189-251B: Algebra 2
Assignment 3

Due: Wednesday, January 29.

1. Solve the linear equation $Ax = y$ over the field $\mathbf{Z}_2$, where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

2. Solve the linear equation $Ax = y$ over the field $\mathbf{Z}_2$,

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

How many solutions does this equation have? Write them all down.

3. Show that the number of distinct solutions of a system of linear equations (in any number of equations, and unknowns) over the field $\mathbf{Z}_p$ is either 0, or a power of p.

4. Show that there is no error-correcting code of dimension 5 in $\mathbf{Z}_2^7$, so that the example constructed in class is in some sense optimal, in the sense that it is the error-correcting code of largest possible dimension in $\mathbf{Z}_2^7$.

5. What is the largest possible dimension of a linear code in $\mathbf{Z}_2^{15}$ that can detect and correct a single (i.e., one-bit) error?

6. Let V be the vector space of polynomials of degree at most 5 over the field $\mathbf{R}$ of real numbers, and let T be the linear transformation from V to V defined by

$$T(f) = \frac{d^3}{dx^3} f + \frac{d^2}{dx^2} f.$$

Describe the kernel of T, and its image. What are the dimensions of these subspaces? What is the subspace of V generated by $\ker(T)$ and $\mathrm{Image}(T)$?

7. A linear transformation $T : V \longrightarrow V$ is called a *projection*, or an *idempotent*, if it satisfies $T^2 = T$. Show that V can be expressed as the *direct sum* of $\ker(T)$ and $\mathrm{Image}(T)$.

8. Let V be a vector space over a field F. A linear transformation $N : V \longrightarrow V$ is said to be *nilpotent* if $N^k = 0$ for some k. Show that if N is nilpotent, then the linear transformation $1 - N$ (where 1 denotes the identity transformation) is invertible. (Hint: think of the power series expansion for $\frac{1}{1-x}$ when $x < 1$.)