189-235A: Basic Algebra I

Final Exam

Monday, December 5, 2005

This exam has ten questions, worth 10 points each. The bonus question is worth 20 points. The final grade will be out of 100*, even though the maximum possible grade is* 120.

1. Show that 64 divides $9^n - 8n - 1$ for every $n \geq 0$, by induction on n.

2. Solve each of the following congruence equations.
   a) $6x = 3 \pmod{30}$,
   b) $x^2 + 4x + 5 = 0 \pmod{7}$.
   c) $x^7 = 1 \pmod{101}$.

3. Let G be an abelian group. Show that, if G is not cyclic, then there a divisor d of $n = \#G$ which is *strictly smaller* than n satisfying $x^n = 1$ for all $x \in G$.

4. Prove that the ring $\mathbf{R}$ of real numbers and the ring $\mathbf{Q}$ of rational numbers are not isomorphic.

5. List the powers $[x]^j$ (for $0 \leq j \leq 7$) of $[x] = x + (x^3 + x + 1)\mathbf{Z}_2[x]$ in the quotient ring $\mathbf{Z}_2[x]/(x^3 + x + 1)$. You should label each result by its unique representative of degree ≤ 2. Using this calculation, write down the multiplicative inverse of $[x]$.

6. Give an example of two *non-isomorphic* rings of cardinality 16 that are *non-commutative*. (You should justify your answer and prove that these two rings are not isomorphic to each other.)

7. Give the definitions of: *maximal ideal* and *prime ideal* in a commutative ring R. Show that if I is a maximal ideal then R/I is a field.

8. Write down the class equation for the symmetric group S_5 on 5 elements. (I.e., list all of the conjugacy classes in S_5, along with their cardinalities.) Use this to give a list of all the normal subgroups of S_5.

9. Show that the groups $\mathbf{GL}_2(\mathbf{Z}_2)$ and S_3 are isomorphic by constructing an isomorphism from one to the other.

10. Write down five non-isomorphic groups of order 8. (You do not need to show that these groups are non-isomorphic.)

Extra Credit Problem

11. Show that there is no surjective homomorphism from $\mathbf{Z}_2[x]$ to the ring $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.