189-251B: Algebra 2
Assignment 7
Due: Wednesday, February 26

1. Let $T : V \longrightarrow V$ be a linear transformation of finite-dimensional vector spaces over a field F which is not necessarily algebraically closed, and let $p(x) = p_1(x) \cdots p_r(x)$ be the factorisation of $p(x)$ into monic irreducible factors.

a) Assume that the factors $p_j(x)$ are all distinct. Show that there is a direct sum decomposition

$$V = V_1 \oplus \cdots \oplus V_r$$

into F-vector subspaces V_j which are stable under T, and such that the restriction T_j of T to V_j has $p_j(x)$ as its minimal polynomial.

b) If $\dim_F(V_j) = \deg(p_j)$, show that V_j cannot be further decomposed into a direct sum of T-stable F-vector subspaces.

2. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots$ be a power series in x with complex coefficients and suppose that $\lambda \in \mathbf{C}$ belongs to the disk of absolute convergence for f. Let $M(\lambda, d)$ be the $d \times d$ Jordan matrix with eigenvalue λ:

$$M(\lambda, d) = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Show that the infinite sum

$$f(M(\lambda, d)) = a_0 + a_1 M(\lambda, d) + a_2 M(\lambda, d)^2 + \cdots$$

converges to a $d \times d$ matrix, and give a closed form expression for this matrix. (You may find it helpful to consider first the simplest cases where $d = 2$ and $d = 3$ in order to guess the general pattern.)

3. Let e^x denote the exponential function defined by the power series

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$$

Using exercise 2 and the theory of the Jordan canonical form, show that the infinite sum defining e^M converges, for any $d \times d$ matrix M with complex entries, and describe an algorithm to compute it. Use your algorithm to compute the exponential of the 2×2 matrix $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.