# 189-346/377B: Number Theory

# Assignment 6

## Due: Wednesday, April 10

1. Show that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s},$$

where $\zeta(s)$ is the Riemann zeta-function, and $\Lambda : \mathbf{Z} \longrightarrow \mathbf{R}$ is the Von Mangoldt function, defined by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^r \text{ with } p \text{ prime, } r \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: Use the fact that $\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds} \log \zeta(s)$.)

*Cultural Note*: The above identity, which expresses the *logarithmic derivative* of the zeta-function as a sum of reciprocals of prime powers, weighted by the factor $\log(p)$, is the starting point for the standard proof of the Prime Number Theorem.

2. Let $q$ be an odd prime, and let $(\frac{n}{q})$ denote the Legendre symbol. Show that the function

$$L(s) = \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) n^{-s}$$

admits the factorisation over primes $p$:

$$L(s) = \prod_{p} \left(1 - \left(\frac{p}{q}\right) p^{-s}\right)^{-1}.$$

3. For all $k \geq 1$, show that the sum

$$\frac{1}{(5k+1)^s} - \frac{1}{(5k+2)^s} - \frac{1}{(5k+3)^s} + \frac{1}{(5k+4)^s}$$

is positive and bounded above by $\frac{s}{(5k+1)^{s+1}}$. Conclude that, when $q = 5$, the function $L(s)$ defined in Exercise 2 converges to a non-zero value when $s \to 1$ from the right.

4. Let $\chi : \mathbf{Z} \longrightarrow \mathbf{C}$ be the function defined by

$$\chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 5; \\ 1 & \text{if } n \equiv 1 \pmod 5; \\ i & \text{if } n \equiv 2 \pmod 5; \\ -i & \text{if } n \equiv 3 \pmod 5; \\ -1 & \text{if } n \equiv 4 \pmod 5, \end{cases}$$

and let $\bar{\chi}$ be the complex conjugate function, defined by

$$\bar{\chi}(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 5; \\ 1 & \text{if } n \equiv 1 \pmod 5; \\ -i & \text{if } n \equiv 2 \pmod 5; \\ i & \text{if } n \equiv 3 \pmod 5; \\ -1 & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

Show that the function $L(\chi, s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ satisfies the same identity as in exercise 2:

$$L(\chi, s) = \prod_p (1 - \chi(p) p^{-s})^{-1},$$

and likewise for $L(\bar{\chi}, s)$. Show that the complex-valued functions $L(\chi, s)$ and $L(\bar{\chi}, s)$ tend to non-zero complex numbers as the real variable $s$ tends to 1 from above.

* 5. Let $L(s)$ be the function defined in exercise 3. Show that there exists a real constant $C$ such that, for all $s$,

$$\left|\log\zeta(s) + \log L(s) + \log L(\chi, s) + \log L(\bar{\chi}, s) - 4 \sum_{p\equiv 1 \pmod 5} \frac{1}{p^s}\right| < C;$$

$$\left|\log\zeta(s) + \log L(s) - \log L(\chi, s) - \log L(\bar{\chi}, s) - 4 \sum_{p\equiv 4 \pmod 5} \frac{1}{p^s}\right| < C;$$

$$\left|\log\zeta(s) - \log L(s) + i\log L(\chi, s) - i\log L(\bar{\chi}, s) - 4 \sum_{p\equiv 3 \pmod 5} \frac{1}{p^s}\right| < C;$$

$$\left|\log\zeta(s) - \log L(s) - i\log L(\chi, s) + i\log L(\bar{\chi}, s) - 4 \sum_{p\equiv 2 \pmod 5} \frac{1}{p^s}\right| < C.$$

Conclude from this that there are infinitely many primes in each of the residue classes of 1, 2, 3 and 4 modulo 5.

6. The infinite sum

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{9} + \frac{1}{11} + \cdots\right),$$

is known to converge to $\pi$. Using the command `sum` in Pari, evaluate the above sum numerically (as a real number) by summing its first first 100 terms. Do the same thing for the first 1000, 100000, and 1000000 terms, comparing each time the value you obtain against the value of $\pi$ given by Pari (which is stored in the constant `Pi`). Do the same experiment with the infinite series

$$\sum_{k=0}^{\infty} \frac{2^{k+1}k!^2}{(2k+1)!},$$

which is also known to converge to $\pi$. What do you observe?

7. Write out $\sqrt{7}$ as an infinite periodic continued fraction, and write down its first few convergents. Use this calculation to find the smallest non-trivial solution of the Pell equation $x^2 - 7y^2 = 1$ by using the continued fraction method (clearly indicate all the steps that you follow).

* 8. Prove the following $p$-adic variant of Liouville's Theorem (Theorem 9.1. of the book of Leveque, as shown in class for real numbers.) Let $\alpha \in \mathbf{Z}_p$ be a $p$-adic integer satisfying an algebraic equation with integer coefficients of minimal degree $k > 1$. Then there exists a constant $C$ such that, for all integers $m \in \mathbf{Z}$,

$$\mathrm{ord}_p(\alpha - m) < C + k\frac{\log(|m|)}{\log(p)},$$

where the logarithms and absolute values on the right hand side are the usual real ones.

9. Use the theorem proved in Exercise 6 to show that the $p$-adic integer

$$1 + p + p^2 + p^6 + p^{24} + p^{120} + \cdots + p^{n!} + \cdots$$

is transcendental. (I.e., that it is not the root of a polynomial with integer coefficients.)