189-235A: Algebra 1

Midterm Exam

Wednesday, October 16

Each of the four questions below is worth 25 points. No calculators or outside materials are allowed during the exam.

1. Solve the following congruence equations (making sure you list *all* the distinct solutions in the given ring). You do not need to justify your answers.

(a) $3x = 4$ in $\mathbf{Z}/13\mathbf{Z}$.
(b) $3x = 4$ in $\mathbf{Z}/12\mathbf{Z}$.
(c) $3x = 9$ in $\mathbf{Z}/12\mathbf{Z}$.
(d) $x^2 = x$ in $\mathbf{Z}/101\mathbf{Z}$.
(e) $x^2 = -1$ in $\mathbf{Z}/303\mathbf{Z}$.

2. a) Let a and $b \in \mathbf{Z}$ be two integers whose only common divisors are 1 and -1. If c is any integer, prove that a divides bc if and only if a divides c.

b) Give a counterexample to the statement of part a) if the ring $\mathbf{Z}$ of regular integers is replaced by the ring $\mathbf{Z}[\sqrt{-5}]$ consisting of all integer linear combinations of 1 and $\sqrt{-5}$.

3. Let n be an integer which is not divisible by 2, 3, 5 or 7. Show that if $n < 121$, then n is a prime number.

4. Compute the value of 2^{15} modulo 15. Explain why your calculation gives a (somewhat perverse[1]) proof that 15 is not a prime number.

[1] There are even more perverse ways of going about it: in the last decade of the 20th century, untold millions of dollars were invested in the (ultimately successful) project to factor the number 15 on a *quantum computer*.