189-251B: Algebra 2
Assignment 10
Due: Wednesday, March 26

1. Let T be a normal operator on a finite-dimensional complex inner product space. Show that T has a square root, i.e., there is a normal operator S such that $S^2 = T$.

2. Show that a self-adjoint operator T on a finite-dimensional real inner product space need not have a square root. Does it always have a cube root, i.e., is there a self-adjoint linear transformation S such that $S^3 = T$?

3. Let $V = M_n(\mathbf{C})$ be the complex vector space of $n \times n$ matrices with complex entries, and define a complex-valued function on $V \times V$ by the rule

$$\langle A, B \rangle = trace(AB^*),$$

where B^* denotes the conjugate transpose of the matrix B, and the trace of a matrix is the sum of its diagonal entries.

(a) Show that this function defines a complex inner product on V.

(b) Given a matrix $M \in M_n(\mathbf{C})$, define a linear transformation $T_M : V \longrightarrow V$ by the rule $T_M(A) = MA$. Show that $T_M^* = T_{M^*}$. (Hint: the identity $trace(AB) = trace(BA)$ may be useful.)

4. A linear operator T on a real inner product space is called an *isometry* (or an orthogonal transformation) if $||T(v)|| = ||v||$ for all $v \in V$. An $n \times n$ matrix M is called *orthogonal* if its columns are an orthonormal set of vectors relative to the standard inner product on $\mathbf{R}^n$. Justify this multiple use of the word "orthogonal " by showing that the matrix of an orthogonal transformation relative to an orthonormal basis for V is an orthogonal matrix. (*We already*

did this, pretty much, in class, but perhaps a bit quickly: this exercise is just to give you a chance to go over your notes...).

5. Show that if T is any linear operator on an inner product space V, then TT^* is self-adjoint.

6. A linear transformation on an inner product space is said to be *skew-adjoint* if $T^* = -T$. Show that ever linear transformation T on a (real or complex) inner product space can be uniquely writen as the sum of a self-adjoint transformation T_1 and a skew-adjoint transformation T_2, and that T_1 and T_2 commute if and only if T is normal.

7. Let V be the vector space of infinite sequences $(a_1, a_2, \ldots)$ with entries in a field F. The goal of this exercise is to lead you to prove that V *never* admits a countable basis. In class, we proved this when F is a finite field, by observing that a vector space over F with a countable basis would have to be countable, while V (as a set) is uncountable. As some of you pointed out to me after class, this argument is not completely satisfying: it does not extend to the case where F is uncountable ($F = \mathbf{R}$ for example) and besides, such a statement really ought to admit a proof that is "independent of the field".

a) Show that, if V admits a countable basis, then there is a nested sequence of subspaces

$$W_1 \subset W_2 \subset \cdots \subset W_n \subset \cdots$$

satisfying

$$\dim(W_j) = j, \qquad \cup_{j=1}^{\infty} W_j = V.$$

(The latter equality asserts that every vector v in V belongs to *some* W_j, for j sufficently large and depending of course on v.)

b) It will be convenient to represent a sequence $(a_n)_{n\geq 1}$ as a concatenation of sequences $(s_2, s_3, s_4, \ldots)$ where s_j is a sequence of length j. For each $j \geq 2$, show that there is a sequence S_j of length j with the property that

$$s_j = S_j \Rightarrow (s_2, \ldots, s_j, \ldots) \notin W_{j-1}.$$

c) Use b) and a diagonalisation argument analogous to Cantor's to construct a vector $v \in V$ which does not belong to any of the W_j's. Conclude that V does not have a countable basis.