189-251B: Honors Algebra 2

Midterm Exam

Wednesday, February 26

Questions 1-4 are worth 25 *points each, for a maximum possible total of* 100 *points. The bonus question is worth* 10 *points.*

1. Let V be a vector space over a field F, and let V_1 and V_2 be two vector subspaces of V. Recall that V is said to be a *direct sum* of V_1 and V_2 (which we write as $V = V_1 \oplus V_2$) if the span of $V_1 \cup V_2$ is equal to V and $V_1 \cap V_2 = \{0\}$.

a) Show that if this is the case, then every vector $v \in V$ can be *uniquely* expressed as a sum $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$.

b) Using nothing more than the basic definition of the dimension of a vector space, show that, if V_1 and V_2 are finite-dimensional, and $V = V_1 \oplus V_2$, then V is also finite-dimensional, and $\dim(V) = \dim(V_1) + \dim(V_2)$.

2. A linear transformation $T : V \to V$ is said to be an *idempotent* if it satisfies the identity $T^2 = T$. Show that T is diagonalisable and that

$$V = \ker(T) \oplus \mathrm{Image}(T).$$

3. Write down the minimal and characteristic polynomials of the following linear transformations, and state whether they are digaonalisable.

a) The transformation $T : F^2 \to F^2$ on the space of column vectors with entries in F given by left multiplication by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, when $F = \mathbf{R}$.

b) Same question as in a), but with $F = \mathbf{Z}/2\mathbf{Z}$.

c) The transformation $T : F^2 \to F^2$ on the space of column vectors with entries in F given by left multiplication by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, when $F = \mathbf{R}$.

d) Same question as in c), but with $F = \mathbf{Z}/5\mathbf{Z}$.

e) The transformation $f(x) \mapsto f'(x)$ on the (20-dimensional) real vector space of polynomials of degree ≤ 19 with real coefficients. (Here $f'(x)$ denotes the derivative of the polynomial f with respect to x.)

4. Let V be a finite-dimensional vector space over a field F and let $T : V \to V$ be a linear transformation of prime order p, i.e., a transformation satisting $T^p = I$, where I denotes the identity tranformation.

a) Show that if F is algebraically closed and $p \neq 0$ in F, the linear transformation T is diagonalisable.

b) If $p = 0$ in F (for instance, if $F = \mathbf{Z}/p\mathbf{Z}$ is the field with p elements), show that T is diagonalisable if and only if it is the identity transformation.

The next problem is a Bonus Question. Only attempt it if you are confident that you've answered the first 4 questions completely. With p a prime number as in question 4, Give an example of a non-identity 2×2 matrix of order p with entries in the field $\mathbf{Z}/p\mathbf{Z}$ with p elements. Show that any two matrices of this kind are necessarily conjugate to each other. (Recall that two matrices M_1 and M_2 are said to be conjugate if there is an invertible matrix P for which $M_2 = P^{-1}M_1P$.)