189-251B: Algebra II

Assignment 1

Due: Wednesday, January 15

1. Show that the set $V = \mathbf{R}^2$, with the addition and scalar multiplication defined by the rules

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2), \quad \lambda \cdot (x, y) := (\lambda x, 0)$$

satisfies all the axioms of a vector space *except* the property that $1 \cdot v = v$ for all $v \in V$.

2. Which of the following subsets of $\mathbf{R}^3$ are $\mathbf{R}$-vector subspaces of $\mathbf{R}^3$?

   1. The set of vectors (x, y, z) with $x \geq 0$.
   2. The set of vectors (x, y, z) satisfying $x + y = 2z$.
   3. The set of vectors satisfying $x = 0$ or $y = 0$.
   4. The set of vectors satisfying $x = 0$ and $y = 0$.
   5. The set of vectors with rational coordinates x, y, z.

3. Let V_1 and V_2 be two vector subspaces of an F-vector space V. Show that the union $V_1 \cup V_2$ can only be a subspace of V if one of the spaces V_i ($i = 1$ or 2) is contained in the other.

4. Let V be the vector space of $n \times m$ matrices with entries in a field F. What is the dimension of V? Give an explicit basis for V over F.

5. Let v_1, v_2, and v_3 be three linearly independent vectors in an $\mathbf{R}$-vector space V. Show that the vectors $v_1 + v_2$, $v_2 + v_3$, and $v_3 + v_1$ are also linearly independent. What if the field $\mathbf{R}$ is replaced by the field $\mathbf{Z}_2$ in this question?

6. Let V be the $\mathbf{R}$-vector space of all infinitely differentiable functions on the real line. Show that the function $T : V \to V$ defined by $T(f) = f'$ (where f' denotes as usual the derivative of f) is a linear transformation from V to itself. Show that T is not injective, and compute its kernel. Show that T is surjective. (Hint: use the fundamental theorem of calculus!) Conclude that V is not finite dimensional.

7. A *generalised vector space* over a field F is a *not necessarily commutative* group V (so that the group operation is written using the multiplicative notation) equipped with a "scalar multiplication"

$$F \times V \to V \quad \text{denoted} \quad (\lambda, v) \mapsto v^{[\lambda]},$$

satisfying the following axioms analogous to those of a usual vector space

M1 $(vw)^{[\lambda]} = v^{[\lambda]} w^{[\lambda]}$ for all $v, w \in V$ and $\lambda \in F$;

M2 $v^{[\lambda_1+\lambda_2]} = v^{[\lambda_1]} v^{[\lambda_2]}$, for all $v \in V$ and $\lambda_1, \lambda_2 \in F$;

M3 $(v^{[\lambda_1]})^{[\lambda_2]} = v^{[\lambda_1 \lambda_2]}$, for all $v \in V$, and $\lambda_1, \lambda_2 \in F$;

M4 $v^{[1]} = v$, for all $v \in V$.

Show that a generalised vector space is just an ordinary vector space: i.e., the group law on V is necessarily commutative. (Hint: Show that $v^{[-1]} = v^{-1}$, the latter expression being the inverse of v in the group V, and consider axiom $M1$ with $\lambda = -1 \in F$.)

8. Let X be a set, and let $\mathcal{P}(X)$ denote the *power set* of X, i.e., the set of all subsets of X. Define the sum of two sets to be

$$A + B := A \cup B - (A \cap B),$$

and define a scalar multiplication of $\mathbf{Z}_2$ on $\mathcal{P}(X)$ by the rule:

$$0 \cdot A := \emptyset, \quad 1 \cdot A = A.$$

Show that $\mathcal{P}(X)$ with these operations is a vector space over $\mathbf{Z}_2$. What is its dimension?