F$. Explain (briefly) why $V$ is a vector space over $F$, and compute its dimension by exhibiting an *explicit basis* for $V$.

b) Suppose that $V$ is the set of all (infinite) sequences with values in $F$, having only finitely many non-zero values. (I.e., a typical element of $V$ is a sequence $(a_1, a_2, \ldots, a_n, \ldots)$ with $a_i \in F$ and such that $a_i = 0$ for all $i \geq M$ for a suitable $M$ depending in general on the sequence $(a_i)$.) Does $V$ have a countable basis? Justify your answer.

2. A *generalised Fibonnaci sequence* is a sequence $(a_0, a_1, \ldots, a_n, \ldots)$ of real numbers satisfying the recursive relation

$$a_{n+1} = a_{n+1} + a_n, \qquad \text{for all } n \geq 0.$$

Show that the set of such sequences is a vector space over the field $\mathbf{R}$ of real numbers, and compute its dimension.

3. a) Let $f(x)$ be a polynomial with coefficients in a field $F$, and let $V = F[x]/(f(x))$ be the quotient of the polynomial ring $F[x]$ by the ideal generated by $f(x)$, viewed as a vector space over $F$. Let $T : V \longrightarrow V$ be the linear transformation given by $T(p(x)) = (x+1)p(x)$.

a) Compute the minimal polynomial of $T$.

b) Compute the characteristic polynomial of $T$.

c) Give a necessary and sufficient condition on $f(x)$ for $T$ to be diagonalisable over $F$.

d) Give a necessary and sufficient conditon on $f(x)$ for $T$ to be invertible.

4. A *projection* is a linear transformation (on a finite-dimensional vector space) satisfying the relation $T^2 = T$. Show that a projection is always diagonalisable.

5. Let $V$ be the real vector space of all functions of the form $g(x)e^x$, where $g$ is a real polynomial of degree $\leq 3$, and let $T : V \longrightarrow V$ be the linear transformation given by

$$T(f) = f + 2f'.$$

Write down a basis for $V$ and the matrix of $T$ relative to this basis.

6. Define the following terms:

a) The *dual space* $V^*$ of a vector space $V$;

b) The *dual linear map* $T^*$ attached to a linear transformation $T : V \longrightarrow W$. Be sure to specify what the domain and target of $T^*$ are, and to write down the formula defining $T^*$.

c) Show that $(T_1T_2)^* = T_2^*T_1^*$ for all $T_1 : V \longrightarrow W$ and $T_2 : U \longrightarrow V$.

7. Let $V$ be the vector space $F^7$ of 7-tuples of elements of a field $F$, and let $W$ be the kernel of the map $M : F^7 \longrightarrow F^3$ described by the following $3 \times 7$ matrix (relative to the standard bases of course)

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

Show that two vectors in $V$ having all but *exactly one* coordinate in common cannot both belong to $W$.

8. Let $\mathbf{R}^3$ be equipped with the standard inner product. Starting with the basis $(1, 1, 0)$, $(2, 2, 3)$, $(3, -1, 2)$ of $\mathbf{R}^3$, compute the orthonormal basis that is derived from it by performing the Gramm-Schmidt orthogonalisation process.

9. A linear transformation $T$ on a finite-dimensional Hermitan inner product space is said to be *self-adjoint* if it satisfies the relation $T^* = T$.

a) If $v_1$ and $v_2$ are eigenvectors for $T$ associated to *distinct* eigenvalues, show that they are orthogonal.

b) Show that a self-adjoint $T$ has an eigenvalue, and that all its eigenvalues are real.

10. Let $V = \mathbf{R}^3$ equipped with the standard dot product and resulting distance function, and let $W$ be the hyperplane (i.e., subspace of dimension 2) defined by the equation

$$W = \{(x, y, z) \text{ with } x + y = 2z\}.$$

Compute the vector in $W$ which is closest to the vector $(1, 1, 1)$, relative to the standard Euclidean distance.