ABELIAN VARIETIES AND p-DIVISIBLE GROUPS

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Goals of this talk: - develop an "intuition" about abelian varieties
- define p-divisible groups
- describe p-divisible groups arising from abelian varieties

1. Abelian varieties

1.1. **The 1-dimensional case: elliptic curves.** A way to approach abelian varieties is to think about them as the higher-dimensional analague of elliptic curves. We may define elliptic curves in various equivalent ways:
- a projective plane curve over k with cubic equation

$$X_1^2 X_2 = X_0^3 + aX_0X_2^2 + bX_2^3,$$

($char\, k \neq 2, 3$ and $4a^3 + 27b^2 \neq 0$),
- a nonsingular projective curve with compatible group structure,
- in the complex case, as a complex torus $T \simeq \mathbb{C}/\Lambda$.

In fact we can generalize the second notion (and the third under some restrictions). Abelian varieties cannot be defined in general by equations.

Definition 1.1.1 (Abelian variety). *An abelian variety X over a field k is a complete algebraic variety with an algebraic group structure, that is, X is a complete variety over k, together with k-morphisms*

$$m : X \times X \to X,$$
$$i : X \to X,$$
$$e : X \to k,$$

satisfying the group axioms.

We will write the group law additively.
Comment on the field of definition: For our purposes, k will usually be $K/\mathbb{Q}_p$ a finite extension.

Example. *The product $E \times \cdots \times E$ of an elliptic curve is an abelian variety.*

1.2. **Questions.** - Structure as an abstract group: is it commutative?
- Is it projective?
- The group structure allows as to define for every n a multiplication by n map

$$[n] : X \to X.$$

Is it surjective? How does its kernel look like?

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1.3. **First properties.**

Lemma 1.3.1 (Rigidity Lemma - case of abelian varieties)**.** *Let X, Y be abelian varieties and let $f : X \times X \to Y$ be a morphism such that there exists a point $x_0 \in X$ and a point $y_0 \in Y$ such that $f : (X \times \{x_0\}) = y_0$. Then $f(X \times X) = y_0$.*

Structure of morphisms of abelian varieties. A *homomorphism* $f : X \to Y$ of abelian varieties is a morphism respecting the group structures. We denote by $T_x : X \to X$ the translation morphism by $x \in X$, $T_x(y) = y + x, \forall y \in X$.

Corollary 1.3.2. *Any morphism $f : X \to Y$ of abelian varieties is the composition of a translation and a homomorphism.*

Proof. By composing (in case) f with a translation, we may always suppose $f(0_X) = 0_Y$. We need hence to show that f is a homomorphism. Define $F : X \times X \to Y$ by $F(x, y) = f(x + y) - f(x) - f(y)$. By the rigidity lemma we conclude that $F \equiv 0$ □

Corollary 1.3.3. *X is commutative.*

Proof. Since a group is commutative if and only if the inversion map is linear, we conclude by the previous corollary. □

1.4. **Abelian varieties over the complex field.**

Fact. *Let X be an algebraic variety over $\mathbb{C}$. We can endow X with the structure of an analytic manifold. We will denote by $X(\mathbb{C})$ the space X with the complex topology. (This is the functor of points).*

If X is an abelian variety, $X(\mathbb{C})$ is a connected, compact analytic manifold with commutative group structure. The space $X(\mathbb{C})$ is a complex torus, that is $X(\mathbb{C}) \simeq \mathbb{C}^g/\Lambda$.

Note. *Differently from the dim 1 case, not every complex torus has structure of an abelian variety.*

Question. *When is a complex Torus a complex abelian variety? In other words, given $T \simeq \mathbb{C}^g/\Lambda$, when does it exists X abelian variety such that $X(\mathbb{C}) \simeq T$?*

1.4.1. *Recalls on invertible sheaves.*

Theorem 1.4.1 (Chow)**.** *Let X be a complete complex algebraic variety and let Y be a closed analytic subset of X. Then there exists and algebraic subvariety Z of X such that $Z(\mathbb{C}) \simeq Y$.*

Corollary 1.4.2. *A complex torus is an abelian variety if and only if it has an ample invertible sheaf.*

Corollary 1.4.3. *An abelian variety over $\mathbb{C}$ is projective.*

In fact this is true for any abelian variety. It is however quite complicated to show in the general case that an abelian variety has always an ample invertible sheaf. We try to describe invertible sheaves on an abelian variety X in order prove properties "similar" to those satisfied by complex tori. (A complex torus is commutative...)

1.5. **Invertible sheaves on abelian varieties.** The first obvious question is: does any abelian variety have an ample invertible sheaf? The answer is yes, but not trivial. Given $\mathscr{L} \in \operatorname{Pic}(X)$ we may define a rational map from X to some projective space. Any $\mathscr{L} \in \operatorname{Pic}(X)$ is characterized by a finite number of local sections $f_0, \dots, f_n$. We put

$$\begin{array}{ccc} X & \dashrightarrow & \mathbb{P}^n \\ P & \mapsto & (f_0 : \dots : f_n) \end{array}$$

Such a map is defined on those $P \in X$ such that there exists i such that $f_i(P) \neq 0$ and P is not a pole for any of the f_i's. Starting from any $\mathscr{L} \in \operatorname{Pic}(X)$ we can obtain an invertible sheaf such that the map is defined on any $P \in X$, that is, the invertible sheaf is very ample.

Corollary 1.5.1. *Every abelian variety is projective.*

Theorem 1.5.2 (of the cube)**.** *Let X, Y be complete varieties, Z any variety and let x_0, y_0, z_0 be points in X, Y, Z respectively. if $\mathscr{L}$ is an invertible sheaf on $X \times Y \times Z$, such that its restrictions to each of $x_0 \times Y \times Z$, $X \times y_0 \times Z$ and $X \times Y \times z_0$ are trivial, then $\mathscr{L}$ is trivial.*

Corollary 1.5.3. *Let X be any variety, Y an abelian variety and $f, g, h : X \to Y$ morphisms. Then for all $\mathscr{L} \in \operatorname{Pic}(Y)$ we have*

$$(f+g+h)^*\mathscr{L} \otimes (f+g)^*\mathscr{L}^{-1} \otimes (g+h)^*\mathscr{L}^{-1} \otimes (f+h)^*\mathscr{L}^{-1} \otimes f^*\mathscr{L} \otimes g^*\mathscr{L} \otimes h^*\mathscr{L}$$

is trivial.

Corollary 1.5.4. *If X is an abelian variety and $\mathscr{L} \in \operatorname{Pic}(X)$ we have*

$$[n]^*\mathscr{L} \simeq \mathscr{L}^{\frac{n^2+n}{2}} \otimes [-1]^*\mathscr{L}^{\frac{n^2-n}{2}}.$$

Proof. We prove the result by induction on n. If we apply the previous corollary with $f = [n], g = [1], h = [-1]$,we obtain

$$[n]^*\mathscr{L} \otimes [n+1]^*\mathscr{L}^{-1} \otimes [n-1]^*\mathscr{L}^{-1} \otimes [n]^*\mathscr{L} \otimes \mathscr{L} \otimes [-1]^*\mathscr{L}$$

is trivial, which is equivalent to saying that

$$[n+1]^*\mathscr{L} \simeq [n-1]^*\mathscr{L}^{-1} \otimes [n]^*\mathscr{L}^2 \otimes \mathscr{L} \otimes [-1]^*\mathscr{L}.$$

Hence, for $n = 1$ the statement above is true. Suppose now true for n, the statement for $n+1$ follows by computations. □

Corollary 1.5.5 (Theorem of the square)**.** *For any $\mathscr{L} \in \operatorname{Pic}(X)$ and any $x, y \in X$, we have*

$$T_{x+y}^*\mathscr{L} \otimes \mathscr{L} \simeq T_x^*\mathscr{L} \otimes T_y^*\mathscr{L}.$$

Proof. Apply Corollary (1.5.3) with $f = x, g = y, g = id$. □

Note that tensoring (1.5.5) with $\mathscr{L}^{-2}$ we obtain

$$T_{x+y}^*\mathscr{L} \otimes \mathscr{L}^{-1} \simeq (T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}) \otimes (T_y^*\mathscr{L} \otimes \mathscr{L}^{-1}),$$

and this defines in particular a homomorphism

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{\mathscr{L}}} & \operatorname{Pic}(X) \\ x & \longmapsto & T_x^*\mathscr{L} \otimes \mathscr{L}^{-1} \end{array}$$

In terms of divisors this means that for $D \in \mathrm{Div}(X)$

$$T_{x+y}^* D + D \sim_l T_x^* D + T_y^* D.$$

1.6. **Applications of these results.** From the theorem of the cube we obtain the following result.

1.7. **Construction of the dual abelian variety.** Let X be an abelian variety. Recall that for any $\mathscr{L} \in \mathrm{Pic}(X)$ we define a homomorphism

$$\phi_{\mathscr{L}} : X \to \mathrm{Pic}(X).$$

Define $\mathrm{Pic}^0(X) := \{\mathscr{L} \in \mathrm{Pic}(X) \mid \phi_{\mathscr{L}} \equiv 0\}$. The theorem of the square tells us that $\phi_{\mathscr{L}}(X) \subseteq \mathrm{Pic}^0(X)$, indeed

$$T_y^*(T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}) \otimes (T_x^*\mathscr{L} \otimes \mathscr{L}^{-1})^{-1} = T_{x+y}^*\mathscr{L} \otimes T_y^*\mathscr{L}^{-1} \otimes T_x^*\mathscr{L}^{-1} \otimes \mathscr{L} = 0.$$

Hence there is an exact sequence

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Pic}^0(X) & \to & \mathrm{Pic}(X) & \to & \mathrm{Hom}(X, \mathrm{Pic}^0(X)) \\ & & & & \mathscr{L} & \mapsto & \varphi_{\mathscr{L}} \end{array}$$

Denote by $K(\mathscr{L})$ the kernel $\mathrm{Ker}(\phi_{\mathscr{L}}) = \{x \in X \mid T_x^*\mathscr{L} \simeq \mathscr{L}\}$.

Proposition 1.7.1. *Let $\mathscr{L} \in \mathrm{Pic}(X)$ be ample:*

- *$K(\mathscr{L})$ is finite,*
- *$\phi_{\mathscr{L}} : X \to \mathrm{Pic}^0(X)$ is surjective.*

That is, $\phi_{\mathscr{L}}$, is an isogeny.

So, as an abstract group $\mathrm{Pic}^0(X) \simeq X/K(\mathscr{L})$ and it turns out that it has a natural structure of abelian variety, which we'll denote by $\widehat{X}$, the *dual abelian variety*.

Definition 1.7.2 (Isogenies)**.** *Let $\phi : X \to Y$ be a homomorphism of abelian varieties. We say that ϕ is an* isogeny*, if it is surjective and* $\mathrm{Ker}(\phi)$ *is finite.*

Theorem 1.7.3. *Let X be an abelian variety of dimension g. It is divisible, that is, the map*

$$[n] : X \leftrightarrow X$$

is surjective for every n. Moreover, for every $n \geq 1$, the kernel $\mathrm{Ker}[n] := X[n]$ *is finite, that is $[n]$ is an isogeny.*

Proof. In order to show that $[n]$ is an isogeny, we just need to show that $\mathrm{Ker}[n]$ is finite (indeed, for any $x \in [n](X)$, one has $dim[n]^{-1}(x) \geq dimX - dim[n](X)$). Consider $\mathscr{L} \in \mathrm{Pic}(X)$ ample: by Corollary (1.5.4) $[n]^*\mathscr{L}$ is again ample. Then its restriction to $\mathrm{Ker}[n]$ is both trivial and ample, and hence $\mathrm{Ker}[n]$ has dimension zero. □

2. Group schemes

Definition 2.0.4. *Fix a base scheme S (ofter we want $S = \operatorname{Spec} \mathscr{O}_K$). We say that $G \to S$ is an S-*group *if it has group structure as an object in the category Sch/S. Explicitly, this means that there are S-maps*

$$m : G \times_S G \to G \text{ multiplication},$$
$$i : G \to G \text{ inversion},$$
$$e : S \to G \text{ neutral element},$$

satisfying the group axioms. We say that a group scheme is commutative *if the commutativity is satisfied by these maps.*

Namely, a group scheme over S corresponds to a contravariant functor from the category of schemes over S to the category of groups.

Basic examples. 1. The *additive group scheme* $\mathbb{G}_a$, corresponding to the additive group structure underlying the affine line.

2. The *multiplicative group scheme* $\mathbb{G}_m$, corresponding to the multiplicative group structure underlying the affine line without the origin.

Remark. *We will always use commutative groups schemes and therefore all the results will refer to these (even though some of them might be true in general).*

Definition 2.0.5. *Given $\pi : R \to S$ a scheme over S, we define the set of R-points of G as*

$$G(R) := \operatorname{Hom}_{Sch/S}(R, G).$$

It has a natural group structure: if $f, g \in G(R)$, then $fg := m \circ (f \times g) \in G(R)$: $id_{G(R)} := e \circ \pi$.
If $R = \operatorname{Spec} A$, we will denote $G(R) = G(A)$.
This makes perfect sense if we think functorially of a group scheme.

Definition 2.0.6. *A* finite group scheme *over S of rank r is a group scheme $f : G \to S$, such that it is finite as a scheme over S and locally free of rank r.*

Definition 2.0.7 (Cartier dual)**.** *Let G be a finite commutative group of rank r. We define its dual as the group of characters*

$$G^* := \operatorname{Hom}_{GrSch/S}(G, \mathbb{G}_m).$$

Examples. 1. The *p^n-th roots of unity* $\mu_{p^n} = \{x \in \mathbb{G}_m \mid x^{p^n} = 1\} \subset \mathbb{G}_m$.

2. Its dual $\mathbb{Z}/p^n\mathbb{Z}$.

3. The *p^n-th roots of zero* $\alpha_{p^n} = \{x \in \mathbb{G}_a \mid x^{p^n} = 0\}$.

2.1. Back to abelian varieties.

Given an isogeny of abelian varieties $f : A \to B$, we have that $\operatorname{Ker}(f)$ is a finite group scheme. ASK. Again, we consider the isogeny $[n] : X \to X$. Denote by $X[n] := \operatorname{Ker}(A(\bar{k}) \to A(\bar{k}))$. For $p = char k$

$$X[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}, \text{ if } p \not| n,$$
$$X[p^m] \simeq (\mathbb{Z}/p^m\mathbb{Z})^i, \forall m, \exists 0 \le i \le g.$$

Clear in the complex case: $X \simeq \mathbb{C}^g/\Lambda$, then

$$X[n] \simeq \frac{1}{n}\Lambda/\Lambda \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

We define the *Tate module* as

$$T_p(X) = \varprojlim_n X[p^n].$$

One has

$$T_p(X) = \begin{cases} (\mathbb{Z}_p)^{2g} & \text{if } char k \neq p \\ (\mathbb{Z}_p)^i & \text{if } char k = p \end{cases}$$

Definition 2.1.1. *An* abelian scheme X *over* $S = \operatorname{Spec} \mathscr{O}_K$ *is a proper smooth group scheme over* S.

The fibers of an abelian scheme are abelian varieties and hence we may see an abelian scheme as a family of abelian varieties parametrized by the base scheme S. Many of the results proved in the case of abelian varieties still hold. For example the rigidity theorem generalized to this case and as a consequence the inverse map is a homomorphism (an abelian scheme is hence commutative).

Given an abelian scheme X we can define the multiplication by n map ($n \in \mathbb{Z}$), like we did in the case of abelian varieties:

$$[n] : X \to X.$$

We denote by $X[n]$ its kernel. We have seen that $\operatorname{Ker}[n]$ is finite on each fiber, so it is finite of rank n^{2g} on X.

One defines the *dual abelian scheme* X^* through a scheme-theoretic generalization of the construction of the dual abelian variety. However, the construction is related to Cartier duality through the following identity $\forall n$

$$(X[n])^* \simeq X^*[n].$$

For p a prime number, again we define the *Tate module* of an abelian scheme X

$$T_pX = \varprojlim X(\overline{K})[p^n]$$

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3. p-divisible groups

Fix p a prime. Let $S = \operatorname{Spec}(\mathcal{O}_K)$, for K finite extension of $\mathbb{Q}_p$.

Definition 3.0.2. *A* p-divisible *group* G *is an inductive system* $\{G_\nu, \iota_\nu\}_\nu$ *where the* G_ν *are finite group schemes over* S *of rank* $p^{\nu h}$ *and such that there is an exact sequence*

$$0 \to G_\nu \xrightarrow{\iota_\nu} G_{\nu+1} \xrightarrow{[p^\nu]} G_{\nu+1}$$

for each $\nu \geq 1$. *The integer* h *is called the* height *of the* p-*divisible group* G.

The exact sequence tells us that G_ν can be identified (via ι_ν) with the kernel of the multiplication by p^ν on $G_{\nu+1}$.

Example. Let X be an abelian scheme over S. In the above notation the kernels $X[p^n]$ form an inductive system with the inclusion map $i_n : X[p^n] \hookrightarrow X[p^{n+1}]$ and the sequence

$$0 \to X[p^n] \xrightarrow{i_n} X[p^{n+1}] \xrightarrow{[p^n]} X[p^{n+1}]$$

is exact. They form hence a p-divisible group, which we'll denote by $X[p^\infty]$.

3.1. **Properties of p-divisible groups.** Fix $\nu \geq 0$. We can compose the maps ι_ν and obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & G_\nu \arrow[r, "\iota_\nu"] & G_{\nu+1} \arrow[r, "{[p^\nu]}"] \arrow[d, "\iota_{\nu+1}"'] & G_{\nu+1} \arrow[d, "\iota_{n+1}"] & \\
 & & G_{\nu+2} \arrow[r, "{[p^n]}"] & G_{\nu+2} \arrow[r, "{[p]}"] & G_{\nu+2}
\end{tikzcd}$$

and hence G_ν can be identified with the kernel of $[p^\nu]$ on $G_{\nu+2}$ and hence, by passing to the limit, G_ν is the kernel of $[p^\nu]$ on G.

We define $\iota_{\nu,\mu} : G_\nu \to G_{\nu+\mu}$ as the iteration of

$$G_\nu \xrightarrow{\iota_n} G_{\nu+1} \xrightarrow{\iota_{\nu+1}} G_{\nu+2} \to \cdots \to G_{\nu+\mu}.$$

In particular, by induction one gets the exact sequence

$$0 \to G_\nu \xrightarrow{\iota_{\nu,\mu}} G_{\nu+\mu} \xrightarrow{[p^\nu]} G_{\nu+\mu},$$

hence through $\iota_{\nu,\mu}$ we identify G_ν with the kernel of $[p^\nu]$ on $G_{\nu+\mu}$.
Now note that $[p^{\nu+\mu}]$ factors through

$$G_{\nu+\mu} \xrightarrow{[p^\mu]} G_{\nu+\mu} \xrightarrow{[p^\nu]} G_{\nu+\mu}$$

and since $[p^{\nu+\mu}]$ kills $G_{\nu+\mu}$ we have that $[p^\mu]$ factors uniquely through G_ν. Denote by $j_{\nu,\mu}$ the map such that $[p^\mu] = \iota_{\nu,\mu} \circ j_{\nu,\mu}$.

Lemma 3.1.1. *There is an exact sequence*

$$0 \to G_\mu \xrightarrow{\iota_{\mu,\nu}} G_{\nu+\mu} \xrightarrow{j_{\nu,\mu}} G_\nu \to 0.$$

<u>Picture</u>:

$$\begin{tikzcd}
0 \arrow[dr] & & & & & \\
 & G_\mu \arrow[dr, "\iota_{\mu,\nu}"'] & & & & \\
 & & G_{\nu+\mu} \arrow[rr, "{[p^\mu]}"] \arrow[dr, "\exists! j_{\nu,\mu}"'] & & G_{\mu+\nu} \arrow[r, "{[p^\nu]}"] & G_{\nu+\mu} \\
 & & & G_\nu \arrow[ur, "\iota_{\nu,\mu}"'] \arrow[dr] & & \\
 & & & & 0 &
\end{tikzcd}$$

By recalling how $j_{\nu,\mu}$ was defined and passing to the limit, we obtain that $[p] : G \to G$ is surjective.

We can summarize the results we obtained in the following way:
The p-divisible group G is *p-torsion*, that is, $G = \varinjlim \operatorname{Ker}([p^\nu])$.
G is *p-divisible*, that is $[p] : G \to G$ is surjective. (Hence "p-divisible" means something more general, sometimes G is called a *Barsotti-Tate group.*) Note that in some texts, p-divisible group are defined as p-torsion, p-divisible groups, such that $\operatorname{Ker}[p]$ is finite group scheme. The above discussion shows that the two definitions are equivalent.

3.2. **The notion of duality.** The G_ν's are finite group schemes, hence we have a notion of duality (Cartier duality) for each ν. We want to construct a *dual p-divisible group* $G^* = \{G^*_\nu, \iota^*_\nu\}$.

Consider $j_{\nu,1} : G_\nu \to G_{\nu+1}$. By dualizing the diagram above, we obtain for every ν an exact sequence

$$0 \to G^*_\nu \xrightarrow{j^*_{\nu,1}} G^*_{\nu+1} \xrightarrow{[p^{n+1}]} G^*_{\nu+1},$$

and hence by putting $\iota^*_\nu = j_{\nu,1}$ we obtain a p-divisible group $G^* = \{G^*_\nu, \iota^*_\nu\}$. ι_ν is the dual of the multiplication by p map $G_{\nu+1} \to G_\nu$.

1. $\mu_{p^\infty} := \varinjlim_n \mu_{p^n}$, where μ_{p^n} denotes the group of p^n-th roots of unity.
2. Its dual $\mathbb{Q}_p/\mathbb{Z}_p = \varinjlim_n \mathbb{Z}/p^n\mathbb{Z}$.
3. If X is an abelian scheme, $X^*[p^\infty] \simeq X[p^\infty]^*$.

3.3. **Structure of p-divisible groups.** Suppose now $S = \operatorname{Spec} \mathcal{O}_K$. There is a general result for finite group schemes on S (We have hence $G = \operatorname{Spec} T$ for some finitely generated $\mathcal{O}_K$-algebra T).

Definition 3.3.1. *We say that $G = \operatorname{Spec} T$ is étale if it is étale as a scheme over S, that is, equivalently, T is a finite étale $\mathcal{O}_K$-algebra.*
We say that $G = \operatorname{Spec} T$ is connected, if T contains no non-trivial idempotents.

Denote by k the residue field of $\mathcal{O}_K$. Saying that $G = \operatorname{Spec} T$ is étale over S means that G is flat over S and that $T \otimes_{\mathcal{O}_K} k$ is a product of finite separable field extensions of k.

Fact. *Given $G = \operatorname{Spec} T$ finite group scheme over S, there exists an exact sequence*

$$0 \to G^0 \to G \to G^{et} \to 0,$$

where G^0 is connected and G^{et} is étale.

It turns out that both G^0 and G^{et} are affine:

$$G^0 = \operatorname{Spec} A^0, \qquad G^{et} = \operatorname{Spec} A^{et}.$$

In particular G^0 is the connected component of the identity in G (A^0 is the component of A which factors through the identity section), and A^{et} is the maximal étale sub-algebra of A (maximal separated sub-algebra).

We say that a sequence of p-divisible group is exact if it is exact at each finite level. Consider the p-divisible group $G = \{G_\nu, \iota_\nu\}$. For every ν one has

$$0 \to G_\nu^0 \to G_\nu \to G_\nu^{et} \to 0.$$

Note that the maps ι_ν induce maps on the connected and étale components and hence there exists two p-divisible groups $G^0 = \{G_\nu^0, \iota_\nu^0\}$ and $G^{et} = \{G_\nu^{et}, \iota_\nu^{et}\}$ and the following sequence of p-divisible groups is exact

$$0 \to G^0 \to G \to G^{et} \to 0.$$

3.4. **Points of a p-divisible group.** Consider R a complete $\mathscr{O}_K$-algebra (not nec. finite) and deonote by $\mathfrak{m} = (\pi S)$, where π is the uniformizer of $\mathscr{O}_K$ (check this out). Denote $\forall i, G(R/\mathfrak{m}^i) = \varinjlim_\nu G_\nu(R/\mathfrak{m}^i)$. We define the R-points of G as

$$G(R) \coloneqq \varprojlim_i G(R/\mathfrak{m}^i).$$

This is really the right definition. Note that by definition of p-divisible group, we have that

$$G_\nu(R/\mathfrak{m}^i) = \mathrm{Ker}(G(R/\mathfrak{m}^i) \xrightarrow{[p^\nu]} G(R/\mathfrak{m}^i)),$$

and hence the kernel of the multiplication by p^ν in $G(R)$ is $\varprojlim_i G_\nu(R/\mathfrak{m}^i) \simeq G_\nu(R)$. (THIS last statement has maybe something to do with formal schemes)

For every ν and for every i, $G_\nu \coloneqq \mathrm{Hom}(R/\mathfrak{m}^i, G_\nu)$ has naturally a group structure. Moreover, the transition maps ι_ν induce transition maps of groups $G_\nu(R/\mathfrak{m}^i) \to G_{\nu+1}(R/\mathfrak{m}^i)$ and hence we can compute the direct limit $\varinjlim_\nu G_\nu(R/\mathfrak{m}^i)$ (this is a finite group). By the properties of p divisible groups $G_\nu(R/\mathfrak{m}^i) = \mathrm{Ker}(G(R/\mathfrak{m}^i) \xrightarrow{[p^\nu]} G(R/\mathfrak{m}^i))$ and hence finally, the kernel of the mult by p^ν on $G(R)$ is $\varprojlim_i G_\nu(R/\mathfrak{m}^i) = G_\nu(R)$. Hence we obtain

$$G(R)_{tors} \simeq \varinjlim_\nu G_\nu(R).$$

We are particularly interested in the case $R = \mathscr{O}_{\mathbb{C}_p}$. It is not hard to see that the exact sequence

$$0 \to G^0 \to G \to G^{et} \to 0$$

induces an exact sequence

$$0 \to G^0(\mathscr{O}_{\mathbb{C}_p}) \to G(\mathscr{O}_{\mathbb{C}_p}) \to G^{et}(\mathscr{O}_{\mathbb{C}_p}) \to 0.$$

(It holds on the fact that the residue field of $\mathscr{O}_K$ is perfect: from this it follows that the exact sequence of p-divisible groups is split..)

In particular we denote the torsion $\mathscr{O}_{\mathbb{C}_p}$-points by

$$\Phi_p G = \varinjlim_\nu G_\nu(\mathscr{O}_{\mathbb{C}_p}) = G(\mathscr{O}_{\mathbb{C}_p})_{tors}.$$

Moreover, denote by

$$T_p G = \varprojlim_\nu G_\nu(\mathscr{O}_{\mathbb{C}_p}),$$

with respects to the maps $j_\nu : G_{\nu+1} \to G_\nu$. If we denote by G_K the absolute Galois group of $\mathscr{O}_K$, then $\Phi_p G$ and $T_p G$ are $\mathbb{Z}_p$-modules and there is a natural continuous action of G_K on them. T_pG is called the *Tate module* of G and Φ_pG is called the *Tate comodule*.

Proposition 3.4.1. *There is a bijection*

$$\mathrm{Hom}_{p\text{-}div/\mathscr{O}_K}(G,G') \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(T_p(G),T_p(G')).$$

Comment: In some references, the Tate module (resp. the Tate comodule) is defined as $T_pG = \varprojlim G_\nu(\mathbb{C}_p)$ (resp. $\Phi_pG := \varinjlim G_\nu(\mathbb{C}_p)$. In the case of abelian schemes this is perfectly equivalent, by the valuative creterion for properness.

3.5. **Back to abelian schemes.** Recall that given an abelian scheme X over K, we can associate naturally to this a p-divisible group $X[p^\infty]$. Recall that we had defined the Tate module of X as

$$T_pX = \varprojlim_n X(\mathbb{C}_p)[p^n].$$

Hence the Tate module of X and of $X[p^\infty]$ coincide.

Let me convince you that studying p-divisible groups attached to abelian schemes is important. Let k be the residue field of $\mathscr{O}_K$. A natural question is:
given an abelian scheme X_0 over k, can we lift it to an abelian scheme over $\mathscr{O}_K$? The first attempt would be to try to lift it to $\mathscr{O}_K/p^n\mathscr{O}_K$. There is the following theorem by Serre and Tate (1968):

Theorem 3.5.1. *Let R be a ring where p is nilpotent, define $R_0 = R/(p)$. Then there is an equivalence of categories*

$$\begin{array}{ccc} \{\textit{Abelian schemes}\} & \longrightarrow & \{(A_0,\Gamma,\epsilon)\} \\ A & \longmapsto & (A_0, A[p^\infty],\epsilon) \end{array}$$

where Γ is a p-divisible group over R, A_0 is an abelian scheme over R_0 and $\epsilon : A_0[p^\infty] \to \Gamma_{R_0}$ is an isomorphism.

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