# 189-251B: Algebra 2
# Assignment 9
## Due: Wednesday, March 19

1. Prove or give a counterexample: the product of any two self-adjoint operators on an inner product space is self adjoint.

2. Let $T \in \mathcal{L}(V)$ be an idempotent linear transformation, (i.e., a transformation satisfying $T^2 = T$) on a finite-dimensional inner product space. Show that $T$ is the orthogonal projection onto its image if and only if $T$ is self-adjoint.

3. Show that a normal operator on an inner product space is self-adjoint if and only if all its eigenvalues are real.

4. Let $V$ be the real vector space of infinitely differentiable $\mathbf{R}$-valued functions $f : [0,1] \longrightarrow \mathbf{R}$ satisfying

$$f(0) = f(1) = f'(0) = f'(1) = \cdots = f^{(j)}(0) = f^{(j)}(1) = \cdots = 0.$$

Equip $V$ with the standard inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Let $T : V \longrightarrow V$ be the linear transformation given by $T(f) = f'$. Show that $T$ is normal.

5. Suppose $V$ is a (real or complex) inner product space, and that $T : V \longrightarrow V$ is self-adjoint. Suppose that there is a vector $v$ with $||v|| = 1$, a scalar $\lambda \in F$, and a real $\epsilon > 0$ such that

$$||T(v) - \lambda v|| < \epsilon.$$

Show that $T$ has an eigenvalue $\lambda'$ such that $|\lambda - \lambda'| < \epsilon$. Discuss the practical significance of this result.

6. Prove that if $T$ is a normal operator on a finite-dimensional inner product space, then it has the same image as its adjoint.

7. Prove that there does not exist a self-adjoint operator $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ (where $\mathbf{R}^3$ is equipped with the standard dot product) satisfying

$$T(1,2,3) = (0,1,0), \quad T(2,5,7) = (1,1,1).$$

8. Let $T$ be a linear transformation on a finite dimensional real vector space $V$. Show that $T$ is diagonalisable if and only if there exists an inner product on $V$ relative to which $T$ is self-adjoint.