189-251B: Algebra 2
Assignment 2
Due: Wednesday, January 22

1. Let $\mathbf{C}$ be the field of complex numbers and let V be the $\mathbf{C}$-vector space $\mathbf{C}^3$. Find the coordinates of the vector $(1, 0, 1)$ in the basis (v_1, v_2, v_3), where

$$v_1 = (2i, 1, 0), v_2 = (2, -1, 1), v_3 = (0, 1 + i, 1 - i).$$

2. Let V denote the $\mathbf{Q}$-vector space of 2×2 matrices with entries in $\mathbf{Q}$. Let A be the matrix $\begin{pmatrix} 2 & -5 \\ 3 & 17 \end{pmatrix}$. Show that the function

$$T : V \longrightarrow V \quad \text{defined by } T(X) = AX - XA$$

is a linear transformation from V to V. Choose a basis for V, and write down the matrix of T with respect to your basis.

3. Let V be a finite-dimensional vector space.

a. Show that any linear transformation $T : V \longrightarrow V$ is injective if and only if it is surjective.

b. Show that neither property implies the other, if the finite-dimensionality assumption on V is dropped.

4. Let V and W be finite-dimensional vector spaces, of dimension n and m respectively, over the field $\mathbf{Z}_2$. How many functions are there from V to W? How many linear transformations? What are the dimensions of these spaces of functions, and linear transformations, respectively?

5. Let V be the set of real-valued sequences $(a_n)_{n\geq 0}$ satisfying

$$a_{n+1} = a_n + a_{n-1} \quad \text{for all } n \geq 1.$$

a. Show that V, equipped with the usual internal addition and scalar multiplication on sequences, is a vector space over $\mathbf{R}$.

b. What is the dimension of V over $\mathbf{R}$?

c. A *geometric progression* is a sequence of the form $a_n = \alpha^n$, for some $\alpha \in \mathbf{R}$. Show that V has a basis consisting of geometric progressions, by producing such a basis.

d. The *Fibonacci sequence* is the unique sequence $(a_n) \in V$ satisfying

$$a_0 = a_1 = 1, \quad \text{so that} \quad a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8, a_6 = 13, \quad \text{etc.}$$

Express this sequence as a linear combination of the basis vectors obtained in part c. Deduce from this a closed form expression for the nth term of the Fibonacci sequence.